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Heterogeneous Speculators and Asset Price Dynamics: Further Results from a One-Dimensional Discontinuous Piecewise-Linear Map

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Abstract In this paper we continue exploring a recently introduced financial market model in which boundedly rational agents follow technical and fundamental trading rules to determine their orders. Amongst other things, our model reveals that interactions between heterogeneous speculators can generate interesting boom-bust cycles. In addition, we provide an extensive analytical treatment of the model's underlying dynamical system, which is given by a one-dimensional discontinuous piecewise-linear map. One result is that we detect a period-adding bifurcation sequence, implying the existence of infinitely many stable cycles. Moreover, we analytically determine the parameter space that yields stable, cyclical and chaotic asset price fluctuations.

Keywords Financial crises · Bull and bear dynamics · Discontinuous piecewise smooth map · Border-collision bifurcation · Adding scheme

1 Introduction

Financial market models with heterogeneous interacting agents have proven to be quite successful in explaining the complex behavior of asset prices. For recent surveys of this burgeoning field of research see, for instance, Chiarella et al. (2009), Hommes and

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Wagener (2009), Lux (2006, 2009) and Westerhoff (2009). These models mainly focus on the interactions between traders who follow different trading strategies. As is well known from survey studies (Menkhoff and Taylor 2007) and laboratory experiments (Hommes et al. 2005), financial market participants rely on both technical and fundamental trading rules to determine their investment positions. Recall that technical trading rules try to profit from extrapolating price trends. By trading in the direction of the current price trend, these rules apparently add positive feedback to the price dynamics and are thus likely to be destabilizing. In contrast, fundamental trading rules bet on a convergence between prices and fundamental values. Since these rules tend to add a negative feedback to the price dynamics, they are often regarded as stabilizing.

Note that most models in this area are nonlinear and thereby have the potential to generate complex endogenous dynamics. In fact, many models provide sound economic arguments for a time-varying market impact of destabilizing technical and stabilizing fundamental trading rules. The dynamics of these models may—in a stylized way—evolve as follows. Suppose that technical traders, also called chartists, dominate the market close to the fundamental value. Their orders drive the price away from the fundamental value, and a bubble path is traced out. As mispricing increases, fundamentalists become increasingly active (e.g. due to higher expected trading profits). Eventually they dominate the market and their orders push prices back towards fundamental values. If mispricing is corrected, however, fundamentalists may become inactive again enabling technical traders to trigger a new bubble again. The price pattern repeats itself, typically in an intricate (chaotic) way. Some prominent models featuring this and related mechanisms include Day and Huang (1990), Chiarella (1992), De Grauwe et al. (1993), Lux (1995), Brock and Hommes (1998), Chiarella et al. (2002), Westerhoff and Dieci (2006) and Franke (2009).

In this paper, we explore an asset pricing model recently proposed by Tramontana et al. (2010b) in which the trading behavior of heterogeneous agents constitutes a simple one-dimensional discontinuous map. To be more precise, the model contains five types of agents. Besides a market maker, who adjusts prices with respect to excess demand, there are two types of technical and two types of fundamental traders, who follow their pertinent trading strategies. The reason for having two types of technical and two types of fundamental traders, a novel and distinguishing feature of this model, is that some of them determine the size of their orders using linear trading rules while others always trade the same amount of assets. It is precisely this assumption that makes the model piecewise linear and preserves its analytical tractability. The shape of the map, i.e. its two slope and two offset parameters, depends on the underlying parameter setting, which, in turn, characterizes the agents' trading behavior. Despite the simplicity of the model, it offers a surprisingly large number of interesting scenarios which may give rise to rich, fascinating and different dynamics.

Here we seek to continue this line of research. One of our goals is to investigate the dynamics of this model from an economic perspective. For instance, we try to understand the emergence of boom and bust cycle dynamics via market participants' trading decisions. Interestingly, the story we can extract from this exercise differs at least to some degree to the (standard) story outlined above. Given the danger emanating from the current financial market crisis, we believe it is quite important to improve our understanding of such disastrous boom-bust cycle phenomena. Another goal is to provide a full analytical treatment of the underlying dynamical system. Note that we are entering a new research area since such maps have not yet been thoroughly studied. We hope that our mathematical insights may prove helpful for other scholars who are studying similar dynamical systems.

Let us finally give a few final technical remarks. The two slope and two offset parameters of our map depend on the aggressiveness of the four trader types we consider in our model. Their precise meaning will be explained in later sections—here it is sufficient to realize that all four parameters are unrestricted, i.e. they can be positive or negative. Tramontana et al. (2010b)[1] focus on situations in which both branches of the piecewise-linear model have either positive or negative slopes while the offsets have opposite signs. In Tramontana et al. (2010a), we started to explore situations where the left branch of the map has a positive slope, the right branch of the map has a negative slope, and both offset parameters are positive. As it turns out, however, the dynamics of the model depend crucially on the relative size of the two offset parameters. In this paper, we continue our study by analyzing an unexplored parameter constellation. At first sight, this may appear as a rather special endeavor. However, our analysis reveals that this constellation is rather fascinating from both an economic and a mathematical perspective.

The cases already covered in Tramontana et al. (2010a) are associated with bifurcation structures of the so-called period-increment type, which are also associated with the phenomenon of bistability, i.e. there are two coexisting attractors. Only unique attractors exist for our new parameter setting. Indeed, the system is characterized by infinitely many stable cycles with periodicity regions that follow the so-called "period-adding" structures, a terminology introduced by Avrutin and Schanz (2006), Avrutin et al. (2006) and Gardini et al. (2010). Besides infinitely many stable cycles, we can also observe a convergence towards a unique steady state or chaotic asset price motion. It is worth noting that we are able to determine analytically for which parameter combinations all these dynamical features occur, i.e. we derive a more or less complete mathematical analysis of the underlying dynamical system (which can then again be interpreted in economic terms).

The main point in the analysis of non-smooth systems is the occurrence of border collision bifurcations (BCB), due to the merging (or collapse) of some invariant set (a fixed point, a periodic point of a cycle, or the boundary of any invariant set) with the kink point at which the function changes its definition. A border collision bifurcation, a term coined by Nusse and Yorke (1992) and Nusse et al. (1994), is a global bifurcation since it depends on the shape of the map on "the other side" of the collision and may lead to several interesting dynamic effects that are impossible in the framework of smooth systems. For example, the dynamics can change directly from an attracting fixed point to an attracting cycle of any period or to chaotic dynamics (Maistrenko et al. 1993, 1995, 1998; Banerjee et al. 2000).[2] Obviously, such a bifurcation can have severe economic consequences. For instance, a stable financial

[1] Other financial market models featuring piecewise linear maps include, e.g., Huang and Day (1993), Day (1997) and Huang et al. (2010).

[2] Moreover, the type of chaos in our paper may be regarded as *robust* (following Banerjee et al. 1998 since *it is persistent as a function of the parameters*.

market may turn into a highly volatile market if speculators change their behavior slightly.

Work associated with discontinuous maps commenced several years ago, and some results have recently been rediscovered. We mention, for example, Mira (1978, 1987), and Gardini et al. (2010). In Avrutin et al. (2010) the authors apply and extend the pioneering works of Leonov, conducted as early as at the end of the 1950s (Leonov 1959, 1962).

Our work is organized as follows. In Sect. 2, we present our financial market model. In Sect. 3, we recall some of our previous results and contrast them with our new findings. In Sect. 4, we analytically establish the BCB curves associated with the period-adding structure, which gives rise to *infinitely many periodicity regions of stable cycles, and show that no coexistence of stable cycles can occur*. Moreover, we shall see that, in the parameter space, a particular set (whose equation is explicitly known) separates the region of regular dynamics (without chaotic behavior) from that of only chaotic behavior. In Sect. 5, we discuss how the model functions and the emergence of boom-bust cycles from an economic perspective. Section 6 concludes.

2 The Financial Market Model

In this section, we recapitulate the model proposed in Tramontana et al. (2010b), which gives rise to a simple one-dimensional discontinuous map. In addition, we clarify the economic meaning of our underlying parameter setting, which is responsible for the shape of the map discussed later in this paper. Overall, the model contains five types of agents: a market maker, two types of chartists and two types of fundamentalists. The main decisive features of our model are as follows. First, some agents (called type 1 chartists and type 1 fundamentalists) determine their orders by applying linear trading strategies while other agents (called type 2 chartists and type 2 fundamentalists) always trade fixed amounts of assets. Second, the agents' trading intensities depend on whether the market is over- or undervalued. The remaining building blocks of the model, describing the (general) behavior of the market participants, are standard: market makers mediate transactions out of equilibrium and adjust prices, chartists chase price trends and fundamentalists place orders on mean reversion.

Let us start with the market maker. As usual, the market maker collects all individual orders from traders and changes prices with a view to excess demand. For instance, if buying orders exceed selling orders, the market maker increases the price (and vice versa). For this reason, the log of price P for period $t+1$ is quoted as

$$P_{t+1} = P_t + a(D_t^{C,1} + D_t^{C,2} + D_t^{F,1} + D_t^{F,2}), \tag{1}$$

where a is a positive price adjustment parameter, $D_t^{C,1}$ and $D_t^{C,2}$ are the orders of the two types of chartists, and $D_t^{F,1}$ and $D_t^{F,2}$ are the orders of the two types of fundamentalists, respectively. For simplicity, we set $a = 1$. Given that a is a scaling parameter, this assumption goes without loss of generality.

Chartists believe in the persistence of bull and bear markets. They therefore optimistically (pessimistically) buy (sell) assets if the current asset price is above (below) its fundamental value. Let F be the log of the fundamental value. Then the orders placed by type 1 chartists are formalized as

$$D_t^{C,1} = \begin{cases} c^{1,a}(P_t - F) & \text{if} \quad P_t - F > 0 \\ c^{1,b}(P_t - F) & \text{if} \quad P_t - F < 0 \end{cases}, \tag{2}$$

where $c^{1,a}$ and $c^{1,b}$ are positive reaction parameters, indicating how aggressively type 1 chartists react to observed trading signals. The orders placed by type 2 chartists are captured by

$$D_t^{C,2} = \begin{cases} c^{2,a} & \text{if} \quad P_t - F > 0 \\ -c^{2,b} & \text{if} \quad P_t - F < 0 \end{cases}, \tag{3}$$

where $c^{2,a} > 0$ and $c^{2,b} > 0$ indicate the amount of traded assets. For instance, in a bull market, type 1 chartists ask for $c^{1,a}(P_t - F)$ assets while type 2 chartists demand $c^{2,a}$ assets.

Fundamentalists believe that prices may disconnect from their fundamental values in the short run but that there is some exploitable mean reversion tendency in the long run. They therefore bet on a convergence between prices and fundamental values. The orders placed by type 1 fundamentalists are given as

$$D_t^{F,1} = \begin{cases} f^{1,a}(F - P_t) & \text{if} \quad F - P_t > 0 \\ f^{1,b}(F - P_t) & \text{if} \quad F - P_t < 0 \end{cases}, \tag{4}$$

where $f^{1,a}$ and $f^{1,b}$ are positive reaction parameters. The orders placed by type 2 fundamentalists are expressed as

$$D_t^{F,2} = \begin{cases} f^{2,a} & \text{if} \quad F - P_t > 0 \\ -f^{2,b} & \text{if} \quad F - P_t < 0 \end{cases}, \tag{5}$$

where $f^{2,a}$ and $f^{2,b}$ are positive reaction parameters. Both types of fundamentalists submit buying (selling) orders when the market is undervalued (overvalued). However, type 1 fundamentalists increase their order size with the observed mispricing while type 2 fundamentalists trade fixed amounts of assets. Note also that fundamentalists' beliefs in the future direction of prices are exactly opposite to what chartists expect, a powerful simplifying model assumption, going back to the pioneering work of Day and Huang (1990), and recently empirically supported by Boswijk et al. (2007) and Westerhoff and Franke (2010).

Although we need eight parameters to describe the behavior of the four different groups of speculators, the model's dynamical system can conveniently be simplified. For this reason, let us first define $s_R = 1 + c^{1,a} - f^{1,b}$, $s_L = 1 + c^{1,b} - f^{1,a}$,

$m_R = c^{2,a} - f^{2,b}$ and $m_L = f^{2,a} - c^{2,b}$. Introducing also the auxiliary variable $\widetilde{P}_t = P_t - F$, the financial market model can be expressed in terms of deviations from fundamental values. Rearranging (1) to (5) and making use of our definitions yields

$$\widetilde{P}_{t+1} = \begin{cases} s_R \widetilde{P}_t + m_R & \text{if} \quad \widetilde{P}_t > 0 \\ s_L \widetilde{P}_t + m_L & \text{if} \quad \widetilde{P}_t < 0 \end{cases}. \tag{6}$$

Finally, using $x' = \widetilde{P}_{t+1}$ and $x = \widetilde{P}_t$, we obtain:

$$x' = T(x) = \begin{cases} f_L(x) = s_L x + m_L & \text{if} \quad x < 0 \\ f_R(x) = s_R x + m_R & \text{if} \quad x > 0 \end{cases} \tag{7}$$

which comprises a family of one-dimensional discontinuous piecewise-linear maps.

The shape and thus the dynamics of (7) depends crucially on the size of the four (aggregated) slope and offset parameters. There is a surprisingly large number of different scenarios associated with (7), leading to intricate dynamics, a few of which have already been covered. In Tramontana et al. (2010b), we focus on a setup in which type 1 chartists are always dominated by type 1 fundamentalists while simultaneously type 2 chartists always dominate type 2 fundamentalists (or viceversa). Formally, this implies that the two slope parameters are either both positive or both negative, and that the offsets have opposite signs.

In Tramontana et al. (2010a) and in this contribution, we break with this kind of symmetry. Now the dominance of one trader type over the other trader type and the "relative size of dominance" depends on economic circumstances, that is, whether the market is in a bear or in a bull state. To be precise, the focus of this paper with respect to the slope parameters is on $s_R < 0 < s_L < 1$. Accordingly, in the bear market the aggressiveness of type 1 fundamentalists is "slightly" higher than the aggressiveness of type 1 chartists (such that $0 < s_L < 1$) while in the bull market it is "much" higher (and such that $s_R < 0$). Moreover, we assume that both offset parameters are positive, which means that in the bear market, type 2 fundamentalists dominate type 2 chartists while in the bull market the opposite occurs.

With respect to the offset parameters, we can furthermore distinguish three subcases: (i) $m_L \geq m_R > 0$, (ii) $0 < x_R^* \leq m_L < m_R$, and (iii) $0 < m_L < x_R^* < m_R$, where $x_R^* = m_R/(1 - s_R)$ stands for the unique fixed point of our model, located in the right side of the map (a derivation will be presented in the next section). The first two subcases were investigated in Tramontana et al. (2010a). Note that $m_L > m_R$ indicates that the dominance of type 2 fundamentalists over type 2 chartists in the bear market may be regarded as larger than the dominance of type 2 chartists over type 2 fundamentalists in the bull market. Of course, $m_L < m_R$ implies the opposite. However, in the latter case it is relevant for the dynamics whether m_L is smaller or larger than x_R^*. Since we assume in this paper that $0 < m_L < x_R^* < m_R$, this corresponds to a situation in which the offset m_L (resp. m_R) is lower (resp. higher) of a certain critical level (x_R^*).

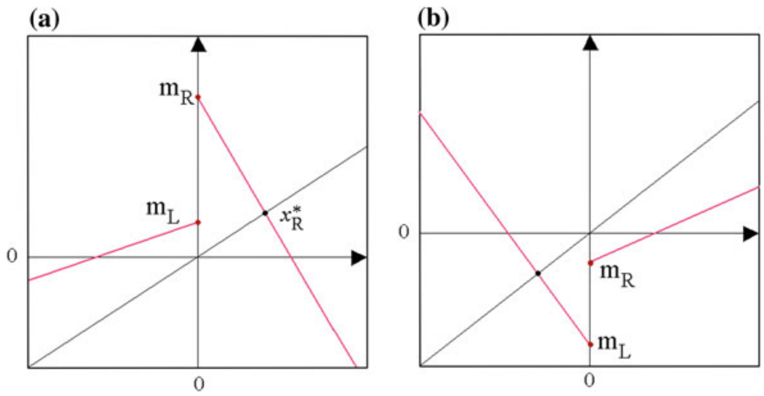

Fig. 1 Qualitative shape of the discontinuous map

3 Some Properties of Our Model

The family of maps considered in this paper is indicated by (7), and the restrictions we impose on our parameters are given by:

$$s_R < 0 < s_L < 1 \quad \text{and} \quad 0 < m_L < x_R^* < m_R, \tag{8}$$

where $x_R^* = m_R/(1 - s_R)$ stands for the fixed point of our model. Accordingly, we have an increasing straight line for $x < 0$, a decreasing straight line for $x > 0$, and the offsets of both branches of the map are positive. One example of the shape of such a map is shown in Fig. 1a, which also enables us to identify what kind of asset price dynamics we can, in principle, expect from our model. Since the slope of the left branch is limited between 0 and 1, there is no equilibrium on the left side and the iterated points are pushed upwards and eventually enter the right side. On the right side, however, the slope is negative and—if the fixed point on the right side is unstable—the trajectory is forced to return to the left side, after a finite number of turns around the unstable fixed point. Then, again on the left side, an increasing price sequence will recommence and the pattern repeats itself. We can also see that price movements are always bounded in a natural way, which makes perfect sense from an economic perspective. We do not observe any exploding price trajectories in either our model or in real markets. Hence, the model appears to be promising with respect to explaining boom-bust cycles and excessively volatile prices, as observed in many real markets.

From Fig. 1a it can be seen again that if we relax the assumption about parameter m_L, we can discriminate the three different cases (i), (ii) and (iii) mentioned in the previous section. Our attention is on case (iii), i.e. the offset of the left branch is positive but located below the fixed point of the model. Before we continue, it is worth pointing out that the results and properties determined in the following also hold when the shape of the map is as depicted in Fig. 1b, due to the symmetry property of $f(x)$, we have that $f(x, s_R, s_L, m_L, m_R) = -f(-x, s_L, s_R, -m_R, -m_L)$.

The equilibrium of our model, determined via $f_R(x_R^*) = x_R^*$, and given with $x_R^* = \frac{m_R}{1-s_R} > 0$, is obviously attracting for $-1 < s_R < 0$. A degenerate flip bifurcation[3] occurs when $s_R = -1$. For $s_R < -1$ a cycle of period 2 must be of symbol sequence LR. We determined this cycle and its bifurcation in cases (i) and (ii). However, such a cycle cannot exist in case (iii). Since $f_L(0) = m_L < x_R^*$, at least two iterations of map f_R are necessary to reach the left side again. Hence, the minimum period for a cycle in case (iii) is 3, with symbol sequence LR^2.

Due to the simplicity of the model, it is also possible to compute the eigenvalue associated with a given cycle. In fact, a periodic orbit with period $k = p + q$, with p points on the L side and q points on the R side, has the eigenvalue $\lambda = s_L^p s_R^q$. For example, the eigenvalue of a 3-cycle with sequence LR^2 is given by $\lambda = s_L s_R^2$.

Let us briefly sketch the main bifurcation scenarios of cases (i), (ii) and (iii). We set $m_R = 3$ and $m_L = 0.15$ in the following. However, it should be noted that our results are generally valid, regardless of the selected numerical values, as long as the main parameter restrictions are met.

In the two-dimensional bifurcation diagram in the parameter plane (s_R, s_L) shown in Fig. 2, we can easily identify two typical bifurcation scenarios. On the one hand, we observe an increasing sequence of periodicity regions of stable $k-$ cycles, for any integer $k \geq 1$ of type $L^k R$, with period increment by 1, belonging to the parameter region of case (ii). On the other hand, in Fig. 2b, which is an enlargement of the right part of Fig. 2a, we have the parameter region belonging to case (iii). For the assumed values of m_L and m_R, the region belonging to case (iii) is the strip between $-19 < s_R < 0$. There is an infinite sequence of periodicity regions of stable cycles, whose periods and periodicity regions follow a period-adding scheme, which can be identified via the *Farey rule*. In other words, between two contiguous periodicity regions of associated with cycles of periods k_1 and $k_2 = k_1 + 1$ another periodicity region associated with a cycle of period $p = k_1 + k_2$ also exists. Moreover, no coexistence of cycles can occur.

In a piecewise linear map, the appearance/disappearance of a cycle can occur only via a border collision bifurcation, so that the boundaries of the existence region of a cycle is given by BCB curves associated with the collision of one periodic point of the cycle with the discontinuity point (here $x = 0$). That is, one boundary is due to the collision of a periodic point of the cycle with $x = 0$ from the right side and the other boundary via a collision from the left side.

The basic cycles shown in Fig. 2b have the symbol sequence $L^k R^2$ for any integer $k \geq 1$, and these cycles are of the so-called first complexity level. For this period-adding scenario we can use a relatively new technique to determine analytically the bifurcation curves. This techniques stems from an idea introduced by Leonov (1959, 1962). It has recently been improved by Gardini et al. (2010) to get an iterative process to calculate families of BCB curves, as we shall see in the next section. The white region above the curve denoted by (S) (explicitly given in Sect. 4) denotes chaos.[4]

Figure 3 shows a two-dimensional bifurcation diagram in the parameter plane (s_R, m_L) from which we can identify two other typical bifurcation scenarios. The

[3] For the definition and properties we refer to Sushko and Gardini (2010).

[4] Here chaos means that there is some invariant set of cyclic intervals having dense unstable periodic points and dense aperiodic trajectories.

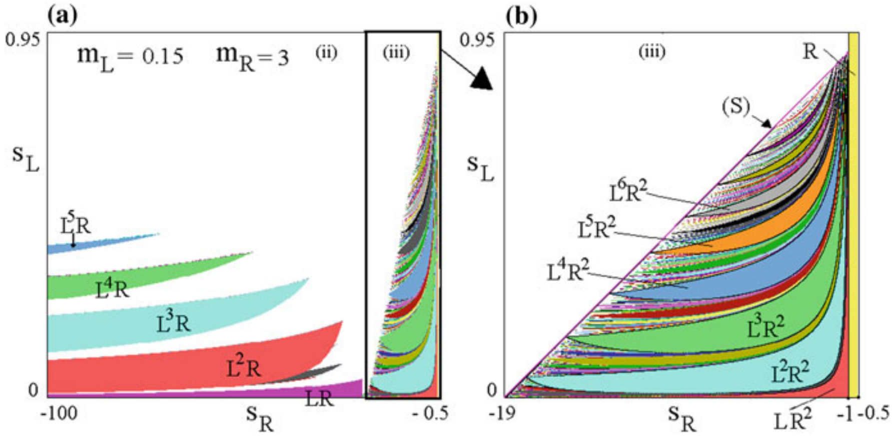

Fig. 2 Two-dimensional bifurcation diagram and its enlarged portion. Different colors (or *grey* tonalities) correspond to stable periodic orbits of different periods

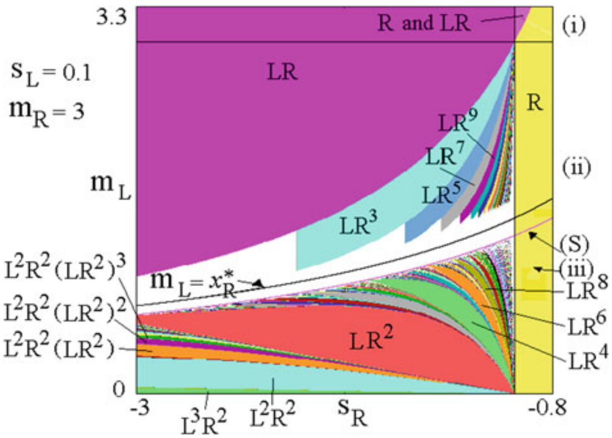

Fig. 3 Two-dimensional bifurcation diagram. Different colors (or *grey* tonalities) correspond to stable periodic orbits of different periods

black curve, given by $m_L = x_R^*$ (that is $m_L = m_R/(1 - s_R)$), separates the region of case (iii) (below it) from case (ii) (above it). The parameter space above line $m_L = 3$ belongs to case (i). In the parameter region belonging to case (ii) we can see an increasing sequence of periodicity regions of cycles of even periods of type LR^{2k+1} for any integer $k \geq 0$, with period increment by 2 (on the R side).[5] The white region belonging to case (iii), between the curve of equation $m_L = x_R^*$ and the curve denoted by (S), represents the parameter space associated with chaotic dynamics. Below curve (S), we can see another period adding scheme, now associated with basic cycles of first complexity level with the symbol sequence LR^{2k} for any integer $k \geq 1$. Elements of the family $L^k R^2$ are also visible at the bottom of the figure.

[5] Despite being invisible, there are regions of bistability between two consecutive cycles.

The analytical equations of the BCB curves associated with the period increment scenarios of cases (i) and (ii) can be found in Tramontana et al. (2010a). In the next section, we determine the analytical BCB curves of the periodicity regions with respect to the period-adding scheme occurring in case (iii).

4 Period-Adding Scheme

The peculiar thing about case (iii), besides the chaotic region, is the region with many stable cycles and the so-called period-adding scheme. In this section we show how to obtain the analytical expressions of the surfaces that, in the parameter space, separate regions characterizing cycles with different periodicities.

4.1 Periodic Orbits of First Complexity Level

Let us first consider Fig. 2b to detect the periodic orbits of first complexity level from which the period-adding scheme can be started. By assumption, we have $m_L < x_R^*$, so that at the bifurcation value (of a point colliding with $x = 0$ from the left side) we have $f_L(0) = m_L < x_R^*$. The existent cycle therefore starts with a periodic point that must do at least two turns around the unstable fixed point before reaching the L side again. That is, such cycles have the symbolic sequence given by $L^k R^2$, for $k \geq 1$. Let us call x_0 the point of the cycle immediately to the left of the discontinuity point $x = 0$. Then the periodic point x_0 of the orbit of symbolic sequence $L^k R^2$ can be obtained by looking for the fixed point of the function ${f_L}^{k-1} \circ f_R^2 \circ f_L(x)$, that is, by solving for ${f_L}^{k-1} \circ f_R^2 \circ f_L(x) = x$. From:

$$\begin{aligned} f_L(x) &= s_L x + m_L \\ f_R^2 \circ f_L(x) &= s_R^2 s_L x + s_R^2 m_L + s_R m_R + m_R \\ {f_L}^{k-1} \circ f_R^2 \circ f_L(x) &= s_L^{k-1}[s_R^2 s_L x + s_R^2 m_L + s_R m_R + m_R] + m_L \frac{1 - s_L^{k-1}}{1 - s_L} \end{aligned}$$

we have

$$x_0 = \frac{s_L^{k-1}}{1 - s_L^k s_R^2}[s_R^2 m_L + s_R m_R + m_R + m_L \phi_{k-1}^L], \tag{9}$$

where $\phi_{k-1}^L = \dfrac{1 - s_L^{k-1}}{s_L^{k-1}(1 - s_L)}$. By setting $x_0 = 0$ we have:

$$BCB_{L^k R^2}^l : s_R = \frac{1}{2m_L}\left[-m_R \pm \sqrt{m_R^2 - 4m_L(m_R + m_L \phi_{k-1}^L)}\right]. \tag{10}$$

Both branches, due to the ± components, are used to draw the BCB curves in Fig. 4, determining the lower boundary of the periodicity regions shown in the stable regime

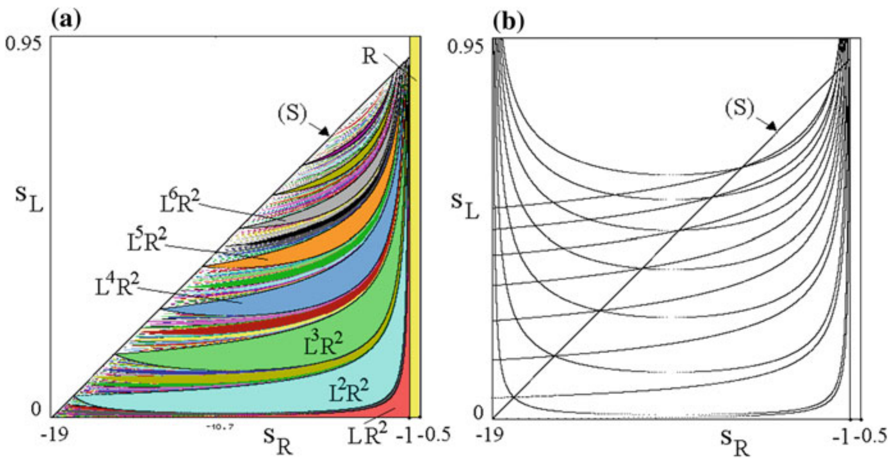

Fig. 4 Numerical periodicity regions in **a** and analytic BCB curves in **b**

(the right side with respect to set (S)), and the upper boundary in the unstable region (on the left side of the locus (S)).

Then such a cycle exists as long as the periodic point which we have called x_0, the first periodic point on the left side of the discontinuity point $x = 0$, merges with the preimage of the origin on the left side, i.e. point $x^L_{-1} = f_L^{-1}(0) = -\frac{m_L}{s_L}$ (and this condition also corresponds to the merging with $x = 0$ of the periodic point on the right side closest to $x = 0$).

The other BCB curves, causing the disappearance of the cycles, are obtained using the following equation:

$$x_0 = -\frac{m_L}{s_L}$$

$$\frac{s_L^{k-1}}{1 - s_L^k s_R^2}[s_R^2 m_L + s_R m_R + m_R + m_L \phi^L_{k-1}] = -\frac{m_L}{s_L}, \tag{11}$$

which leads to the following BCB curves:

$$BCB^r_{L^k R^2} : s_R = -1 - \frac{m_L}{m_R s_L^k} - \frac{m_L}{m_R}\phi^L_{k-1} \tag{12}$$

A few of these curves (for $k = 1, \ldots, 8$) are shown in Fig. 4, bounding the regions for the existence of cycles $L^k R^2$.

We can see from the same figure that the BCB curves bounding the regions of the cycles $L^k R^2$ intersect each other on a straight line of equation

$$(S) : m_L(1 - s_R) - m_R(1 - s_L) = 0$$

This bring us to the following property:

Property 1 *(S) is the locus in which the eigenvalues of all cycles become equal to* 1.

For example, let us consider the intersection point of the BCB curves of equations given in (10) and (12), bounding the existence region of cycle $L^k R^2$ (whose eigenvalue is given by $\lambda = s_L^k s_R^2$). Parameters that satisfy (10) are such that (from (9)):

$$s_R^2 m_L + s_R m_R + m_R + m_L \phi_{k-1}^L = 0$$
$$s_R^2 m_L + s_R m_R = -m_R - m_L \phi_{k-1}^L$$
$$s_R^2 \frac{m_L}{m_R} + s_R = -1 - \frac{m_L}{m_R} \phi_{k-1}^L$$

and substituting into (12) we obtain:

$$s_R = -1 - \frac{m_L}{m_R s_L^k} - \frac{m_L}{m_R} \phi_{k-1}^L$$
$$s_R = -\frac{m_L}{m_R s_L^k} + s_R^2 \frac{m_L}{m_R} + s_R$$
$$s_L^k s_R^2 = 1$$

This proves the property for the cycles of the first complexity level. The statement holds also for the BCB curves of the other periodicity regions, proved in a similar manner, by using the analytical expressions of the BCB curves computed via the Leonov approach, as described in the next subsection.

4.2 Periodic Orbits of Higher Complexity Levels

As can be seen from the bifurcation diagrams (and as can be also rigorously proved), the periodicity regions in which stable cycles $L^k R^2$ exist are disjoint, and there are cycles with different periods in between. The Farey rule also works here. Let us remark that in the description of the periodicity regions we can allocate a number to each region, which may be called *rotation number*, in order to classify all periods and several cycles with the same period. In this notation, a periodic orbit of period k is characterized not only by the period but also by the number of points in the two branches separated by the discontinuity point $x = 0$, already denoted by L and R, respectively. We can say that a cycle has a rotation number $\frac{q}{k}$ if a k-cycle has q points on the R side and the others $(k - q)$ on the L side. Then, between any pair of periodicity regions associated with the rotation numbers $\frac{q_1}{k_1}$ and $\frac{q_2}{k_2}$ there also exists the periodicity region associated with the rotation number $\frac{q_1}{k_1} \oplus \frac{q_2}{k_2} = \frac{q_1+q_2}{k_1+k_2}$, where $\oplus$ stands for the so-called Farey composition rule, or summation rule (see, for example, Hao 1989).

Then, by using a technique already proposed in Leonov (1959, 1962) and Mira (1978), (see also Mira 1987, pp. 56–61 and pp. 80–84), we call *regions of first level of complexity* those associated with the basic cycles $L^k R^2$ for $k \geq 1$. Between any pair of consecutive *regions of first level of complexity*, say with rotation numbers $\frac{2}{k_1}$ and $\frac{2}{k_1+1}$, we can then construct *two infinite families* of periodicity regions, called *regions of second level of complexity*, via the sequence obtained by adding using the Farey composition rule $\oplus$ iteratively the first one or the second one, i.e. $\frac{2}{k_1+1} \oplus \frac{2}{k_1} = \frac{4}{2k_1+1}$, $\frac{4}{2k_1+1} \oplus \frac{2}{k_1} = \frac{6}{3k_1+1}$, $\frac{6}{3k_1+1} \oplus \frac{2}{k_1} = \frac{8}{4k_1+1}$, …and so on, that is:

$$\frac{2q}{qk_1+1} \quad \text{for any } q > 1$$

and $\frac{2}{k_1} \oplus \frac{2}{k_1+1} = \frac{4}{2k_1+1}$, $\frac{4}{2k_1+1} \oplus \frac{2}{k_1+1} = \frac{6}{3k_1+2}$, $\frac{6}{3k_1+2} \oplus \frac{2}{k_1+1} = \frac{8}{4k_1+3}$ …, that is:

$$\frac{2q}{qk_1+n-1} \quad \text{for any } q > 1,$$

which give two sequences of regions accumulating on the boundaries of the two starting sequences.

Clearly, this mechanism can be repeated: we can construct two infinite families of periodicity regions, called *regions of third level of complexity*, between any pair of contiguous *regions of second level of complexity*, for example $\frac{2q}{qk_1+1}$ and $\frac{2(q+1)}{(q+1)k_1+1}$, the sequence obtained by adding using the composition rule $\oplus$ iteratively the first one or the second one, and so on. In this way, we obtain all of the infinitely many periodicity regions.

We notice that, although we see periodicity regions filling the section on the right side of the set (S) up to the stability region of the fixed point, the region is not filled by the existence of periodicity regions. Some curves in between are left, the complementary set, which is a set of zero Lebesque measure. Quasiperiodic trajectories (not chaotic, as no Cantor set of points can exist) correspond to such values of the parameters. Similarly, for parameters on the set (S): at a point belonging to the intersection of two BCB curves, the map is conjugated with a linear rotation with rational rotation number. In the residual set of parameter values, the map is conjugated with a linear rotation, which has an irrational rotation number.

Hence, set (S) denotes the change of stability of all cycles on the right side of the set: although these cycles also exist on the left side, between the curves with the same equations given in (10) and (12), they are unstable.

4.3 The Leonov Technique

The Leonov technique for maps with positive slopes, which has been improved by Gardini et al. (2010) (and extended by Avrutin et al. (2010), can also be used in our context to get an iterative map in the coefficients. This also leads to the analytical equations of the border collision bifurcation curves of second complexity level and further levels. To demonstrate the application of the process, it suffices to notice that, locally, we are in the same situation. If we consider a parameter point between two consecutive periodicity regions of cycles of periods $L^k R^2$ and $L^{k+1} R^2$, in a neighborhood of the origin the graph of function $F_L(x) = f_L{}^k \circ f_R^2 \circ f_L(x)$ for $x < 0$ and the graph of function $F_R(x) = f_L{}^k \circ f_R^2(x)$ for $x > 0$ is that shown in Fig. 5, which is the standard situation in which the adding scheme works. Thus, considering map

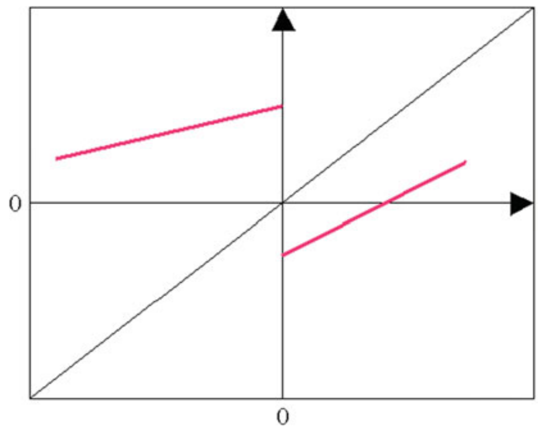

Fig. 5 Qualitative shape of the iterated functions in a neighborhood of the origin, $F_L(x) = f_L{}^k \circ f_R^2 \circ f_L(x)$ for $x < 0$ and $F_R(x) = f_L{}^k \circ f_R^2(x)$ for $x > 0$

$F(x)$ as defined accordingly, we can apply the iterative process described by Gardini et al. (2010).

That is, consider *the operator for the coefficients* defined by

$$x' = F(x) = \begin{cases} F_L(x) = A_L x + M_L, & \text{if} \quad x < 0 \\ F_R(x) = A_R x + M_R, & \text{if} \quad x > 0 \end{cases} \tag{13}$$

where, to determine the BCB curves of the second level, we consider $F_L(x) = f_L{}^k \circ f_R^2 \circ f_L(x)$ and $F_R(x) = f_L{}^k \circ f_R^2(x)$, so that we have

$$\begin{aligned} A_L &= s_L^{k+1} s_R^2 \\ A_R &= s_L^k s_R^2 \\ M_L &= s_L^k [s_R^2 m_L + s_R m_R + m_R + m_L \phi_{k-1}^L + m_L \phi_k^L] \\ M_R &= s_L^k [s_R m_R + m_R + m_L \phi_{k-1}^L + m_L \phi_k^L] \end{aligned} \tag{14}$$

We then obtain one second-level family by considering functions $F_R^n \circ F_L(x) = A_R^n A_L x + M_L A_R^n + M_R \frac{1-A_R^n}{1-A_R}$ for $n \geq 1$. The periodic point x^* of $T(x)$ (of the cycle with symbolic sequence $(L^k R^2)^n \, L^{k+1} R^2$), which is the first on the left of the origin, is given by:

$$M_R \leq x^* = \frac{1}{1 - A_R^n A_L} \left[M_L A_R^n + M_R \frac{1 - A_R^n}{1 - A_R} \right] \leq 0, \tag{15}$$

and we obtain the BCB curves from equations $M_R = x^*$ and $x^* = 0$.

The second family is obtained in a similar manner, by considering functions $F_L^n \circ F_R(x) = A_L^n A_R x + M_R A_L^n + M_L \frac{1-A_L^n}{1-A_L}$ for $n \geq 1$. The periodic point x^* of $T(x)$ (of the cycle with symbolic sequence $(L^{k+1} R^2)^n \, L^k R^2$), which is the first on the right of the origin, is given by:

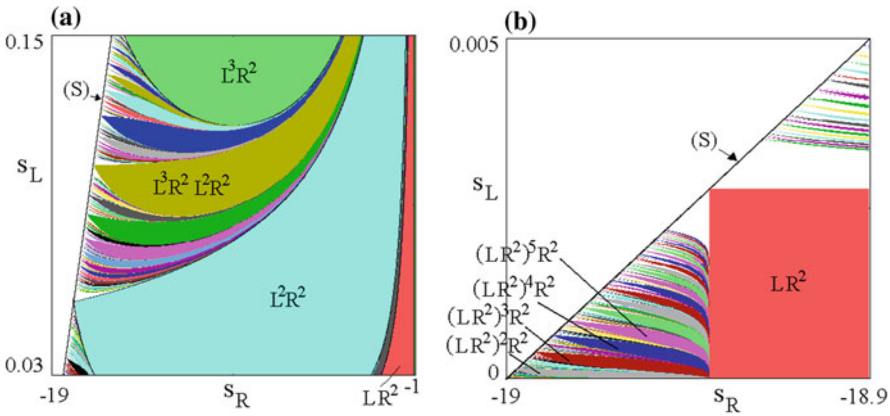

Fig. 6 In **a** periodicity regions of the second complexity level. In **b** enlarged part of the leftmost corner of Fig. 2b

$$M_L \geq x^* = \frac{1}{1 - A_L^n A_R}\left[M_R A_L^n + M_L \frac{1 - A_L^n}{1 - A_L}\right] \geq 0 \tag{16}$$

and we have the BCB curves from equations $M_L = x^*$ and $x^* = 0$.

The two second-level families can be seen in the enlargement of Fig. 6a for $k = 2$, the first one accumulating to the periodicity region of cycle $L^k R^2$ and the second family accumulating on $L^{k+1} R^2$, and so on, iteratively. We can construct two infinite sequences of periodicity regions between any two pairs of consecutive regions in a similar manner.

Moreover, as we have seen from Fig. 3, there are other families of stable regions following the period-adding scheme, all of which can be detected using the procedure described in this section. Another example is in the enlarged part of the leftmost corner of Fig. 2b, shown in Fig. 6b: it reveals that a sequence of infinitely many regions also exists below the periodicity region of the 3− cycle LR^2, with cycles of the first complexity level which have the symbol sequence $(LR^2)^k R^2$ for $k \geq 1$, and the related adding scheme.

4.4 The Locus S

We note that the existence of set (S) and its special role has already been described by Gardini et al. (2010), associated with the same map, but in a regime with positive slopes only, in which the adding scheme applies to the periodicity regions of principal (or maximal) cycles. In that case, it was a separator between regions with only stable cycles or quasiperiodic orbits or only chaos. Set (S) plays the same role in our case (iii), since all cycles are stable on one side of (S) and all are unstable on the other side. We can thus conclude that *on one side of (S) we have stable cycles or quasiperiodic trajectories; on the other side of (S) we have robust chaos*.

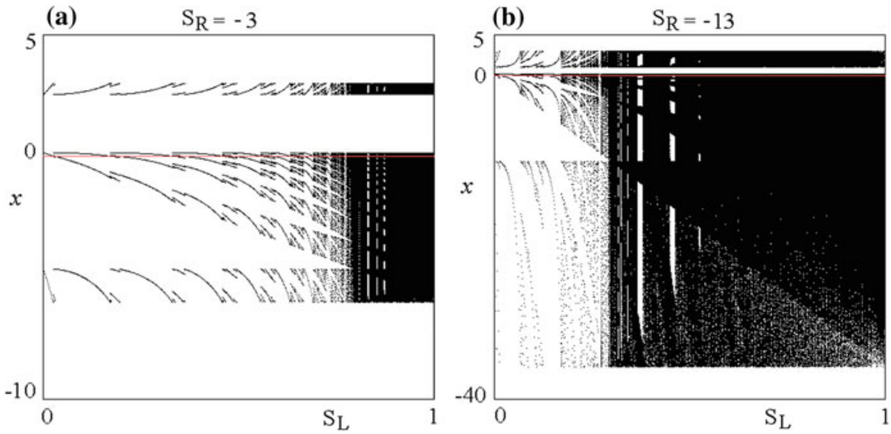

Fig. 7 Two bifurcation diagrams at fixed values $m_L = 0.15$ and $m_R = 3$. In **a** $s_R = -3$; in **b** $s_R = -13$

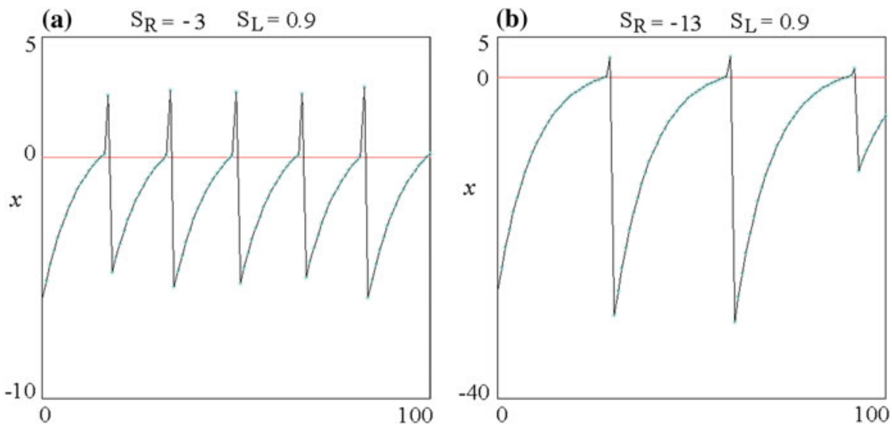

Fig. 8 Versus time trajectories at fixed values $m_L = 0.15, m_R = 3$ and $s_L = 0.9$. In **a** $s_R = -3$; in **b** $s_R = -13$

We close this section by showing two bifurcation diagrams of the state variable x as a function of the left slope s_L (Fig. 7). A comparison of the two diagrams reveals that the lower slope s_R, the shorter the interval of regular dynamics and the earlier chaotic dynamics occur. However, some regularity can also be detected in the chaotic regime. For instance, once the state variable is pushed into the L region, it always increases. This can also be seen in Fig. 8, which we discuss in more detail in the next section.

5 Boom-Bust Price Cycles

The dynamics depicted in Fig. 8 are quite appealing from an economic perspective and, fortunately, our model allows us to comprehend them. We focus for concreteness on the left panel and recall that our financial market model is formulated in terms of deviations from the fundamental value. As can be seen, the panel reveals the emergence of significant boom-bust cycles. We are now ready to explore step-by-step what is driving the dynamics within our model.

Let us start our analysis with a situation in which the market is deep in bear territory. While chartists are depressed and consequently submit selling orders, fundamentalists believe that the market is undervalued and perceive profitable buying opportunities. Since both type 1 and type 2 chartists are dominated by type 1 and type 2 fundamentalists, positive demand pressure drives the price upwards. This process continues for a few time steps, and the strong underpricing is corrected. However, the price increase starts to slow down as the market converges towards the fundamental value. Since the price adjustment of the market maker is proportional to excess demand, the reason for this is also quite clear. Due to the reduction of mispricing, both type 1 chartists and—in particular—type 1 fundamentalists receive weaker trading signals, and their orders diminish, easing the upwards price pressure.

However, the orders of type 2 traders remain constant. A dominance of type 2 fundamentalists over type 2 chartists eventually pushes the price into the bull region. It should be noted that at this moment the behavior of type 2 fundamentalists is destabilizing. Clearly, their orders trigger an overshooting of the fundamental value. Once the price is above its fundamental value, chartists change their attitude from pessimistic to optimistic. Since we have assumed that type 2 chartists dominate type 2 fundamentalists in such a market environment, a positive excess demand accumulates and prices strongly increase. This leads to a collapse in the market. Since type 1 fundamentalists strongly dominate type 1 chartists and since the market is now more overvalued than in the previous time step, the excess demand of type 1 traders is (quite) negative and clearly overcompensates the still positive excess demand of type 2 traders. We therefore observe a substantial crash. The more type 1 fundamentalists dominate type 1 chartists, the deeper the crash. The behavior of fundamentalists thus once more appears ambiguous with respect to market efficiency, which is, in general, a rather surprising and notable finding. After the market has crashed, chartists sell assets while fundamentalists buy assets. As just described, the market recovers, first quickly but then at a slower pace.

Note that the strong price increase immediately prior to the collapse of the bubble is typical for many financial market crises witnessed in the past. We find it quite remarkable that our model is able to mimic this feature and that it offers an explanation for this phenomenon: just before the crash, there is a strong buying pressure from optimistic chartists while at the same time there are basically no stabilizing orders from fundamentalists which would be able to balance the excess demand and counter the price increase. Also the consequent abrupt, sharp market crash can be observed in the real world. Within our model, such extreme price drops are caused by fundamentalists who bet (too) aggressively on mean reversion.

6 Conclusions

In this paper we considered a piecewise linear discontinuous map with an increasing branch on the left side, a decreasing branch on the right side, and two positive offsets, representing the interactions of heterogeneous traders in a simple financial market model. We determined the border collision bifurcation curves leading to the existence of infinitely many stable cycles, and described period-adding schemes. In addition, we demonstrated that there can only be one attracting set, which may either be a cycle (whose period may be associated with infinitely many rotation numbers) or chaotic motion.

From an economic perspective, we conclude that our model delivers a plausible story for the emergence of boom-bust cycles which differs, at least to some degree, from the (standard) story reported in the introduction. What is important to note is that the model does not only incorporate interactions between "standard" chartists and fundamentalists but interactions between different types of chartists and fundamentalists. Although there are only four deterministic trading rules, the model posses quite rich dynamics.

Of course, our model is stylized and many relevant aspects are missing, but, given the importance of this topic, we consider it important to further our knowledge of what may drive the dynamics of financial markets. Amongst other things, our model highlights the ambiguous role of fundamentalists during the course of boom-bust cycles and the appealing implications of discontinuous dynamical systems brought about by simple heterogeneous trading rules of boundedly rational agents, leading to potentially momentous regime shifts. It would be interesting to calibrate this model such that it matches the stylized facts of financial markets more closely. This will probably require the inclusion of exogenous disturbances. We hope that the analysis of the deterministic skeleton of such a stochastic model may prove helpful for this important challenge.

References

Avrutin, V., & Schanz, M. (2006). Multi-parametric bifurcations in a scalar piecewise-linear map. *Nonlinearity, 19*, 531–552.

Avrutin, V., Schanz, M., & Banerjee, S. (2006). Multi-parametric bifurcations in a piecewise-linear discontinuous map. *Nonlinearity, 19*, 1875–1906.

Avrutin, V., Schanz, M., & Gardini, L. (2010). Calculation of bifurcation curves by map replacement. *International Journal of Bifurcation and Chaos, 20*(10), 3105–3135.

Banerjee, S., Karthik, M. S., Yuan, G., & Yorke, J. A. (2000). Bifurcations in one-dimensional piecewise smooth maps—theory and applications in switching circuits. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 47*, 389–394.

Banerjee, S., Yorke, J. A., & Grebogi, C. (1998). Robust chaos. *Physical Review Letters, 80*, 3049–3052.

Boswijk, P., Hommes, C., & Manzan, S. (2007). Behavioral heterogeneity in stock prices. *Journal of Economic Dynamics and Control, 31*, 1938–1970.

Brock, W., & Hommes, C. (1998). Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control, 22*, 1235–1274.

Chiarella, C. (1992). The dynamics of speculative behavior. *Annals of Operations Research, 37*, 101–123.

Chiarella, C., Dieci, R., & Gardini, L. (2002). Speculative behaviour and complex asset price dynamics: A global analysis. *Journal of Economic Behavior and Organization, 49*, 173–197.

Chiarella, C., Dieci, R., & He, X.-Z. (2009). Heterogeneity, market mechanisms, and asset price dynamics. In T. Hens & K. R. Schenk-Hoppé (Eds.), *Handbook of Financial Markets: Dynamics and Evolution* (pp. 277–344). Amsterdam: North-Holland.

Day, R. (1997). Complex dynamics, market mediation and stock price behavior. *North American Actuarial Journal, 1*, 6–23.

Day, R., & Huang, W. (1990). Bulls, bears and market sheep. *Journal of Economic Behavior and Organization, 14*, 299–329.

De Grauwe, P., Dewachter, H., & Embrechts, M. (1993). *Exchange rate theory—chaotic models of foreign exchange markets*. Oxford: Blackwell.

Franke, R. (2009). A prototype model of speculative dynamics with position-based trading. *Journal of Economic Dynamics and Control, 33*, 1134–1158.
Gardini, L., Tramontana, F., Avrutin, V., & Schanz, M. (2010). Border collision bifurcations in 1D PWL map and the Leonov approach. *International Journal of Bifurcation and Chaos, 20*(10), 3085–3104.
Hao, B.-L. (1989). *Elementary symbolic dynamics and chaos in dissipative systems*. Singapore: World Scientific.
Hommes, C. H., Sonnemans, J., Tuinstra, J., & van de Velden, H. (2005). Coordination of expectations in asset pricing experiments. *Review of Financial Studies, 18*, 955–980.
Hommes, C. H., & Wagener, F. (2009). Complex evolutionary systems in behavioral finance. In T. Hens & K. R. Schenk-Hoppé (Eds.), *Handbook of financial markets: Dynamics and evolution* (pp. 217–276). Amsterdam: North-Holland.
Huang, W., & Day, R. (1993). Chaotically switching bear and bull markets: the derivation of stock price distributions from behavioral rules. In R. Day & P. Chen (Eds.), *Nonlinear dynamics and evolutionary economics* (pp. 1–49). Oxford: Oxford University Press.
Huang, W., Zheng, H., & Chia, W.-M. (2010). Financial crises and interacting heterogeneous agents. *Journal of Economic Dynamics and Control, 34*, 1105–1122.
Leonov, N. N. (1959). Map of the line onto itself. *Radiofisica, 3*, 942–956.
Leonov, N. N. (1962). Discontinuous map of the straight line. *Dohk Ahad Nauk SSSR, 143*, 1038–1041.
Lux, T. (1995). Herd behavior, bubbles and crashes. *Economic Journal, 105*, 881–896.
Lux, T. (2006). Financial power laws: Empirical evidence, models and mechanisms. Economics Working Paper No 2006-12, University of Kiel.
Lux, T. (2009). Stochastic behavioural asset-pricing models and the stylize facts. In T. Hens & K. R. Schenk-Hoppé (Eds.), *Handbook of Financial Markets: Dynamics and Evolution* (pp. 161–216). Amsterdam: North-Holland.
Maistrenko, Y. L., Maistrenko, V. L., & Chua, L. O. (1993). Cycles of chaotic intervals in a time-delayed Chua's circuit. *International Journal of Bifurcation and Chaos, 3*, 1557–1572.
Maistrenko, Y. L., Maistrenko, V. L., & Vikul, S. I. (1998). On period-adding sequences of attracting cycles in piecewise linear maps. *Chaos, Solitons & Fractals, 9*, 67–75.
Maistrenko, Y. L., Maistrenko, V. L., Vikul, S. I., & Chua, L. O. (1995). Bifurcations of attracting cycles from time-delayed Chua's circuit. *International Journal of Bifurcation and Chaos, 5*, 653–671.
Menkhoff, L., & Taylor, M. (2007). The obstinate passion of foreign exchange professionals: technical analysis. *Journal of Economic Literature, 45*, 936–972.
Mira, C. (1978). Sur les structure des bifurcations des diffeomorphisme du cercle. *Comptes Rendus de l'Academie Des Sciences Paris, Series A, 287*, 883–886.
Mira, C. (1987). *Chaotic dynamics*. Singapore: World Scientific.
Nusse, H. E., Ott, E., & Yorke, J. A. (1994). Border-collision bifurcations: An explanation for observed bifurcation phenomena. *Physical Review E, 49*, 1073–1076.
Nusse, H. E., & Yorke, J. A. (1992). Border-collision bifurcations including period two to period three for piecewise smooth systems. *Physica D, 57*, 39–57.
Sushko, I., & Gardini, L. (2010). Degenerate bifurcations and border collisions in piecewise smooth 1D and 2D maps. *International Journal of Bifurcation & Chaos, 20*(7), 2045–2070.
Tramontana, F., Gardini, L., & Westerhoff, F. (2010a). Intricate asset price dynamics and one-dimensional discontinuous maps. In T. Puu & A. Panchuck (Eds.), *Nonlinear Economic Dynamics*. New York: Nova Science Publishers.
Tramontana, F., Westerhoff, F., & Gardini, L. (2010b). On the complicated price dynamics of a simple one-dimensional discontinuous financial market model with heterogeneous interacting traders. *Journal of Economic Behavior and Organization, 74*, 187–205.
Westerhoff, F. (2009). Exchange rate dynamics: A nonlinear survey. In J. B. Rosser, Jr. (Ed.), *Handbook of research on complexity* (pp. 287–325). Cheltenham: Edward Elgar.
Westerhoff, F., & Dieci, R. (2006). The effectiveness of Keynes–Tobin transaction taxes when heterogeneous agents can trade in different markets: a behavioral finance approach. *Journal of Economic Dynamics and Control, 30*, 293–322.
Westerhoff, F. & Franke, R. (2010, in press). Converse trading strategies, intrinsic noise and the stylized facts of financial markets. *Quantitative Finance*.