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On the changes of periodicities in a piecewise linear rotation model

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Abstract

Based on the results obtained for the Hicksian multiplier-accelerator model with a consumption flor in [T. Pu, L. Gardini, I. Shusko, On the change of periodicities in the Hicksian multiplier-accelerator model with a consumption floor, Chaos, Solitons & Fractals 29 (3) (2006) 681–696], in this paper we show the appearance of a change of the periodicities according to a simple rule for a similar model given by a piecewise-linear discontinuous map defined on the plane which can be faced not only in some applications to Economics, but also in more general models related to Engineering leading to border collision bifurcations.

Keywords: Discrete dynamical systems; Local bifurcations; Periodic orbits

1. Introduction

Multiplier-accelerator model for business cycles was first formulated by Samuelson [8], but later Hicks [4] suggested bounds, named "floor" and "ceiling", giving some explanations about this new event. Many others authors, like Hommes [5], Gandolfo [2], Duesemberry [1], Rau [7], Goodwin [3], have discussed about these bounds. But, summing up, the Hicksian nonlinear model can be formulated as the recurrence equation in the income Y as follows:

$$Y_t = cY_{t-1} + \max\{a(Y_{t-1} - Y_{t-2}), -D\},$$

where a, c, D represent economic parameters.

In [6] it is introduced another constraint to the model, called *consumption flor* what transforms the model into the new one

$$Y_t = \max\{cY_{t-1}, 0\} + \max\{a(Y_{t-1} - Y_{t-2}), -D\}. \quad (1)$$

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For this model, there is just one fixed point given by $Y_t = Y_{t-1} = Y_{t-2} = 0$ and the authors focus on the case where this fixed point loses stability, at $a = 1$, showing a detailed local study of what happens to small amplitude oscillations of this centre-like bifurcation.

Concretely, assuming that D is positive, in some small neighbourhood of the fixed point, the second order equation in (1) can be written as the piecewise linear two-dimensional map on the plane

$$F_1(Y_t, Z_t) : \begin{cases} Y_{t+1} = (1+c)Y_t - Z_t \\ Z_{t+1} = Y_t \end{cases} \quad \text{for } Y_t \geqslant 0$$

and

$$F_2(Y_t, Z_t) : \begin{cases} Y_{t+1} = Y_t - Z_t \\ Z_{t+1} = Y_t \end{cases} \quad \text{for } Y_t < 0$$

The functions F_1 and F_2 are both linear maps and can be represented in terms of matrices, respectively, by

$$M = \begin{pmatrix} 1+c & -1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The map F_1 produces an m/n periodic orbit from any initial condition, provided we choose c so as to result in a rational rotation number m/n,

$$c = 2\cos(2\pi m/n) - 1.$$

In this case,

$$\begin{pmatrix} 1+c & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2\cos(2\pi m/n) & -1 \\ 1 & 0 \end{pmatrix}$$

and this new matrix is called $M_{m/n}$. Observe that $R = M_{1/6}$.

In this situation, in [6] it is shown that the fixed point of the piecewise linear map F is a centre-like bifurcation point with new rotation number $2m/(6m+n)$.

This unexpected event stimulates an equivalent problem arising from two rotations in the discontinuous case. That is, it is interesting to understand if a new rotation number exists or the same persists when we have discontinuity on the border line $x = 0$, since the bifurcation value provides two different rotations on the two sides of this discontinuity line.

The question is that this is a problem which can be faced not only in some applications to Economics, but also in more general models related to Engineering (specially Electronics or Mechanics) leading to border collision bifurcations.

In this work we give some results related to this bifurcation case. It was quite surprising for us to detect the same "rotation rule" given in [6] for the discontinuous case, although only under suitable conditions. Likewise, we found out possibilities of coexistence of different periodicities (or rotation numbers). However, in this last context, after checking some different cases by computer simulation, we were not able to understand under what conditions these dynamics occur. So, we give some examples which show these different kinds of possibilities.

Concretely, we are concerned about the behaviour of the discontinuous two-dimensional piecewise-linear rotation map F which is defined by

$$F(x,y) = \begin{cases} F_1(x,y) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x \geqslant 0, \\ F_2(x,y) = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x < 0. \end{cases} \tag{2}$$

Each of the functions F_1 and F_2 in (2) corresponds to a rotation of the plane with rotations numbers α and β, respectively. For our purposes we suppose that $\alpha,\beta \leqslant \pi$ in order to have preserving orientation maps.

Let us denote

$$\alpha = 2\pi\frac{m}{n} \quad \text{and} \quad \beta = 2\pi\frac{m'}{n'}.$$

In this paper, we prove (Theorem 1) that if $\frac{m'}{n'} = \frac{1}{2k}$, $k \in \mathbb{N}^*$ and $\frac{m}{n} < \frac{m'}{n'}$, then every orbit of the piecewise rotation map is either eventually periodic or exactly periodic with rotation number

$$\frac{2m}{2km+n}.$$

The two assumptions taken above are necessary because, otherwise the result may not be true, as we show at the end of the work.

The plan of the sections in this paper is as follows. In second section, we will deal with the special case $\beta = 2\pi\frac{1}{6}$, what help to understand the generic case treated in the third section. Finally, to complete the study, in the last section we show some other particular cases in which the result does not work.

2. Special case $\beta = 2\pi\frac{1}{6}$

In this section, we consider the map F given by the expression (2) in the particular case $\beta = 2\pi\frac{1}{6}$ and prove the result using some graphics to clarify the reasoning.

Hence, we are going to show that for any $\alpha = 2\pi\frac{m}{n}$ with $\frac{m}{n} < \frac{1}{6}$, every orbit of the piecewise-linear rotation map F is either eventually periodic or (exactly) periodic with rotation number

$$\frac{2m}{6m+n}.$$

To shorten notation, we will call $M_{m/n}$ the matrix associated to F_1 and R the matrix associated to F_2.

First of all, as the movement of any point in the right half-plane is always related to the angle $2\pi\frac{m}{n}$, we are interested in the number of different zones of this amplitude in which we can divide this half plane. It is easy to check that this number is $\frac{n}{2m}$.

We will take into account this number along the proof. Concretely, we know that there exist $q, r \in \mathbb{N}$ such that

$$n = 2m \cdot q + r, \quad 0 \leqslant r < 2m. \tag{3}$$

Observe that if $r = 0$, then $\frac{m}{n} = \frac{1}{2q}$. So, we would have the even fundamental resonances. The proof in this case is not difficult, even if $\frac{m}{n} \geqslant \frac{1}{6}$, because of the reflection property of the associated matrix $M_{m/n}$, what means that

$$M^n_{m/n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

On the contrary, if $r \neq 0$, we can still divide the right half-plane in (sub-)zones of the same amplitude. Certainly, due to (3) we have q zones of amplitude $2\pi\frac{m}{n}$ and a residual zone of amplitude $2\pi\frac{r}{2n}$.

We name each zone by the number of iterations of the matrix $M_{m/n}$ needed to arrive in the left half-plane, i.e., the zone nearest to the line $x = 0$ in the Northeast quadrant will be Z_1, the next Z_2 and finally the residual zone nearest to the line $x = 0$ in the Southeast quadrant will be Z_{q+1} as schematically shown in Fig. 1.

Also, we have $2m/r$ (sub-)zones of amplitude $2\pi\frac{r}{2n}$ inside any non-residual zone Z_j of amplitude $2\pi\frac{m}{n}$ for all $j = 1, 2, \ldots, q$. In this way, we are going to distinguish between two cases:

– r divides $2m$: Then we can divide the right half-plane in $\frac{2m}{r}q + 1$ (sub-)zones of amplitude $2\pi\frac{r}{2n}$. So, we have $\frac{n}{r}$ (sub-)zones of amplitude $2\pi\frac{r}{2n}$. For each zone Z_j, $j = 1, 2, \ldots q$, we denote each (sub-)zone with the (ordinal) number in the anticlockwise sense. That is, in the residual zone Z_{q+1} we only have a subzone, coincident with Z_{q+1}, which is now denoted by the upper index 1, i.e., Z^1_{q+1}. In all the other zones we have $\frac{2m}{r}$ subzones, denoted by the upper indexes $1, 2, \ldots, \frac{2m}{r}$ as schematically shown in Fig. 2.
  As we shall see, any point of the plane, in one round or at most in two, goes to either zone Z_{q+1} or to one of the subzones Z^i_q with i between 1 and $(\frac{2m}{r} - 1)$. So, first of all, we are going to study the movement of their points, belonging to Z^1_{q+1} and Z^i_q, $i = 1, 2, \ldots, (\frac{2m}{r} - 1)$.

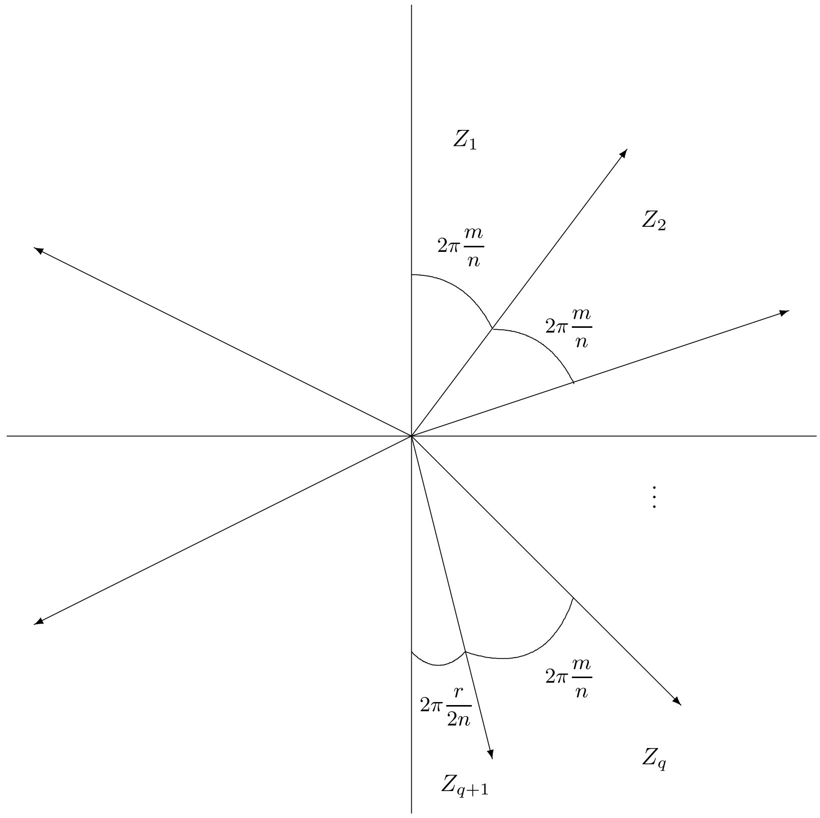

Fig. 1. Zones of the right half-plane related to the number of iterations needed to arrive in the left half-plane.

- If the initial point belongs the zone Z_{q+1}, coincident with Z^1_{q+1}, one can apply $(q+1)$-times the matrix $M_{m/n}$ and arrives in the left half-plane. Then, the matrix R applies 3-times and the point arrives in the subzone $(\frac{2m}{r}-1)$ of Z_q. Then, applying $M_{m/n}$ q-times and R 3-times, the point arrives in zone $(\frac{2m}{r}-2)$. Likewise, at each application of $M_{m/n}$ q-times and R 3-times, the point moves from the zone Z^i_q to the zone Z^{i-1}_q. Finally, we get the starting zone Z^1_{q+1} at the same point. Actually, we have that the previous movements are represented by

$$(R^3M^q_{m/n})^{2m/r-1}R^3M^{q+1}_{m/n} = -R^{3(2m/r-1)}M^{q(2m/r-1)+q+1}_{m/n} \tag{4}$$

  Note that, we can assume m,n have no common factor. Hence, as r divides n and $2m$, then r divides 2. So r can be only 1 or 2.

  * When $r=1$ the expression (4) above reduces to

$$M^{q2m+1}_{m/n} = M^n_{m/n} = I.$$

  * When $r=2$, n is even and m is odd. Therefore, the expression in (4) reads

$$-M^{qm+1}_{m/n} = -M^{n/2}_{m/n} = I.$$

- If the initial point belongs to the zone Z^i_q for $i=1,2,\ldots,(\frac{2m}{r}-1)$ the movements are similar to those in the case before. That is, in each turn, by $R^3M^q_{m/n}$, the point goes down a subzone. In some turns, it arrives in zone Z_{q+1}, where we can apply $R^3M^{q+1}_{m/n}$ and return to subzone $\frac{2m}{r}-1$. As it is easy to check, the turns around the origin are given again by R^3M^q in $(\frac{2m}{r}-1)$ and R^3M^{q+1} in one time. So, due to the property related to the matrix R, we obtain in each case the same power of the matrix $M_{m/n}$ which, as we have shown, reduces to the identity.

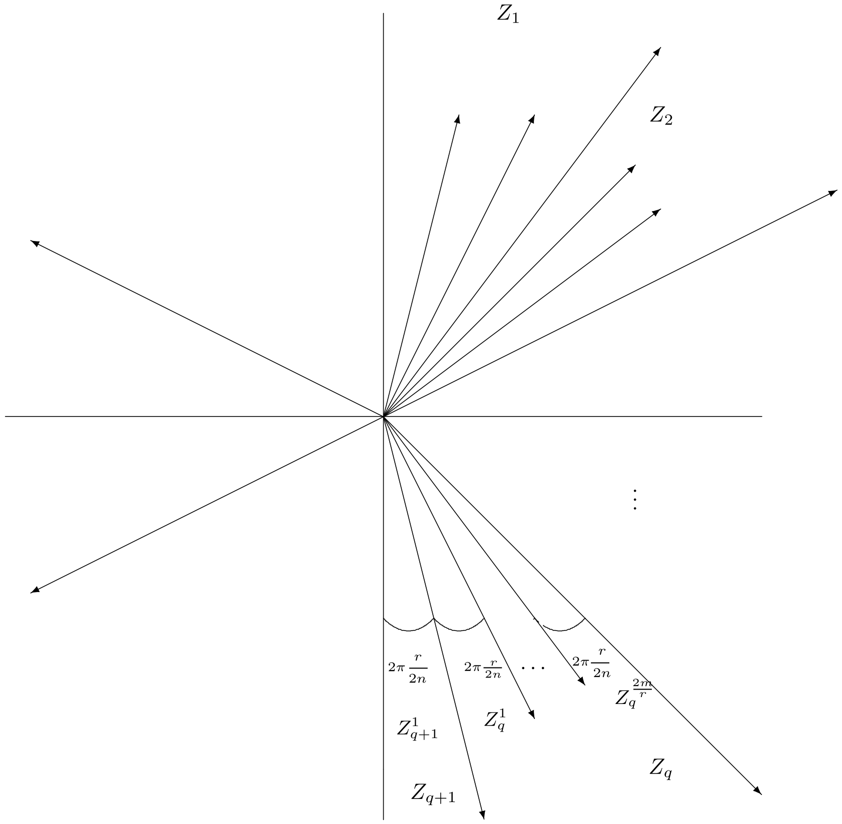

Fig. 2. Zones and subzones of the right half-plane when r divides $2m$.

- On the other hand, if the initial point belongs to some other subzone on the right half-plane, then it is the image of one of the points considered before. As those points are periodic, their images, which are in the periodic orbit, are also periodic points of the same period. In order to see that, we are going to describe precisely how the movements of these points are. If the initial point belongs to any other zone of the right half-plane, Z_j, then with the application of $R^3M^j_{m/n}$ it reaches one of the points treated before and the go-down movement begins again.
  - * If the point belongs to a zone Z^i_j for $i = 1, 2, \ldots, (\frac{2m}{r} - 1)$, the turns around the origin are given again by R^3M^q in $(\frac{2m}{r} - 2)$ times and R^3M^{q+1} in one time. After that, we can apply M^{q-j} and we return to the original point.
  - * If the point belongs subzone $\frac{2m}{r}$ of zone Z_j, then the first point we get among those studied before by means of $R^3M^j_{m/n}$ is in subzone $(\frac{2m}{r} - 1)$ inside zone Z_q. So, the turns around the origin are given now by R^3M^q in $(\frac{2m}{r} - 1)$ times and R^3M^{q+1} in no time. But, after that we can to apply M^{q+1-j} which compensate the turn with R^3M^{q+1}.

The above clarifies the dynamics of any point in the right half-plane. Now, let us consider the points belonging to the left half-plane.

It is obvious that any of these points reaches a point in the right half-plane in a finite number of iterations. So, it follows that each one is either periodic or eventually periodic (of the same period given before). In fact, the points which are images of (periodic) points of the right half-plane are periodic. The rest of them are eventually periodic, because their iterations never return to the zone where they were at the beginning, but get the right half-plane where every point is periodic. Thus, the region $x < 0$ is made up of wedges of periodic points alternated with wedges of eventually periodic ones.

The presence of eventually periodic orbits (in the left half-plane) proves that the discontinuous piecewise-linear map given in (2) cannot be topologically conjugated to a rotation with that mentioned rotation number. Differently, in the continuous case (1) considered in [6] the equivalent case is demonstrated to be topologically conjugated to a rotation.

– r does not divide $2m$: Now, we also have $2m/r$ (sub-)zones of amplitude $2\pi\frac{r}{2n}$ inside any non-residual zone of amplitude $2\pi\frac{m}{n}$. But, unfortunately, as r does not divide $2m$, $2m/r$ is not an integer number. Then the idea is to divide the zones $Z_1, Z_2, \ldots, Z_q$ and the residual zone Z_{q+1}, also in this case, in different subzones of the same amplitude. So, we have to look for a number x such that the numbers given by

$$\frac{2\pi\frac{r}{2n}}{2\pi x} \quad \text{and} \quad \frac{2\pi\frac{m}{n}}{2\pi x} \tag{5}$$

are positive integers.

We can choose $x = 1/2n$ and we will have $2m$ subzones of amplitude $2\pi\frac{1}{2n}$ inside each zone $Z_1, Z_2, \ldots, Z_q$ and r subzones of the same amplitude inside the residual zone Z_{q+1}. We are going to construct our proof taking into account those subzones in a similar way we did when r divides $2m$.

For each zone, we denote each (sub-)zone with the (ordinal) number in the anticlockwise sense. That is, in the residual zone Z_{q+1} we have r subzones, $Z^1_{q+1}, Z^2_{q+1}, \ldots, Z^r_{q+1}$, and in any non residual zone Z_j for $j = 1, 2, \ldots, q$ we have $2m$ subzones, denoted by upper indexes, $Z^1_j, Z^2_j, \ldots, Z^{2m}_j$, as it is schematically represented in Fig. 3.

As in the previous case, we have that any point of the plane (except some points of the left half-plane) goes, in the first round (these do it on the second) to either one of the subzones of Z_{q+1} or to one of the subzones of zone Z_q between 1 and $(2m - r)$. So, first of all, we are going to study the movement of their points.

- If the initial point belongs the zone Z_{q+1}, e.g. subzone Z^1_{q+1}, one can apply $(q + 1)$-times the matrix $M_{m/n}$ and the point gets the left half-plane. Then, the matrix R applies 3-times and the point arrives in the subzone $Z_q^{2m-2r+1}$ of Z_q. Then, applying $M_{m/n}$ q-times and R 3-times, the point arrives in the subzone $Z_q^{2m-3r+1}$. Likewise, applying again $M_{m/n}$ q-times and R 3-times, the point arrives in zone $Z_q^{2m-4r+1}$. After some turns, we get zone Z_{q+1} again but not in the same subzone and hence not at the same point. So, we are not in a periodic point yet and we have to continue the reasoning. As can be check, thanks to this "go-down" movement of r subzones in each turn, we have to do $\mathrm{lcm}(2m, r)/r$ turns around the origin to reach the same subzone of zone Z_{q+1}. Among these turns, $\mathrm{lcm}(2m, r)/2m$ are obtained by applying R^3M^{q+1} and the rest by R^3M^q. So, due to the property of the matrix R, we have that the previous movement can be represented by

$$(R^3M^q_{m/n})^{\mathrm{lcm}(2m,r)/r-\mathrm{lcm}(2m,r)/2m}(R^3M^{q+1}_{m/n})^{\mathrm{lcm}(2m,r)/2m} \tag{6}$$

But, we know that

$$\mathrm{lcm}(2m, r)/r = 2m/\gcd(2m, r) \quad \text{and} \quad \mathrm{lcm}(2m, r)/2m = r/\gcd(2m, r).$$

So, the expression in (6) reads

$$(R^3M^q_{m/n})^{2m/\gcd(2m,r)-r/\gcd(2m,r)}(R^3M^{q+1}_{m/n})^{r/\gcd(2m,r)} \tag{7}$$

Besides, note that we can assume m,n have no common factors. Hence, neither have r,m and $\gcd(2m, r)$ is either 1 or 2.

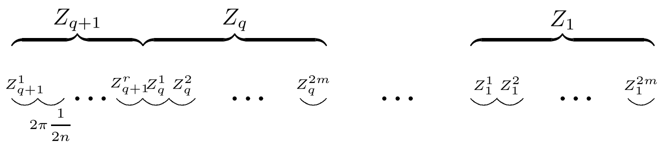

Fig. 3. Zones and subzones of the right half-plane when r does not divide $2m$.

  * When $\gcd(2m, r) = 1$, r is odd and the expression (7) reduces to

    $$(R^3 M^q_{m/n})^{2m-r} (R^3 M^{q+1}_{m/n})^r$$

    what can be simplified to

    $$(R^3)^{2m} M^{2mq+r}_{m/n} = I M^n_{m/n} = I.$$

  * When $\gcd(2m, r) = 2$, r is even and m is odd. Therefore, expression (7) reads

    $$(R^3)^m M^{mq+r/2}_{m/n} = -I M^{n/2}_{m/n} = (-I)^2 = I.$$

- If the initial point belongs zone Z^i_q with i between 1 and $(2m - r)$ the movement is similar to the case before. That is, in each turn, by $R^3 M^q_{m/n}$, the point goes down r subzones. In some turns, it arrives in zone Z_{q+1}, where we can apply $R^3 M^{q+1}_{m/n}$ and return to one of this subzones inside the zone Z_q. The turns around the origin are given again by $R^3 M^q$ in $\mathrm{lcm}(2m, r)/r - \mathrm{lcm}(2m, r)/2m$ and $R^3 M^{q+1}$ in $\mathrm{lcm}(2m, r)/2m$. So, thanks to the property related to the matrix R, we obtain in each case the same power of the matrix $M_{m/n}$ which, as we have shown, reduces to the identity.
- On the other hand, if the initial point belongs to any other zone of the right half-plane, then it is the image of one of the points treated before. So, it is periodic with the same period. In order to see that, we are going to describe briefly the movement of these points. If the initial point belongs to another zone of the right half-plane, e.g. Z_j, then, with the application of $R^3 M^j_{m/n}$, it gets one of the points treated before and the "go-down" movement begins again.
  * If the point belongs to zone Z^i_j with i between 1 and $(2m - r)$, the turns around the origin are given again by $R^3 M^q$ in $\mathrm{lcm}(2m, r)/r - \mathrm{lcm}(2m, r)/2m - 1$ times and $R^3 M^{q+1}$ in $\mathrm{lcm}(2m, r)/2m$. After that, we can apply M^{q-j} and we return to the original point.
  * If the point belongs to a subzone Z^i_j with i between $(2m - r + 1)$ and $2m$, then the first point we get, among those studied before by means of $R^3 M^j_{m/n}$, is in a subzone of Z_q between $Z^{2m-2r+1}_q$ and Z^{2m-r}_q. So, the turns around the origin are given now by $R^3 M^q$ in $\mathrm{lcm}(2m, r)/r - \mathrm{lcm}(2m, r)/2m$ times and $R^3 M^{q+1}$ in $\mathrm{lcm}(2m, r)/2m - 1$ times. But, after that, we can apply M^{q+1-j} what compensates the turn with $R^3 M^{q+1}$.

  From this reasoning it results that any point in the region $x \geqslant 0$ is periodic. This is not the case in the open half-plane $x < 0$ as we will explain below.

It is obvious that any of the points in the left half-plane reaches a point in the right half-plane in a finite number of iterations. So, it follows that each one is either periodic or eventually periodic (of the same period given before). In fact, the points which are images of the (periodic) points of the right half-plane are periodic. The rest of them are eventually periodic: their iterations never return to the zone where they were at the beginning, but get the right half-plane where every point is periodic. Thus, the region $x < 0$ is made up of wedges of periodic points alternated with wedges of eventually periodic ones.Also in this case, the presence of eventually periodic orbits (in the left half-plane) proves that the piecewise linear map F cannot be topologically conjugated to a rotation with that mentioned rotation number. However, in the continuous case considered in [6] the equivalent case is demonstrated to be topologically conjugated to a rotation.

3. The rule for the change of rotation numbers

In this section, we give the result for the generic case, summed up in the following Theorem 1.

Theorem 1. *Assuming that* $\frac{m'}{n'} = \frac{1}{2k}$, $k \in \mathbb{N}^*$ *and* $\frac{m}{n} < \frac{m'}{n'}$. *Then the fixed point of the piecewise rotation map given by* (2) *is called quasicentre-like with rotation number*

$$\frac{2m}{2km + n}$$

i.e, every orbit of the piecewise rotation map is either eventually periodic or periodic with this rotation number.

Proof. The proof made before in Section 2, for the particular case $\frac{1}{6}=\frac{1}{2\cdot 3}$, can be translated to the generic case $\frac{1}{2k}$. In fact, to prove Theorem 1 it is enough to write k in place of 3. □

The two assumptions of Theorem 1 are necessary because, as we will show in the next section, in other cases the result may be not true.

4. Other special cases

Here, we are going to provide some examples showing that the assumptions of Theorem 1 are both necessary.

When we consider a value $\frac{m}{n}>\frac{1}{2k}$, on the contrary of the continuous case given in (1), the result may not persist for our discontinuous map F expressed by (2). For example, if we consider the case $\frac{m}{n}=\frac{1}{5}$ and $\frac{1}{2k}=\frac{1}{6}$ the result is still true and the new number of rotation is $\frac{2}{11}=\frac{2m}{6m+n}$ as one can verify following the reasoning made in Section 2.

But, if we take the case $\frac{m}{n}=\frac{1}{3}$ and $\frac{m'}{n'}=\frac{1}{6}$, we find that every point is periodic or eventually periodic with rotation number 1/4 different from 2/9 which is the corresponding rotation number given by the rule.

This last example can be also used to prove that if we consider $\frac{m'}{n'}\neq\frac{1}{2k}$, the rule does not remain, even if $\frac{m}{n}<\frac{m'}{n'}$.

Although this rule does not remain in every case, when we have preserving orientation maps in both sides of the plane, the periodic points would have all the same period, as it seems by computer simulation. However, if we consider $\frac{m}{n}>\frac{1}{2}$ we have found a new interesting phenomenon: different periods can appear. This can be easily check for the case $\frac{m'}{n'}=\frac{1}{6}$ and $\frac{m}{n}=\frac{3}{4}$, where there exist periodic and eventually periodic points with rotation number 2/5 and nevertheless all the points of the form $(0,y)\in\mathbb{R}^2$, $y>0$ are periodic with rotation number 3/4.

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