

Original article

Endogenous cycles in discontinuous growth models

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Received 13 August 2009; received in revised form 7 December 2010; accepted 7 December 2010

Available online 21 December 2010

Abstract

In this paper we consider a discontinuous one-dimensional piecewise linear model describing a neoclassical growth model. These kind of maps are widely used in the applied context. We determine the analytical expressions of border collision bifurcation curves, responsible for the observed dynamics, which consists of attracting cycles of any period and of quasiperiodic trajectories in exceptional cases.

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Keywords: Piecewise-linear maps; Border collision bifurcations; Discontinuous maps; Economic growth models

1. Introduction

Bifurcations in a piecewise-smooth system are quite different from those occurring in a smooth one. It is nowadays well known that in smooth systems the dynamics may evolve from a regular dynamic behavior to a complex one via a sequence of bifurcations (as, for example, routes to chaos via Feigenbaum cascades of period doubling bifurcations), while in piecewise smooth systems *border-collision bifurcations* (BCB for short) may occur. Of interest in this paper are piecewise-linear systems for which only BCB occur. Border-collision refers to any contact between an invariant set of a map with the border of its region of definition, and this generally gives rise to a *bifurcation*. The term *border-collision bifurcation* was used for the first time by Nusse and Yorke [33] (see also [34]) and it is now widely used in this context (i.e. for piecewise smooth maps). These bifurcations have been widely studied in recent years, mainly because of their relevant applications in physics and engineering [1–3,6,13,14,45]. However, the study and description of such border collision bifurcations started a long time ago. We remark that the bifurcations associated with piecewise smooth maps are studied in [27–29,31,32] even if the bifurcations were not called of border-collision. In particular, in [14] some results by Feigin are republished, these were already printed in 1978¹ (but not known widely). We may also go further back, citing the works by Leonov in the 60s, [25,26]. In his works, Leonov described several bifurcations, giving a recurrence relation to find the analytic expression of the family of bifurcations occurring in a one-dimensional piecewise linear map with one discontinuity point, which also is still mainly unknown. Some of his results have been

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¹ It is worth noticing that the clear and simple analysis performed by Feigen in 1978 is the first one for *n-dimensional piecewise linear continuous maps*, with $n > 1$.

improved recently [4,17] and will also be recalled in this work, because they are our starting point, in order to understand and describe the bifurcations occurring in our model.

It is particular to the economic context that piecewise smooth models may arise, and in fact economic models described by one-dimensional piecewise smooth functions, continuous or discontinuous, are already present in the literature. A pioneer in this context is Richard Day. In fact, his first studies of piecewise smooth models started more than 25 years ago, and his results, found independently on the authors cited above, on the bifurcation curves in such models are still extremely up to date (although they were not called border-collision bifurcation curves). We mention, in particular, Day's results in [9–12], which have been used also recently in [30]. Clearly other economic applications of piecewise-smooth system can be found. We mention for example [18–21] of several years ago. But the BCBs have been widely considered more recently, in several economic models described by continuous functions, for example in Business Cycles models in [36,37,42]; in growth models in [16,40,41]. While BCBs in discontinuous models also occur in Duopoly models in [36,44].

Also the model considered in this paper comes from the economic context, as it will be described in the next section, it deals with the dynamics of a growth model proposed in [8].

The case which the authors Böhm and Kaas were interested in corresponds to the one here called of regular dynamics. In [8] the authors prove that all the existing cycles are globally attracting, and refer to a remarkable paper by Keener [24] to state that for a set of parameter values of zero Lebesgue measure the attracting set is a Cantor set, which however they have never observed. Indeed it is correct, because the attracting set in regular dynamics cannot be a Cantor set. As we shall see below, in such a case the asymptotic dynamics are either periodic or quasiperiodic (and dense in an interval). In the paper by Keener [24] there is an important result: it is shown that for increasing piecewise smooth discontinuous maps (not only linear), there exists a sharp transition from regular dynamics (no chaotic set can exist) to chaotic one (no stable cycle can exist). Moreover, in [8] the authors remark a surprisingly rich occurrence of periods in the cycles: the rich structure of periodicity regions with cycles of any period and with periodic points in several positions between the L and R regions will be here fully described.

The plan of the work is as follows. In Section 2 we shall describe the model, showing that in the framework of the economic application, only the stable regime is of interest. In Section 3 we shall determine the equations of the BCB curves at the boundaries of the periodicity regions associated with the so called k -cycles of maximal periods, having one periodic point on one side and $(k - 1)$ points in the other side, here called (following Leonov) of first complexity level. In Section 4 we shall describe the so-called period adding scheme, and in Section 5 the Leonov approach, which leads us to obtain the analytical equations of the BCB curves of many other periodicity regions (in principle we can find all the analytical equations) of second, third, . . . , complexity levels (they are infinite in number). Section 6 concludes.

2. The model

One of the main issues of economic growth theory is the research for the conditions that permit the model to exhibit endogenous growth cycles. Neoclassical one-sector models of optimal growth like the Ramsey model [38] and the Solow–Swan model [39,43] deal with the analysis of the monotonically converging paths of capital and output per capita towards a long run unique steady state. Among the conditions that ensure the existence of a unique steady state, the weakest one is probably the concavity of the aggregate saving function. This property was a consequence of the assumption of constant and identical saving propensity characterizing the representative consumer of the population. Kaldor [22,23], Pasinetti [35] and others, take into consideration a subdivision of the population in workers and shareholders. In their opinion it was not realistic to assume the same saving propensities for the representative agents of both groups. In fact, they assume that shareholders are more inclined to save than workers. These economists were interested in the effect of these alternative assumption on the growth path of the economy and they did not analyze all the consequences of such an assumption. A saving function obtained by the aggregation of saving functions of the two groups of agents, no longer needs to be concave² and this fact has the important implication that multiple (stable and/or unstable) steady states may occur.

After Dixit [15], Böhm and Kaas have proposed the unique treatment of the role of differential savings behaviors for stability of stationary states. In [8] these authors move from a standard neoclassical one-sector growth model in

² In fact, the amount of saving depends now on the income distribution between workers and shareholders.

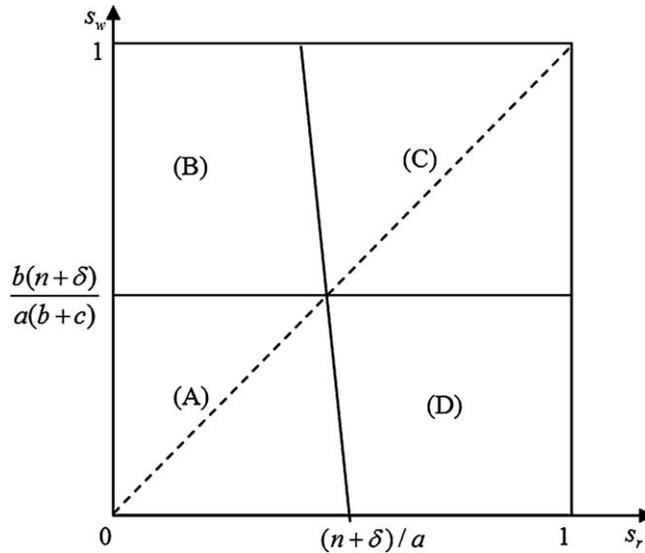


Fig. 1. Qualitative description of the regions in the parameter plane (s_r, s_w) .

which workers and shareholders are endowed with constant but different saving propensities (S_w and S_r , respectively). They assume a production function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ mapping capital per worker k into output per worker y and a capital depreciation rate $0 < \delta \leq 1$. The wage rate is determined by the following (usual) assumption:

$$w(k) = f(k) - kf'(k)$$

Shareholders receive the marginal product of capital $f'(k)$, so the total capital income per worker is $kf'(k)$. The capital accumulation is given by

$$k_{t+1} = G(k_t) := \frac{1}{1+n} ((1-\delta)k_t + s_w w(k_t) + s_r k_t f'(k_t)) \tag{1}$$

where n is the labor force growth rate.

In order to close the model an explicit productive function must be introduced. One of the functions used in [8] is the so-called Leontief production function:

$$f(k) = \min(ak, b) + c, \quad a, b, c > 0 \tag{2}$$

Assuming a Leontief technology, the capital accumulation is described by the following discontinuous map:

$$k' = G(k) = \begin{cases} G_L(k) = \frac{1}{1+n} ((1-\delta + s_r a)k + s_w c), & \text{if } k \leq \frac{b}{a} \\ G_R(k) = \frac{1}{1+n} ((1-\delta)k + s_w(b+c)), & \text{if } k > \frac{b}{a} \end{cases} \tag{3}$$

where G_L and G_R are linear maps whose slopes G'_L and G'_R are positive and such that:

$$G'_R < G'_L, \quad G'_R < 1 \tag{4}$$

The regions shown in Fig. 1 summarize the four possible cases, for which the shape of the map $G(k)$ is as shown in Fig. 2.

The special feature of this model consists in being piecewise linear and discontinuous. However, the dynamics occurring in cases (A), (B) and (C) are immediately clear (in [8] the authors prove that a necessary condition for the coexistence of multiple fixed points is $S_w > S_r$), while case (D) is much more interesting: no equilibria, but eventually different cycles of many different periods. Nevertheless, case (D) represents a case in which the saving propensities

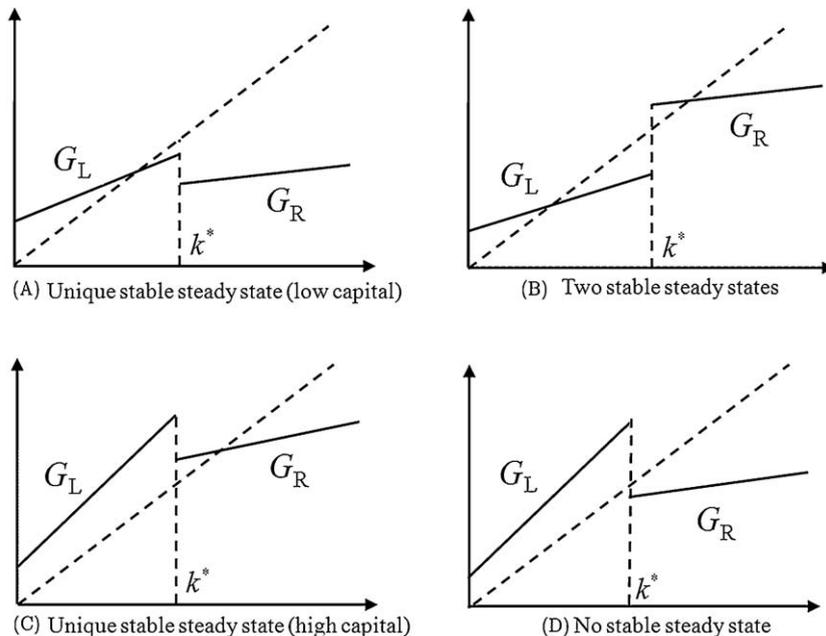


Fig. 2. Qualitative description of the possible shapes for the model in the four regions described in Fig. 1.

satisfy the economically meaningful condition $s_r > s_w$. The constraint in order to have case (D) is as follows:

$$G_L\left(\frac{b}{a}\right) > \left(\frac{b}{a}\right) > G_R\left(\frac{b}{a}\right) \tag{5}$$

In [8,24] it is proved that when an attracting cycle exists then it is globally attracting, showing that attracting $k - 1$ cycles can exist of any period $k \geq 2$, and several different periods have been observed in the region for the parameters (S_r, S_w) on which the interest is focussed, showing a numerical result of periodicity regions in a very complex structure, as shown in Fig. 3a (different color or grey tonalities in the regions correspond to different periods).

The goal of the present paper is to describe the bifurcation curves which fill the parameter plane (S_r, S_w) . They can be analytically determined as a function of the parameters, and a detailed description of all the dynamics which may occur in case (D) is given.

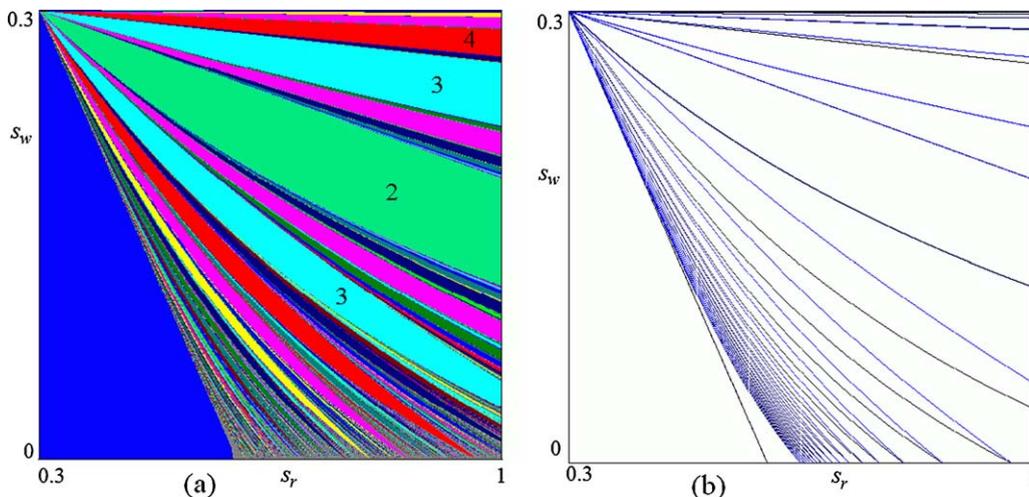


Fig. 3. Periodicity regions in the parameter plane (s_r, s_w) of interest. Detected numerically in (a) and analytically in (b) different colors (grey tonalities) correspond to different periods.

Clearly, by rescaling the independent variable we can reduce the parameters by one unit. However, this does not simplify our analysis, so that we keep all the parameters as defined above, using, for computational convenience, a stylized form as follows:

$$x' = G(x) = \begin{cases} G_L(x) = \mu_L x + q_L, & \text{if } x \leq \frac{b}{a} \\ G_R(x) = \mu_R x + q_R, & \text{if } x > \frac{b}{a} \end{cases} \tag{6}$$

with $x = k$ and the obvious definitions for μ and q as the slopes and offsets of the two linear functions:

$$\mu_L = \frac{1 - \delta + s_r a}{1 + n}, \quad q_L = \frac{s_w c}{1 + n}, \quad \mu_R = \frac{1 - \delta}{1 + n}, \quad q_R = \frac{s_w(b + c)}{1 + n} \tag{7}$$

i.e.

$$\mu_L = \mu_R + \frac{s_r a}{1 + n}, \quad q_L = \frac{s_w c}{1 + n}, \quad \mu_R = \frac{1 - \delta}{1 + n}, \quad q_R = q_L + \frac{s_w b}{1 + n} \tag{8}$$

with the constraints (from (4) and (5)):

$$\mu_R < \mu_L, \quad \mu_R < 1, \quad \mu_L \frac{b}{a} + q_L > \frac{b}{a} > \mu_R \frac{b}{a} + q_R \tag{9}$$

and we can restrict the study of this map in the invariant absorbing interval I :

$$I = \left[G_R \left(\frac{b}{a} \right), G_L \left(\frac{b}{a} \right) \right] = \left[\mu_R \frac{b}{a} + q_R, \mu_L \frac{b}{a} + q_L \right] \\ = \left[\frac{1}{1 + n} \left((1 - \delta) \frac{b}{a} + s_w c \right) + \frac{s_w b}{1 + n}, \frac{1}{1 + n} \left((1 - \delta) \frac{b}{a} + s_w c \right) + \frac{s_r b}{1 + n} \right] \tag{10}$$

whose width is given by $(b/(1 + n))(s_r - s_w)$ and thus a necessary condition for case (D) is $S_r > S_w$: the reverse one with respect to the case of two coexisting fixed points.

Notice that while one slope is “stable”, $0 < \mu_R < 1$, we can have $0 < \mu_L < 1$ or $\mu_L > 1$, and thus unstable cycles may occur. When μ_L is higher than 1, we have an unstable fixed point on the left side of the discontinuity point, given by $x^* = -(q_L/(\mu_L - 1)) < 0$ so that it is always in the region out of interest in the applied context (being $x > 0$). So, independently on the existence of the unstable fixed point, we have that the basin of the invariant absorbing interval is given by $\mathcal{B}(I) =]0, +\infty[$, i.e. it is the whole region of interest.

Moreover, when $\mu_L > 1$ we may expect both regular dynamics (with attracting cycles or quasiperiodic trajectories) and chaotic dynamics. The most striking result is that *we cannot have them both simultaneously* (as it occurs in the smooth case, for example in the standard logistic map when the parameter is in a periodic window after the Feigenbaum point³), and *chaos is always robust* (following [7]), which means persistent as a function of the parameters. From an economic point of view this result implies that the initial value of the capital per capita does not have any influence on its asymptotic value: that is, even a quite poor country (i.e. countries endowed with low levels of capital per capita) shall approach the (sequence of) levels of capital per capita of the richest countries, even if this process may need a lot of time.

The most obvious expectation in the case of one slope higher than 1 and one lower than 1 is a kind of progressive destabilization of the possible cycles, while this does not occur. The transition from a regular regime to a chaotic one occurs instantaneously for all the possible cycles. That is to say, in a suitable region of the parameter space all the existing cycles are stable, while outside it either the cycles do not exist or all the cycles are unstable. This is clearly a very particular bifurcation. This was already proved in [24] for a discontinuous map $F(x)$ increasing of the interval $[0, 1]$ into itself, with a discontinuity in $x = d \in]0, 1[$ with jump equal to 1, for which there are regular (resp. chaotic) dynamics for $F(0) > F(1)$ (resp. $F(0) < F(1)$) and the transition stability/chaos occurs at $F(0) = F(1)$. The same result is given in [19] for the piecewise linear map in canonic form.

³ As it is well known that an invariant chaotic set always exists.

For the map we are interested in, this leads to the condition

$$G_L \circ G_R \left(\frac{b}{a} \right) > G_R \circ G_L \left(\frac{b}{a} \right) \tag{11}$$

which may be rewritten, by using the model’s parameters, $G_L \circ G_R(b/a) - G_R \circ G_L(b/a) = q_L(1 - \mu_R) - q_R(1 - \mu_L) > 0$. This condition is satisfied for $S_w > 0$, while for $S_w = 0$ we have $q_L(1 - \mu_R) - q_R(1 - \mu_L) = 0$ which means (see [37]) that all the BCB curves intersect in pair in the locus $S_w = 0$ (as it can be seen in Fig. 3). All the BCB curves are issuing from the same point $(S_r, S_w) = (0.3, 0.3)$, they do not exist in the other region of the phase space, and intersect in pair on the line $S_w = 0$, crossing it and still existing below, corresponding to the region in which the dynamics are chaotic. That is, below that curve only chaos can occur, for $S_w < 0$, when the condition in (11) is not satisfied (although out of interest in this work).

Notice that the simple structure of the piecewise linear map allows for a simple computation of the eigenvalue associated with a k -cycle. In fact, if a cycle has m points on one side, say L and n points on the other side R , then its eigenvalue is given by $\lambda_k = \mu_L^m \mu_R^n$. In [19,37] it is proved that as long as the condition in (11) is satisfied any k -cycle has eigenvalue $\lambda_k = \mu_L^m \mu_R^n < 1$, independently on the numbers m and n of points in the two sides of the discontinuity. Thus, as long as the condition in (11) holds, no chaos can occur, because there is no sensitivity with respect to the initial conditions (initial conditions close to each other will have iterated points forever close to each other). Also it is not possible to have a Cantor set as attractor (as it was claimed in [17]), because a Cantor set requires the existence of infinitely many coexisting unstable cycles, which cannot occur in this regime of stable dynamics. We shall see that only attracting cycles can exist, or quasiperiodic trajectories.

So, in the framework of the economic application, we are interested only in the stable regime, and the complex structure observed in the periods of the existent attracting cycles is shown numerically in Fig. 3a. In the next section we shall determine the equations of the BCB curves at the boundaries of the periodicity regions associated with the so called k -cycles of first complexity level, shown in Fig. 3b. But, as we shall see, between any two of them there exist infinitely many other periodicity regions, in a complex mechanism called *period adding scheme* (which corresponds to the Farey composition rule and the devil’s staircase structure), which also can be analytically determined by using the Leonov technique. So the two key points, which are associated with the *period adding scheme*, is that coexistence of stable cycles is impossible (all the existing ones are globally attracting), and no invariant chaotic set can exist. So, if not periodic, a trajectory can ultimately be only quasiperiodic.

We close this section noticing that the values of the parameters which are used in Fig. 3, $a = 1.5$, $b = 2.9$, $c = 2.9$, $n = 0.45$ and $\delta = 0.45$ are the same used for all the figures in this paper.

3. BCB curves of first level

Following Leonov [25,26], and the recent extensions of his technique [17], the different periodicity regions can be conveniently organized in different complexity levels. The *first complexity level* is the most simple, and refers to periodic orbits having a quite simple symbolic sequence, that is, cycles of period $(p + 1)$ having one point in the L region, on the left of the discontinuity point, and p points on its right side, which we label by R (also called *maximal cycles* or *principal cycles* in the recent literature). While in continuous maps a BCB leading to the appearance of cycles always gives rise to a pair of cycles (of saddle-node type, or saddle-saddle), in discontinuous maps a BCB leads to the appearance of a unique cycle. So that inside the periodicity region associated with a periodic orbit, no other cycle can exist (no stable and no unstable). The reason why this occurs can be immediately observed in the graph of the function. As an example, we show in Fig. 4 the BCB leading to the appearance/disappearance of a cycle of period 3. As we can see in Fig. 4, labelled as x_0, x_1 and x_2 the periodic point of a 3-cycle with one point in the L side, and two points in the R side the cycle exists as long as the periodic point x_0 is within the interval $]G_R(b/a), (b/a)[$ the border conditions bifurcations are given by the collision of the periodic points with the discontinuity point, that is $x_0 = (b/a)$ (as in Fig. 4b) and $x_2 = (b/a)$ (as in Fig. 4a). Note also that this second condition can be expressed as $G_R(x_2) = G_R(b/a)$, that is: $x_0 = G_R(b/a)$. So that indeed the two BCBs leading the cycle to exist/disappear are also associated with the maximum values attainable inside the (globally) absorbing interval I , as in fact the two conditions correspond to a contact of the cycle with the boundary of the interval I : in $x_0 = G_R(b/a)$ (Fig. 4a) and in $x_1 = G_L(b/a)$ (Fig. 4b).

So in order to determine the existence condition of a cycle having, in the generic case, one point on the L side and p points in the R side, we consider the periodic points $x_0 < (b/a)$, and $x_1 > \dots > x_p$ so that the existence condition is

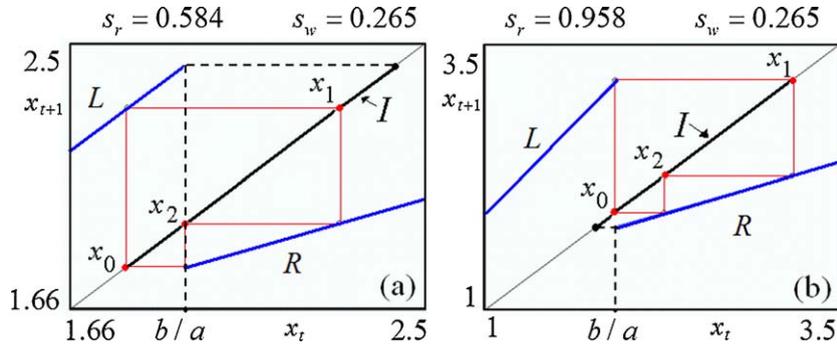


Fig. 4. Border collision bifurcations of a cycle of period 3.

given as fixed point of the composite map $G_R^p \circ G_L(x_0) = x_0$. Writing this condition we are helped from the iterative application of a linear function, which can be easily done (by using the simplified formalism of a geometric sequence):

$$\begin{aligned}
 G_L(x_0) &= \mu_L x_0 + q_L \\
 G_R^1 \circ G_L(x_0) &= \mu_R(\mu_L x_0 + q_L) + q_R \\
 &\dots \\
 G_R^p \circ G_L(x_0) &= \mu_R^p(\mu_L x_0 + q_L) + q_R \sum_{j=1}^{p-1} \mu_R^j
 \end{aligned}
 \tag{12}$$

so that, as we have $\mu_R < 1$ by assumption, by setting:

$$\phi_p^R = \sum_{j=0}^{p-1} \mu_R^j = \frac{1 - \mu_R^p}{1 - \mu_R}
 \tag{13}$$

the periodic point x_0 is obtained by using the equation $G_R^p \circ G_L(x_0) = x_0$:

$$\mu_R^p(\mu_L x_0 + q_L) + q_R \phi_p^R = x_0
 \tag{14}$$

and, as remarked in the previous section, under our assumptions we know that the eigenvalue of the cycle is $\lambda = \mu_R^p \mu_L < 1$, so we have:

$$x_0 = \frac{\mu_R^p q_L + q_R \phi_p^R}{1 - \mu_R^p \mu_L}
 \tag{15}$$

and the region of existence of this cycle is given by:

$$\mu_R \frac{b}{a} + q_R = G_R \left(\frac{b}{a} \right) < \frac{\mu_R^p q_L + q_R \phi_p^R}{1 - \mu_R^p \mu_L} < \frac{b}{a}
 \tag{16}$$

the equations of the two BCB curves associated with its appearance/disappearance are denoted by ξ_{LR}^l (as the periodic point collides with the discontinuity point, from the left) and ξ_{LR}^r (as the periodic point collides with the discontinuity point from the right), and are given analytically by:

$$\begin{aligned}
 \xi_{LR}^l &: \frac{\mu_R^p q_L + q_R \phi_p^R}{1 - \mu_R^p \mu_L} = \frac{b}{a} \\
 \xi_{LR}^r &: \mu_R \frac{b}{a} + q_R = \frac{\mu_R^p q_L + q_R \phi_p^R}{1 - \mu_R^p \mu_R}
 \end{aligned}
 \tag{17}$$

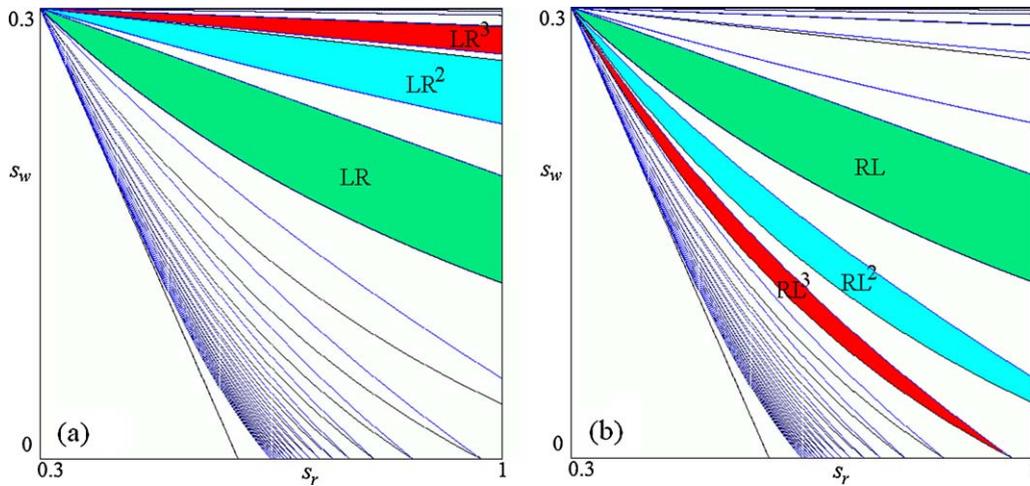


Fig. 5. Parameter plane \$(s_r, s_w)\$. Border collision bifurcation curves of the first level of complexity.

which may be rewritten as follows:

$$\xi_{LR^p}^l : \mu_L = \frac{\frac{b}{a} - (\mu_R^p q_L + q_R \phi_p^R)}{\mu_R^p \frac{b}{a}} =: \Phi_p^R \tag{18}$$

$$\xi_{LR^p}^l : s_r = \frac{1+n}{a} (\Phi_p^R - \mu_R)$$

and

$$\xi_{LR^p}^r : \mu_L = \frac{1}{\mu_R^p} - \frac{\mu_R^p q_L + q_R \phi_p^R}{(\mu_R(b/a) + q_R) \mu_R^p} =: \Psi_p^R \tag{19}$$

$$\xi_{LR^p}^r : s_r = \frac{1+n}{a} (\Psi_p^R - \mu_R)$$

The BCB curves determined in (18) and (19), for \$p=1, 2, \dots\$, give us the borders of the periodicity regions in the upper part of Fig. 3b, above the region of the 2-cycle, associated with cycles of period \$(p+1)=2, 3, \dots\$. See also Fig. 5a, where a few of them are colored.

It is clear that we can also look for simple periodic orbits, always of the *first complexity level*, having “a symmetric” structure. That is, having one periodic point \$x_0\$ in the \$R\$ side, in the interval \$(b/a, G_L(b/a))\$, and the other \$p\$ points in the \$L\$ side, with \$x_1 < \dots < x_p < (b/a)\$. Reasoning symmetrically with respect to the computations reported above, we can repeat all the steps, but it is immediately clear that, to get the periodic point \$x_0\$ of such cycles and to get the equations of the BCB curves, it is enough to exchange the letter \$L\$ and \$R\$, and to reverse the inequalities. So we can immediately write the periodic point \$x_0\$ which satisfies the equation \$G_L^p \circ G_R(x_0) = x_0\$:

$$\mu_L^p (\mu_R x_0 + q_R) + q_L \phi_p^L = x_0 \tag{20}$$

where

$$\phi_p^L = \sum_{j=0}^{p-1} \mu_L^j = \frac{1 - \mu_L^p}{1 - \mu_L} \text{ if } \mu_L \neq 1, \quad \phi_p^L = p \text{ if } \mu_L = 1. \tag{21}$$

From our assumptions we know that the eigenvalue of the cycle is \$\lambda = \mu_L^p \mu_R < 1\$, so we have:

$$x_0 = \frac{\mu_L^p q_R + q_L \phi_p^L}{1 - \mu_L^p \mu_R}. \tag{22}$$

The region of existence of this cycle is thus

$$\mu_L \frac{b}{a} + q_L = G_L \left(\frac{b}{a} \right) > x_0 = \frac{\mu_L^p q_R + q_L \phi_p^L}{1 - \mu_L^p \mu_R} > \frac{b}{a}. \tag{23}$$

The equations of the two BCB curves associated with its appearance/disappearance are denoted by $\xi_{RL^p}^r$ (as the periodic point collides with the discontinuity point, from the right) and $\xi_{RL^p}^l$ (as the periodic point collides with the discontinuity point from the left), and they are given analytically by:

$$\begin{aligned} \xi_{RL^p}^r &: \frac{\mu_L^p q_R + q_L \phi_p^L}{1 - \mu_L^p \mu_R} = \frac{b}{a} \\ \xi_{RL^p}^l &: \mu_L \frac{b}{a} + q_L = \frac{\mu_L^p q_R + q_L \phi_p^L}{1 - \mu_L^p \mu_R} \end{aligned} \tag{24}$$

which may be rewritten as follows:

$$\begin{aligned} \xi_{RL^p}^r &: q_L = \frac{(b/a)(1 - \mu_L^p \mu_R) - \mu_L^p q_R}{\phi_p^L} =: \Phi_p^L \\ \xi_{RL^p}^r &: s_w = \frac{\Phi_p^L(1+n)}{c} \end{aligned} \tag{25}$$

and

$$\begin{aligned} \xi_{RL^p}^l &: q_L = \left(\frac{\mu_L^p q_R}{1 - \mu_L^p \mu_R} - \mu_L \frac{b}{a} \right) \left(\frac{1 - \mu_L^p \mu_R}{1 - \mu_L^p \mu_R - \phi_p^L} \right) =: \Psi_p^L \\ \xi_{RL^p}^l &: s_w = \frac{\Psi_p^L(1+n)}{c} \end{aligned} \tag{26}$$

The BCB curves determined in (25) and (26), for $p = 1, 2, \dots$ give us the borders of the periodicity regions in the lower part of Fig. 3b, below the region of the 2-cycle, associated with cycles of period $(p + 1) = 2, 3, \dots$. See also Fig. 5b, where a few of them are colored. Clearly, the borders of the periodicity region associated with the 2-cycle are detected with both families for $p = 1$, as the cycle symbol sequence RL and LR is the same, with only a shift in the sequence used to denote the cycle. This property is maintained also in the families of higher level of complexity, as we shall see in Section 5.

4. Rotation numbers

Comparing the periodicity regions shown in Fig. 3 with the periodicity regions obtained with the BCB curves of first level, it is immediately clear that we are just at the beginning of a story which can be repeated infinitely many times. The first important observation is that each periodicity tongue is disjoint from the closest ones, i.e. all are disjoint from each other. They are issuing from points belonging to degenerate cases and each pair of boundaries (BCB curves associated with the appearance/disappearance of the same cycle) are intersecting in one point of the set in which the transition stable regime/chaotic regime occurs, which is on the border ($s_w = 0$) of the region which interests us.

Moreover, let us remark that in the description of the periodicity regions we can associate a number to each region, which may be called the *rotation number*, in order to classify all the periods and several cycles with the same period. In this notation a periodic orbit of period k is characterized not only by the period but also by the number of points in the two branches separated by the discontinuity point a/b , denoted by L and R , respectively. We can say that a cycle has a rotation number q/k if a k -cycle has q points on the L side and the others $(k - q)$ on the R side. Then, between any pair of periodicity regions associated with the rotation numbers q_1/k_1 and q_2/k_2 there exists also the periodicity region associated with the rotation number $(q_1/k_1) \oplus (q_2/k_2) = ((q_1 + q_2)/(k_1 + k_2))$ (where \oplus stands for the so-called Farey composition rule, or summation rule, see for example in [5]).

Then, following [25,26] and [31] (see also [32] pp. 56–61 and pp. 80–84), between any pair of consecutive regions of first level of complexity, say with rotation numbers $(1/k_1)$ and $(1/(k_1 + 1))$, we can construct two infinite families of periodicity regions, called regions of second level of complexity via the sequence obtained by adding

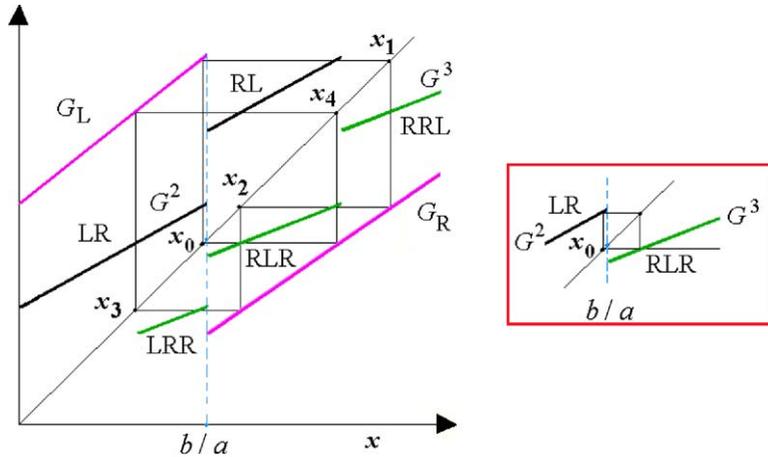


Fig. 6. Qualitative shape of the map for parameters belonging to the region in which a 2-cycle no longer exists and a 3-cycle is close to appear, but still not existing.

with the Farey composition rule \oplus iteratively the first one or the second one, i.e. $(1/(k_1 + 1)) \oplus (1/k_1) = (2/(2k_1 + 1))$, $(2/(2k_1 + 1)) \oplus (1/k_1) = (3/(3k_1 + 1))$, $(3/(3k_1 + 1)) \oplus (1/k_1) = (4/(4k_1 + 1))$, ... and so on, that is:

$$\frac{q}{qk_1 + 1} \text{ for any } q > 1$$

$$\text{and } \frac{1}{k_1} \oplus \frac{1}{k_1 + 1} = \frac{2}{2k_1 + 1}, \quad \frac{2}{2k_1 + 1} \oplus \frac{1}{k_1 + 1} = \frac{3}{3k_1 + 2}, \quad \frac{3}{3k_1 + 2} \oplus \frac{1}{k_1 + 1} = \frac{4}{4k_1 + 3} \dots, \text{ that is :}$$

$$\frac{q}{qk_1 + n - 1} \text{ for any } q > 1$$

which give two sequences of regions accumulating on the boundaries of the two starting ones.

Clearly, this mechanism can be repeated: between any pair of contiguous “regions of second level of complexity”, for example $(q/(qk_1 + 1))$ and $((q + 1)/((q + 1)k_1 + 1))$, we can construct two infinite families of periodicity regions, called “regions of third level of complexity” via the sequence obtained by adding with the composition rule \oplus iteratively the first one or the second one. And so on. All the rational numbers are obtained in this way, giving all the infinitely many periodicity regions.

We also notice that the eigenvalue of a cycle obtained by the composition rule \oplus is the product of the eigenvalues of the two cycles entering in the composition. That is, let us consider for example the cycle with rotation number $(q/(qk_1 + 1))$ (obtained combining the cycle with rotation number $(1/(k_1 + 1))$ and $(q - 1)$ times the cycle with rotation $1/k_1$), then the eigenvalues of the new cycle is

$$\lambda = (\lambda_{k_1})^{(q-1)} \lambda_{k_1+1} \tag{27}$$

5. BCB curves of higher complexity levels

All this is quite known nowadays. What is not well known is that also the BCB curves of the second, third, ..., complexity levels can be obtained analytically in quite a simple way, due to a kind of iterative scheme. The first to propose this “iterated map in the coefficients” was Leonov [25,26], followed by Mira [31,32]. However, in their mechanism there is a change of coordinate introduced in the system which leads to heavy formulas. Instead, in [17] its technique has been improved, and this change of coordinate is no longer necessary, leading to a quicker and more useful process.

Let us introduce the process reasoning on a picture. Fig. 6 shows the map $G(x)$ at fixed parameters, taken between two periodicity regions of first level, associated with the cycles of periods 2 (of symbol sequence LR) and 3 (of symbol sequence LRR). At the chosen parameters the map has an attracting cycle of period 5 (due to 2 + 3). This 5-cycle is

shown in Fig. 6 and its points are labelled starting from the first periodic point on the left side of the discontinuity point $x = (b/a)$, called x_0 . In the same picture we have also drawn the graph of the map G^2 (which has no longer two points on the diagonal, as the parameters are outside the periodicity region of the 2-cycle) and the graph of the map G^3 (which has not yet three points on the diagonal, as the parameters are outside the periodicity region of the 3-cycle). In the box of Fig. 6 we have isolated the shape of the functions G^2 and G^3 immediately on the left and on the right, respectively, of the discontinuity point, which are given by the functions $G_R \circ G_L(x)$ and $G_R \circ G_L \circ G_R(x)$ respectively. It is immediate to see that the shape of the map $T(x)$ constructed in this way, i.e. with $T_L(x) = G_R \circ G_L(x)$ on the left of the discontinuity point and $T_R(x) = G_R \circ G_L \circ G_R(x)$ is in the same situation as the map $G(x)$ from which we started to look for periodicity regions of the first level of complexity. Indeed, for the parameter used in that figure we can see that T has a 2-cycle, and the periodic point x_0 of this 2-cycle is exactly the periodic point of the 5-cycle of the original map $G(x)$. It is clear that we may look also for all the cycles of first level of this map T , by using the conditions to find the first periodic point x_0 on the left of the discontinuity point by using the family of functions $T_R^m \circ T_L(x)$ for any $m \geq 1$.

It is clear that for the original map $G(x)$ we are looking for periodic points associated with the fixed points of the functions $T_R^m \circ T_L(x) = [G_R \circ G_L \circ G_R]^m [G_R \circ G_L](x)$. That is, starting from cycles with symbol sequence LR and LR^2 , we obtain the periodic point and the periodicity region associated with the cycle of symbol sequence $(LR^2)^m LR$.

We can also search for the periodic point x_0 on the right of the discontinuity point of the family of functions $T_L^m \circ T_R(x)$ for any $m \geq 1$, which represent, for the original map $G(x)$, periodic point and periodicity regions associated with the functions $T_L^m \circ T_R(x) = [G_R \circ G_L]^m [G_R \circ G_L \circ G_R](x)$. That is, starting from cycles with symbol sequence LR and LR^2 , we obtain the periodic point and the periodicity region associated with the cycle of symbolic sequence $(LR)^m LR^2$.

Once we have obtained these two families of periodicity regions of the map $T(x)$, it is clear that we have got two infinite families of periodicity regions of the second complexity level for the map $G(x)$, and those two families give periodicity regions which are accumulating on the boundaries of the two starting regions of the 2-cycle and of the 3-cycle.

It is clear that we can repeat the process between any two consecutive periodicity regions of the first level of complexity. That is, by using the functions $T_L(x) = G_R^p \circ G_L(x)$ and $T_R(x) = G_R^p \circ G_L(x) \circ G_R$ for any $p \geq 1$ (associated with the regions of cycles of first level LR^p and LR^{p+1}), and also between any pair $T_R(x) = G_L^p \circ G_R(x)$ and $T_L(x) = G_L^p \circ G_R(x) \circ G_L$ for any $p \geq 1$ (associated with the regions of cycles of first level RL^p and RL^{p+1}).

All what we have described above can easily be put in formulas, the advantage is due to the fact that any kind of composition of affine maps is always an affine map. So we have the required operator $T(x)$ with $T_L(x) = G_R^p \circ G_L(x)$ and $T_R(x) = G_R^p \circ G_L(x) \circ G_R$ for any $p \geq 1$ as follows:

$$x' = T(x) = \begin{cases} T_L(x) = A_L x + M_L, & \text{if } x < \frac{b}{a} \\ T_R(x) = A_R x + M_R, & \text{if } x > \frac{b}{a} \end{cases} \tag{28}$$

where:

$$\begin{aligned} A_L &= \mu_L \mu_R^p \\ A_R &= \mu_L \mu_R^{p+1} \\ M_L &= q_L \mu_R^p + q_R \frac{1 - \mu_R^p}{1 - \mu_R} \\ M_R &= A_L q_R + M_L \end{aligned} \tag{29}$$

Now it is only a matter of applications of the results of the first level. By using the equality $x_0 = T_R^m \circ T_L(x_0)$ we obtain the periodic point of $G(x)$ which is the first one on the left of the discontinuity point:

$$A_R \frac{b}{a} + M_R < x_0 = \frac{1}{1 - A_R^m A_L} \left[M_L A_R^m + M_R \frac{1 - A_R^m}{1 - A_R} \right] < \frac{b}{a} \tag{30}$$

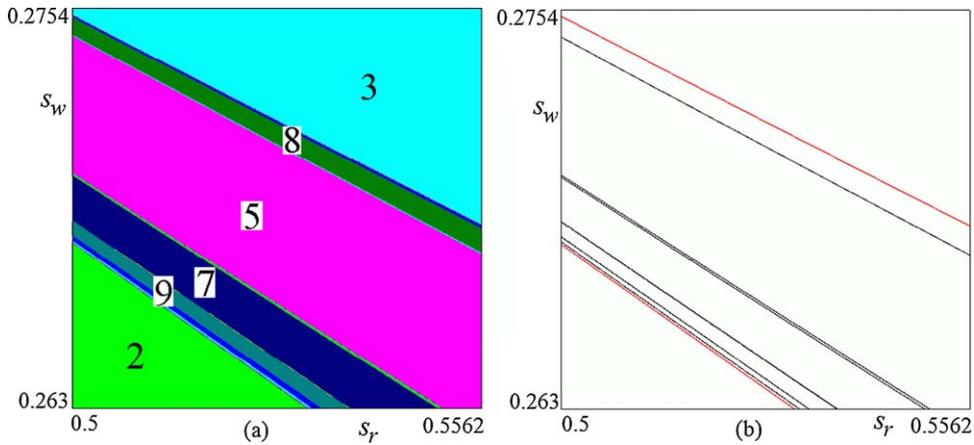


Fig. 7. Enlarged portion of the parameter plane (s_r, s_w) . Periodicity regions numerically computed in (a). In (b) analytical border collision bifurcation curves of the second complexity level between the two curves of first level bounding the region of the 2-cycle and 3-cycle and accumulating on the region of the 2-cycle.

and the BCB curves of the cycles obtained by using $T_R^m \circ T_L(x)$ correspond to the BCB curves of the periodicity regions of the second complexity level for the map $G(x)$. From (17) with obvious changes we obtain:

$$\begin{aligned} \xi_{LR^p(RLR^p)^m}^l &: \frac{A_R^m M_L + M_R \phi_m^R}{1 - A_R^m A_L} = \frac{b}{a} \\ \xi_{LR^p(RLR^p)^m}^r &: A_R \frac{b}{a} + M_R = \frac{A_R^m M_L + M_R \phi_m^R}{1 - A_L^m A_R} \end{aligned} \tag{31}$$

where

$$\phi_m^R = \sum_{j=0}^{m-1} A_R^j = \frac{1 - A_R^m}{1 - A_R} \tag{32}$$

Clearly the equations can now be put in explicit form by using the replacements due to the definitions in (29) and in (7).

Similarly we proceed with the second family from (23) and (24), obtaining:

$$A_L \frac{b}{a} + M_L > x_0 = \frac{A_L^m M_R + M_L \phi_m^L}{1 - A_L^m A_R} > \frac{b}{a} \tag{33}$$

where

$$\phi_m^L = \sum_{j=0}^{m-1} A_L^j = \frac{1 - A_L^m}{1 - A_L} \tag{34}$$

$$\begin{aligned} \xi_{RL^p(LRL^p)^m}^r &: \frac{A_L^m M_R + M_L \phi_m^L}{1 - A_L^m A_R} = \frac{b}{a} \\ \xi_{RL^p(LRL^p)^m}^l &: A_L \frac{b}{a} + M_L = \frac{A_L^m M_R + M_L \phi_m^L}{1 - A_R^m A_L} \end{aligned} \tag{35}$$

Fig. 7a shows an enlarged portion of Fig. 3a between the periodicity regions of the 2-cycle and of the 3-cycle, filled with other periodicity regions, and Fig. 7b shows a few BCB curves of the second complexity level computed with the expressions given in (31) and (35).

5.1. Level of complexity higher than 2

It is clear now that we can repeat the process, considering any pair of consecutive intervals of existence of cycles of the second complexity level, we have an empty space inside which we can reiterate the same reasoning, obtaining two families of intervals of existence of cycles of the third complexity level, and so on.

So the third level of complexity comes in a similar way: given two consecutive functions involved in the second level for example with symbolic sequence $T_L T_R^m$, and $T_L T_R^{m+1}$ (i.e. $LR^p(RLR^p)^m$ and $LR^p(RLR^p)^{m+1}$), we consider the functions $T'_L(x) = T_R^m \circ T_L(x)$ on the L side of $x=(b/a)$ and $T'_R(x) = T_R^m \circ T_L \circ T_R(x)$ on the R side. Assuming that T_L has coefficients given by A_L and M_L and T_R with coefficients A_R and M_R , we apply the same operator in (28) and (29) which we denote as T' substituting to μ_L, q_L, μ_R, q_R , the coefficients A_L, M_L, A_R, M_R to obtain the new ones A'_j, M'_L, A'_R, M'_R , that is: the operator T' represents a map for the coefficients:

$$x' = T'(x) = \begin{cases} T'_L(x) = A'_L x + M'_L, & \text{if } x < \frac{b}{a} \\ T'_R(x) = A'_R x + M'_R, & \text{if } x > \frac{b}{a} \end{cases} \tag{36}$$

where:

$$\begin{aligned} A'_L &= A_L A_R^m \\ A'_R &= A_L A_R^{m+1} \\ M'_L &= M_L A_R^m + M_R \frac{1 - A_R^m}{1 - A_R} \\ M'_R &= A'_L M_R + M'_L = A_L A_R^m M_R + M_L A_R^m + M_R \frac{1 - A_R^m}{1 - A_R} \end{aligned} \tag{37}$$

Then substituting A'_L, M'_L, A'_R, M'_R to A_L, M_L, A_R, M_R and k to m in (31) and (35) we have the BCB curves (with the equation) of the family of cycles of the third level of complexity obtained from the function $T_R^k \circ T'_L(x)$ for any $k \geq 1$. From the function $T_R^k \circ T'_L(x) = A_R^k A'_L x + M'_L A_R^k + M'_R ((1 - A_R^k)/(1 - A_R))$ we obtain the periodic point of $G(x)$ which is the first on the left of the discontinuity point:

$$A'_R \frac{b}{a} + M'_R \leq x_0 = \frac{1}{1 - A_R^k A'_L} \left[M'_L A_R^k + M'_R \frac{1 - A_R^k}{1 - A'_R} \right] \leq \frac{b}{a} \tag{38}$$

and the two BCB curves are obtained with the equalities, for any $k \geq 1$.

The second family of third level of complexity of $G(x)$ associated with these two cycles is obtained exchanging L and R and the inequalities, that is, the periodic point and the BCB curves of the family of cycles of third level of complexity obtained in the form $T'_L \circ T'_R(x)$ for any integer $k \geq 1$ are given by:

$$A'_L \frac{b}{a} + M'_L \geq x_0 = \frac{1}{1 - A_L^k A'_R} \left[M'_R A_L^k + M'_L \frac{1 - A_L^k}{1 - A'_L} \right] \geq \frac{b}{a} \tag{39}$$

where the equalities give the BCB curves.

So from the four families of the second level obtained in the previous sections, we can have eight families of the third level of complexity. And so on, iteratively. It is clear that the number of families is doubled from one level to the subsequent one. This self similar process, called adding structure or Farey composition rule, which can be observed in Fig. 8, is related to the devil’s staircase structure when we consider the graph of the periods of the cycles as a function of the parameter. In particular, for $s_w = 0$ as a function of s_r the map is conjugate to a linear rotation.

Remark. We deserve a few comments to the results obtained related with the existence of periodic or quasiperiodic trajectories. As we have seen, the region in which stable cycles are allowed to exist, is separated by a line (a surface in the whole phase space) between a “stability region” (or regular regime) on one side, and “chaotic region” on the other side. In the stability region we have seen that by a self similar process we can determine periodicity regions which densely fill the stability region. This means that a generic parameter taken inside that region will correspond for the map

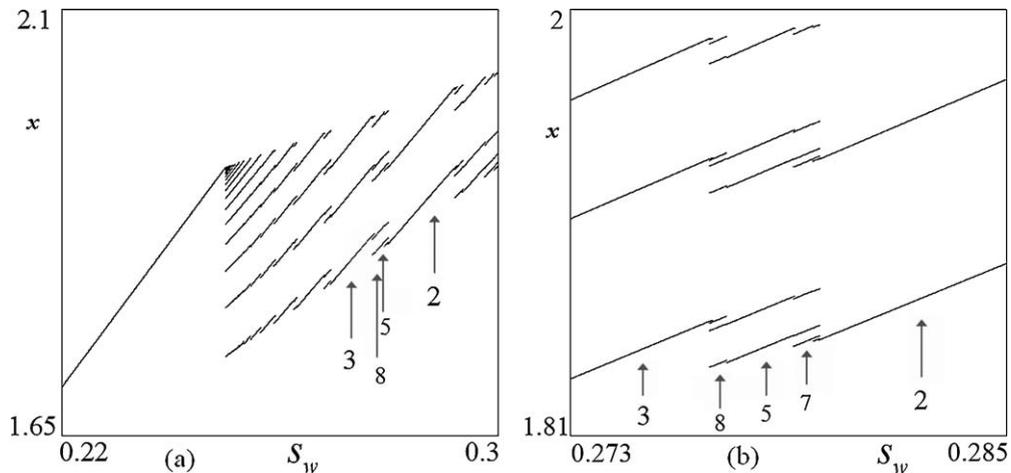


Fig. 8. One-dimensional bifurcation diagram showing x as a function of s_w at $s_r = 0.35$ fixed. The adding structure is evidenced in the enlargement shown in (b).

to trajectories convergent to a unique attracting cycle (of some period and with a specific rotation number). However, it is not true that for any set of parameter values taken there we are *inside* a periodicity region or *on its boundary*, and thus that a stable cycle always exists. In fact, each periodicity region is a *limit set of other periodicity regions*, thus also the reunion of all the existing periodicity regions gives not the whole points in the parameter region. Some points are left: the complementary set, which in this case is a set of zero measure in the parameter space. If the parameters are taken exactly in this complementary set, for the dynamics of the map we have no attracting cycle, but quasiperiodic trajectories which are dense in the absorbing interval. To give an example of quasiperiodic trajectories, we can think that the map behaves as a linear rotation with irrational rotation number. This kind of dynamics, either periodic or quasiperiodic, may be compared with the dynamic behavior of the periodicity regions (or Arnold's tongues) close to a Neimark-Sacker bifurcation curve. Differently, in the chaotic region, no stable cycle can exist, and the dynamics are chaotic in cyclic invariant intervals. This region is, however, not in the range of interest here.

6. Conclusions

In this paper we have investigated the model proposed by Böhm and Kaas in the range of interest associated with stable cycles. The rich bifurcation structure has been completely explained. We have shown how Leonov's technique, improved following [17], can be very useful to calculate the border collision bifurcation curves analytically. In particular, in our model we have a generic discontinuity point $x = (b/a)$, and we have shown how the BCB curves of any complexity level can be analytically detected directly, as a function of the model's parameters.

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