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On the change of periodicities in the Hicksian multiplier-accelerator model with a consumption floor

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Abstract

The Hicksian multiplier-accelerator model with "floor" and "ceiling" continues to be the most successful machine generating business cycles. This is, no doubt, due to its capability of explaining both downturn and upswing through one single model. The "ceiling" is due to a full employment constraint, whereas the "floor" is due to a limit to disinvestment when no worn out capital at all is replaced. However, another "floor" to consumption at zero level seems never to have been discussed. Hence, net disinvestments, even if they are bounded downwards, may also give rise to negative consumption, which is absurd. As we will show, the effect of an additional constraint to avoid this is easy to analyze, and results in a change of the periodicities according to a simple rule.

1. Introduction

It is known that the multiplier-accelerator model for business cycles was first formulated by Samuelson [8], and further elaborated by Hicks [5], who suggested bounds, "floor" and "ceiling", to limit the motion of an otherwise explosive linear model. Hicks offered substantial explanations for these bounds: If investors follow the linear principle of acceleration, then, in periods of sharp income decrease, investments may become, not only negative, i.e., disinvestments, but may even exceed the disinvestment which occurs when no worn out capital is replaced at all. As this means active destruction of capital, it must be prevented through imposing a "floor" at the depreciation level.

Likewise, if income grows very fast, then other inputs than capital, labour or raw materials, may become limiting, and a "ceiling" must be imposed. It can be incorporated in the investment function, along with the floor, which means that it is the investors who abstain from further expenditures once they realize that output cannot be increased due to limitations in the availability of other inputs.

Scientists who incorporated both ceiling and floor in the investment function include Goodwin [4], who, however, preferred a smooth nonlinear investment function with asymptotes to the Hicksian linear with inequality bounds. There has been produced a sizeable literature in Goodwin's tradition.

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An alternative is to impose the ceiling on the total of all expenditures, without making explicit whether it is consumers, investors, or somebody else who cut their expenditures once the ceiling is hit. Hicks was much concerned with what actually happens at the ceiling when the system runs into capacity constraints. The most likely real life experience is inflation, but Hicks, like Samuelson, and all Keyenesians, preferred to think of the model in real terms.

The Hicksian model, where the ceiling was put as a constraint on total expenditures, seems to have been first formalized by Rau [7], as Hicks never formalized the complete model with floor and ceiling. A first exhaustive modern analysis of this setup was presented by Hommes [6], where it was shown that the model had a rather complex dynamics which could produce orbits of any periodicity.

2. The original multiplier-accelerator model

The Samuelson model of 1939 [8], in Hicks's version of 1950 [5], where the principle of acceleration was applied to all expenditures (not just to consumption), was built on two simple components: A given fraction c of past incomes was consumed, i.e., $C_t = cY_{t-1}$, and productive capital was acquired in a fixed proportion a to aggregate income, i.e., $K_t = aY_{t-1}$, provided we incorporate a reaction delay. Given that net investments are defined as the net change of the stock of capital, i.e., $I_t = K_t - K_{t-1}$, we obtain $I_t = a(Y_{t-1} - Y_{t-2})$, which is the traditional form for the principle of acceleration. In addition we only need the accounting identity $Y_t = C_t + I_t$. Eliminating the consumption and investment variables, a single recurrence equation in income alone is obtained:

$$Y_t = (a + c)Y_{t-1} - aY_{t-2}. \tag{1}$$

As the propensity to consume is a fraction, we, of course, have $0 < c < 1$, whereas the capital coefficient is just positive, i.e., $a > 0$. Eq. (1) is a simple second order linear difference equation, and there is just one fixed point $Y_t = Y_{t-1} = Y_{t-2} = 0$, which is stable if, and only if, $a < 1$. As is well known, Eq. (1) has a simple closed form solution. Provided $(a + c)^2 > 4a$, the solution is

$$Y_t = A\lambda_1^t + B\lambda_2^t, \tag{2}$$

where A, B are arbitrary coefficients to accommodate initial conditions, and

$$\lambda_{1,2} = \frac{a+c}{2} \pm \frac{\sqrt{(a+c)^2 - 4a}}{2}, \tag{3}$$

are the real eigenvalues of the characteristic equation.

When $(a + c)^2 < 4a$, which is the case we concentrate on in the sequel, $\lambda_{1,2}$ become complex conjugates with

$$\operatorname{Re}\lambda_{1,2} = (a+c)/2 \quad \text{and} \quad \operatorname{Im}\lambda_{1,2} = \sqrt{4a - (a+c)^2}/2.$$

The general solution can be written as

$$Y_t = \rho^t(A\cos\omega t + B\sin\omega t), \tag{4}$$

where

$$\rho = \sqrt{\operatorname{Re}^2\lambda_{1,2} + \operatorname{Im}^2\lambda_{1,2}} = \sqrt{a} \tag{5}$$

and

$$\omega = \arccos\frac{\operatorname{Re}\lambda_{1,2}}{\rho} = \arccos\frac{a+c}{2\sqrt{a}}. \tag{6}$$

The arbitrary coefficients, A and B, are again chosen so as to fit given initial conditions. The value $a = 1$ is the bifurcation value at which the fixed point becomes a centre.

The solution (4) is a product of an exponential growth factor, increasing when $a > 1$, decreasing when $a < 1$, and a simple harmonic oscillation. Note that the frequency of oscillation as a rule is an irrational multiple of 2π, so the oscillatory motion is quasiperiodic, not periodic in terms of our basic predefined unit time period. It becomes periodic only if

$$\frac{a+c}{2\sqrt{a}} = \cos\left(2\pi\frac{m}{n}\right), \tag{7}$$

where m/n is an irreducible fraction which defines the rotation number of the periodic motion.

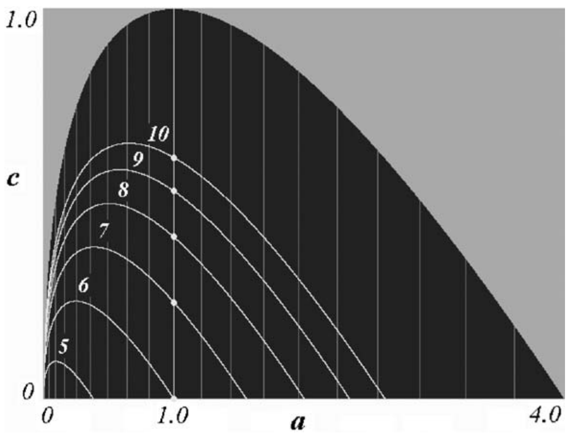

Fig. 1. Parameter plane with periodicity curves and growth lines.

In Fig. 1 we see that this happens on the set of parabolas. We drew the curves for the basic resonances ($m = 1$), and $n = 5, \ldots, 10$. Lower resonances $n < 5$ do not fall within the admissible positive parameter range. As $n \to \infty$, the periodic oscillation curves accumulate towards the curve:

$$\frac{a + c}{2\sqrt{a}} = 1, \tag{8}$$

which is the same as $(a + c)^2 = 4a$, i.e., the borderline between real and complex eigenvalues $\lambda_{1,2}$ (3).

Fig. 1 displays the box $(a, c) \in [0, 4] \times [0, 1]$, further a vertical line at $a = 1$, and the parabola (8), touching the box top, towards which the periodicity curves accumulate. They are shown for n from 5 up to 10, and are labelled accordingly. We stopped at $n = 10$, because for higher n the stack of resonance curves accumulates and seems to fill the entire area, so we no longer see any distinct curves. We chose the fundamental resonances, with $m = 1$ in (7). But 1/6 is the same as 2/12, and 1/7 the same as 2/14, so we should not be surprised to find 2/13, i.e., a 13-period resonance curve between these labelled 6 and 7. In fact, according to the summation rule for rotation numbers, between any two rotation numbers m_1/n_1 and m_2/n_2 there also exists the rotation number $(m_1 + m_2)/(n_1 + n_2)$. Indeed $\cos(2\pi/6) = 0.5$, and $\cos(2\pi/7) = 0.62$, whereas $\cos(4\pi/13) = 0.57$, so the 13-period resonance fits between the 6-period and the 7-period ones according to (7). And so it continues.

Above the parabola (8) in Fig. 1, the zero fixed point for (1) is a node, below it is a focus. To the left of the line $a = 1$ it is stable, to the right it is unstable.

The meaning of the parabola (7) was already explained: It represents parameter combination such that the oscillatory part of the solution (4), i.e., frequency ω in (6), is *periodic*. The significant fact is that they are all *curves*, with zero area measure. Later on we will see that in the *piecewise linear* Hicksian model, the curves swell to thick tongues (so called Arnold tongues).

Considering all rational numbers m/n in (7), we need not choose any particularly high numbers m and n to see the entire screen area completely filled. (In reality both numerator and denominator range to infinity.) This is deceptive and due to the finite resolution of the screen. In reality, though the rational numbers are infinitely many, they are still outnumbered by the irrationals, so if we pick parameter values at random we *never* hit a rational point, i.e., any of the infinitely many periodic curves of Fig. 1.

The picture also contains another feature, the vertical lines placed at growth rates $\rho = \sqrt{a}$, ranging from 0 to 2, in intervals of 0.1. The line $a = 1$ obviously represents the case of constant amplitude oscillations, so called centres.

3. The Hicksian nonlinear model

Now, consider the complete (nonlinear) Hicksian model with floor and ceiling, as formalized by Rau [7]. We then need two changes to the model. The consumption function remains $C_t = cY_{t-1}$ as before, whereas the investment function is changed to $I_t = \max\{a(Y_{t-1} - Y_{t-2}), -D\}$, where D denotes maximum disinvestment, equal to the natural depreciation on capital. There is also a change to the accounting identity, which reads $Y_t = \min\{C_t + I_t, Y^*\}$, where Y^* denotes the maximum production capacity at full employment.

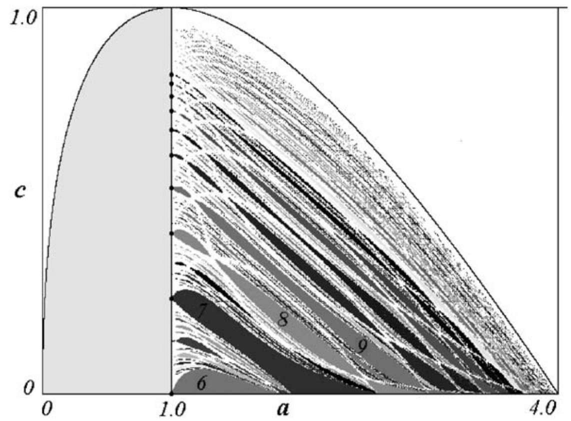

Fig. 2. Parameter plane with periodicity tongues (nonlinear one floor model).

The resulting recurrence equation for income after eliminating the consumption and investment variables can now be written:

$$Y_t = \min\{cY_{t-1} + \max\{a(Y_{t-1} - Y_{t-2}), -D\}, Y^*\}.$$

This roughly represents the case studied by Hommes [6].

Hicks's original discussion indicates that the bounds, the depreciation on capital and the maximum capacity income, are growing. Several later models, for instance Gandolfo [3], unlike Hommes, interpret the bounds as growing. If the growth rates for the bounds are equal, then the growth trend can be eliminated, and the residual model can be recast in a form which is equivalent to the Hommes version (see [2]).

To focus on a particular amendment we will put the ceiling, i.e., the upper bound for income very high, which in practice is equivalent to deleting it altogether. It may be recalled that Duesenberry [1] actually argued that the ceiling was *not* necessary, because, each time the motion hits the floor, it will start a new free oscillatory motion with new initial conditions, and will hence not necessarily hit the ceiling at all. Duesenberry's argument is in fact true whenever we deal with oscillatory motion. So, to sum up, our recurrence equation reads:

$$Y_t = cY_{t-1} + \max\{a(Y_{t-1} - Y_{t-2}), -D\}. \tag{9}$$

The dynamic behaviour of Eq. (9) is very different from that of Eq. (1), as is explained in Appendix A. Due to the nonlinearity present, the periodic solution becomes the main frame, and the thin curves in Fig. 1 swell to thick, so called Arnold tongues, as we see in Fig. 2. Note that the periodicity 5 is no longer present, as it does not reach the vertical line $a = 1$ which is the centre bifurcation line from which the periodicity tongues issue. But, if we compare Figs. 1 and 2 carefully, we find that the points, at which the periodicity curves of Fig. 1 intersect this vertical bifurcation line, exactly coincide with the points of issue of the Arnold tongues of Fig. 2. We are not going to comment the facts for this picture, as they are known from previous studies (see [2]).

4. The consumption floor

As mentioned, there is one disadvantage of the model as it stands now, a disadvantage it shares with all other Hicksian models as well. When the cycle goes in the downswing, income can become negative, as already discussed. As we deal with net income (NNP), this is not so absurd in itself, because the negative net investments, or disinvestments are a fully legitimate accounting entry.

What is absurd, however, is that, according to the linear consumption function, $C_t = cY_{t-1}$, *the negative investments also trigger negative consumption*, so, to avoid this, we should at least write:

$$C_t = \max\{cY_{t-1}, 0\}. \tag{10}$$

In this way gross income (GNP), with depreciation added, *will never become negative*, due to the floor to investments. Recall that this goal was attained in the original models through exogenously adding "autonomous expenditures" to consumption and investment, and redefining the variable in the reduced recurrence equation as the *deviation* of income

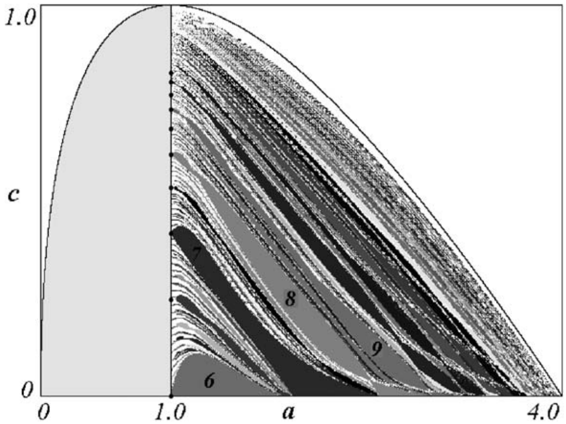

Fig. 3. Parameter plane with periodicity tongues (nonlinear two floor model).

from its equilibrium resulting from these autonomous expenditures alone. Though a negative income is absurd, a negative deviation from some positive value is not. However, the remedy was only partial, as, for instance, with growing oscillatory solutions, not only the deviation, but income itself, even gross income, sooner or later is bound to become negative. The advantage of the suggested constraint is that this never happens.

The complete model now reads:

$$Y_t = \max\{cY_{t-1}, 0\} + \max\{a(Y_{t-1} - Y_{t-2}), -D\}. \tag{11}$$

The consumption floor introduces a new complication, which we are going to concentrate on. For the time being, we will limit the detailed analysis to a local study of what happens to small amplitude oscillations on the bifurcation line. The global dynamics of the system (11) is discussed in Appendix B. It is worth mentioning that we still get periodicity as the main scenario (see Fig. 3).

Intriguing is, however, the fact that all the periodicity tongues in Fig. 3 are *displaced*, as compared to Fig. 2. The starting point of the 6-tongue is still where it should be according to Fig. 1, but that of the 7-tongue is lifted (one step) to the place of the 8-tongue, that of the 8-tongue (two steps) to the place of the 10-tongue, and that of the 9-tongue (three steps) to the place of the 12-tongue, and so forth.

5. Local analysis of the displacement

To understand why the tongues are lifted, it is convenient to focus on the bifurcation value $a = 1$, corresponding to the loss of stability of the fixed point.

Provided we assume a positive D, Eq. (11) locally, in some neighbourhood of the fixed point $Y_t = Y_{t-1} = 0$, can be written as a piecewise linear map F, defined by

$$F_1: \begin{cases} Y_{t+1} = (1+c)Y_t - Z_t \\ Z_{t+1} = Y_t \end{cases} \quad \text{for } Y_t \geqslant 0 \tag{12}$$

and

$$F_2: \begin{cases} Y_{t+1} = Y_t - Z_t \\ Z_{t+1} = Y_t \end{cases} \quad \text{for } Y_t < 0, \tag{13}$$

where we have defined a new variable for lagged income $Z_t = Y_{t-1}$.

The linear maps F_1 and F_2 can be represented in terms of matrices:

$$M = \begin{pmatrix} 1+c & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \tag{14}$$

respectively. It might be worth recording that R is a particular case of M obtained for $c = 0$.

Any point of the orbit can be calculated as a product of powers of these matrices, using some arbitrarily chosen initial vector for the state variables, such as $v_0 = (Y_0, Z_0) = (0, -1)$. As $Y_0 = 0$ in the initial condition, the matrix M applies initially (indeed, due to continuity of the map F, for an initial vector with $Y_0 = 0$ we can equivalently apply the matrix R). The application of M goes on until a power of this matrix multiplied by the initial vector produces a negative first element. Whenever this happens, the matrix R applies. We stick to these two symbols M and R whenever we do not want to specify the particular periodicity with which we are concerned. Otherwise, we denote the matrix corresponding to the periodic orbit with the rotation number m/n by $M_{m/n}$.

6. Some more notation

Let us now specify the matrix $M_{m/n}$ for the m/n-periodic orbit. Note that at the bifurcation value $a = 1$, the map F_1 produces an m/n-periodic orbit from *any* initial condition, provided we choose c so as to result in a rational rotation number m/n. Substituting $a = 1$ in (7), the coefficient c reads:

$$c = c_{m/n} \stackrel{\text{def}}{=} 2\cos(2\pi m/n) - 1. \tag{15}$$

This means that in m rounds around the origin n different points are visited. From (15) we have $c + 1 = 2\cos(2\pi m/n)$. Thus, substituting in (14):

$$M_{m/n} = \begin{pmatrix} 2\cos\left(2\pi\frac{m}{n}\right) & -1 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

Note that, as $\cos(\pi/3) = 1/2$, we have $M_{1/6} = R$. Picking any arbitrary periodicity m/n and any initial point $v_0 = (Y_0, Z_0)$, we just iterate according to $M_{m/n}$ as long as $Y_i \geqslant 0$, and then shift to $M_{1/6} = R$ whenever $Y_i < 0$.

We can obtain a closed form formula for any integer power of the matrix $M_{m/n}$, whatever the period:

$$M^i_{m/n} = \frac{1}{\sin\left(2\pi\frac{m}{n}\right)}\begin{pmatrix} \sin\left(2\pi\frac{m}{n}(i+1)\right) & -\sin\left(2\pi\frac{m}{n}i\right) \\ \sin\left(2\pi\frac{m}{n}i\right) & -\sin\left(2\pi\frac{m}{n}(i-1)\right) \end{pmatrix}. \tag{17}$$

If we put $i = n$, then we get the identity matrix, i.e.,

$$M^n_{m/n} = \frac{1}{\sin\left(2\pi\frac{m}{n}\right)}\begin{pmatrix} \sin\left(2\pi\frac{m}{n}\right) & 0 \\ 0 & \sin\left(2\pi\frac{m}{n}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{18}$$

7. The geometric shape of the orbits

Let us now consider just the linear map F_1 defined in (12), and describe the geometrical shape of the invariant curves in the phase plane (Y, Z).

The m/n-periodic orbit for any given rotation number m/n is located on an ellipse. To find the formulas for such an ellipse, suppose we start from $v_0 = (Y_0, Z_0) = (0, -1)$, and apply the matrix $M_{m/n}$ i times. Then

$$\begin{pmatrix} Y_i \\ Z_i \end{pmatrix} = \frac{1}{\sin\left(2\pi\frac{m}{n}\right)}\begin{pmatrix} \sin\left(2\pi\frac{m}{n}i\right) \\ \sin\left(2\pi\frac{m}{n}(i-1)\right) \end{pmatrix}. \tag{19}$$

After n iterations $M^n_{m/n}v_0 = Iv_0 = v_0$, i.e., the trajectory is back at the initial point v_0.

We can easily eliminate i in the right hand sides of Eq. (19), and, deleting the indices for the phase variables, obtain:

$$Y^2 + Z^2 - 2YZ\cos\left(2\pi\frac{m}{n}\right) = 1. \tag{20}$$

This is the equation for an ellipse with the main axes in the diagonal directions. It is easy to check that all ellipses for different m, n pass through the point $Y = 0$, $Z = -1$. There are infinitely many such closed orbits of different sizes. We obtain the whole family by letting the right hand constant take other values than unity.

We can check out what the ellipses look like for different m and n. Obviously, for $m = 1$, $n = 4$, $\cos(\pi/2) = 0$, so the ellipse $Y^2 + Z^2 = 1$ becomes a circle. The smaller the number m/n, the more oblong the ellipse becomes. As $n \to \infty$, the left hand side of the equation approaches a perfect square, and in the limit the equation $Y^2 + Z^2 - 2YZ = 1$ represents the degenerate case of two parallel straight lines.

Given the periodicity m/n and the initial point, we get a unique ellipse on which the orbit is located. If we consider all different initial points on one given ellipse, we get a dense covering of the ellipse itself.

As we deal with the center bifurcation case, then for each periodicity m/n there exist a family of concentric ellipses of all amplitudes. In terms of our formula (20) this means that the right hand side takes other values than unity.

We just need one word of caution: If the orbits become large enough, then the investment floor constraint may indeed become involved. According to the investment function $I_t = \max\{a(Y_{t-1} - Y_{t-2}), -D\}$, this happens whenever $a(Y - Z) + D < 0$.

Then only the initial points which belong to an invariant polygonal area (see Appendix A) behave as described, whereas initial points exterior to it are mapped to its boundary in a finite number of steps.

8. The important 1/6 orbit

The facts told in the previous section relate to the single linear map F_1. But the map F we consider is a composite of two different linear maps F_1 and F_2, involving, respectively, the matrix $M_{m/n}$ for $Y \geqslant 0$ and $M_{1/6} = R$ for $Y < 0$. This implies that every orbit we consider also has some points located on ellipses corresponding to the 1/6 orbits. An example of such ellipse through the point $(Y_0, Z_0) = (0, -1)$ is

$$Y^2 + Z^2 - 2YZ = 1. \tag{21}$$

Let us now retrieve one important fact from (17) for the matrix R. Put $i = 3$. Then we have:

$$R^3 = M_{1/6}^3 = \frac{1}{\sin(\pi/3)}\begin{pmatrix} \sin(4\pi/3) & -\sin(\pi) \\ \sin(\pi) & -\sin(2\pi/3) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{22}$$

This means that three iterations by R result in the negative of the identity matrix. Hence, whenever the trajectory arrives at a negative phase variable Y in iterating any matrix $M_{m/n}$, then we need just three iterations of the matrix R, or $M_{1/6}$, in order to retrieve a positive value of Y, hence in order to be able to apply the matrix $M_{m/n}$ anew. Accordingly, these three forward iterations play an important role.

For the piecewise linear map F the iterated points belong to an invariant curve which consists of portions of three ellipses, one given in (20) and its two forward images by the matrix R. These image ellipses can be easily derived, and the equations be given as

$$Y'^2 + 2\left(1 - \cos\left(2\pi\frac{m}{n}\right)\right)Z'^2 - 2\left(1 - \cos\left(2\pi\frac{m}{n}\right)\right)Y'Z' = 1 \tag{23}$$

and

$$2\left(1 - \cos\left(2\pi\frac{m}{n}\right)\right)Y''^2 + Z''^2 - 2\left(1 - \cos\left(2\pi\frac{m}{n}\right)\right)Y''Z'' = 1, \tag{24}$$

respectively. We let Y', Z' denote the first iterate, and Y'', Z'' denote the second iterate. We do not need any equation for the third iteration, because it gives us Eq. (20) back.

In Fig. 4 we illustrate the three ellipses (20), (23) and (24) for the particular case of $m = 1$ and $n = 8$. In the picture we have drawn the relevant parts of the three curves in black, as not the entire ellipses, only segments of them, are locations for the iterated points.

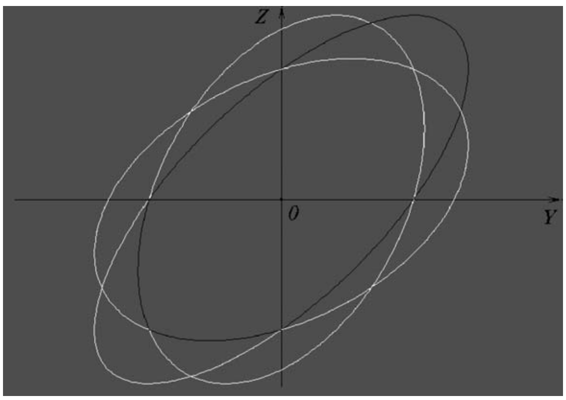

Fig. 4. The ellipses in the phase plane for $m = 1$, $n = 8$.

Before giving some general proofs we will present two detailed examples as illustrations of what is involved.

8.1. Example 1: How the 1/8 orbit becomes an 1/7 orbit

It is easiest to deal with the fundamental resonances (i.e., with $m = 1$) of even period. So, let us check out $n = 8$ for the linear map F_1. We find the corresponding c-value from (15): $c_{1/8} = \sqrt{2} - 1$. The first matrix (14) then becomes

$$M_{1/8} = \begin{pmatrix} \sqrt{2} & -1 \\ 1 & 0 \end{pmatrix}. \tag{25}$$

It is now easy to calculate that

$$M_{1/8}^4 = \begin{pmatrix} \sqrt{2} & -1 \\ 1 & 0 \end{pmatrix}^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{26}$$

The rightmost matrix is the negative of the identity matrix. If we multiply this with any initial state vector, then the signs of the entries are reversed. Hence we see that, if we started with the map (12), we will subsequently have to apply the map (13), i.e., the matrix R as given in Eq. (14). We could also check that all powers lower than 4 for the matrix (25), applied to an initial vector in the southeast quadrant, still result in positive first elements.

Substituting from (22) we get the composition:

$$M_{1/6}^3 M_{1/8}^4 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^3 \begin{pmatrix} \sqrt{2} & -1 \\ 1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{27}$$

As we land at the identity matrix, this means that the trajectory is *back to an initial point in state space*, so indeed, though we have the two-piece map (12) and (13), the orbits are still periodic. But, from (27) we find that the trajectory returns in 7 steps. Thus the period of the orbit is $n' = 7$, though the parameter c was calculated for $n = 8$.

By this we understand why the point of issue for the 7-period orbit in Fig. 3 is lifted to the place where the 8-period orbit issues in Fig. 2. The facts are illustrated in Figs. 5 and 6, where the first picture shows the original 8-period cycle of the map F_1 without the consumption floor applied, and the latter shows the 7-period resulting from the combination of the two maps F_1 and F_2. The pictures are more or less self-explanatory. We may just underline one more feature: There is the positive diagonal, and a "staircase" construction showing how the new iterated points are obtained by reflection in this diagonal. This is possible once we know the shape of the locus for the periodic orbits, because in each new iteration Z is always replaced by the old value of Y.

In the same way we could explain why the point of issue for the 9-period tongue is lifted to the place where the 12-period tongue bifurcates, and why nothing at all happens to the 6-period tongue. Anticipating the general considerations, the relation of the new period n' for the model with a consumption floor seems to be related to the period n of the original model through the relation $n' = (n + 6)/2$. Apparently this works in the case of fundamental resonances ($m = 1$) for even periods n.

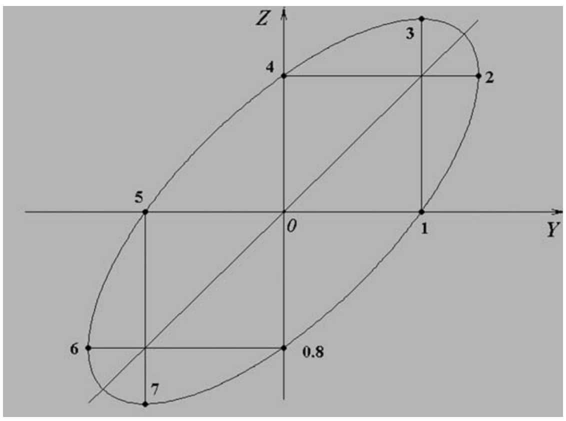

Fig. 5. One 1/8-orbit of the linear map F_1 on its proper ellipse.

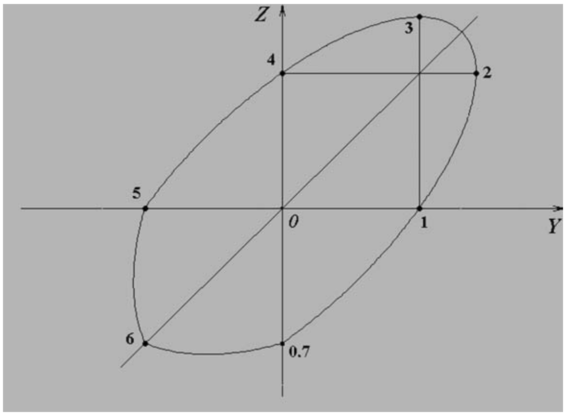

Fig. 6. Emergence of the 1/7-orbit in the map F.

8.2. Example 2: How the 1/13 orbit becomes a 2/19 orbit

Suppose we again start from the initial point: $v_0 = (Y_0, Z_0) = (0, -1)$. We iterate v_0 through continued multiplication by $M_{1/13}$ as long as $Y_i \geqslant 0$ holds. After six steps this is still true, so we can apply $M_{1/13}$ in a total of 7 steps. However, $v_7 = M_{1/13}^7 v_0$ returns a negative first element, so we have to apply R in stead, as always in three steps. Next, $v_{10} = R^3 M_{1/13}^7 v_0$ again returns a nonnegative first element, so we now apply $M_{1/13}$ anew in six iterations, until $v_{16} = M_{1/13}^6 R^3 M_{1/13}^7 v_0$. Finally, three more iterations with R yield

$$v_{19} = R^3 M_{1/13}^6 R^3 M_{1/13}^7 v_0. \tag{28}$$

From (22) we know that $R^3 = -I$, so

$$v_{19} = M_{1/13}^6 M_{1/13}^7 v_0 = M_{1/13}^{13} v_0. \tag{29}$$

However, from (18) $M_{1/13}^{13} = I$, so

$$v_{19} = v_0, \tag{30}$$

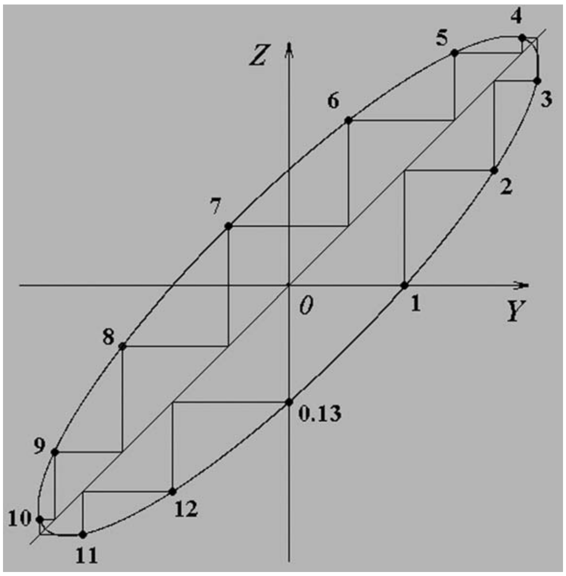

Fig. 7. The 1/13-cycle of the linear map F_1.

and the orbit is back at the starting point. As we see from (28), the period is now $n = 19$. This is shown in Fig. 8, where we labelled all the periodic points. As comparison, the original 1/13-cycle is shown in Fig. 7.

We have thus seen how the cycle of period 13 becomes a cycle of period 19, but, as we need two rounds in the phase diagram, it is the cycle with rotation number 2/19 that takes the place of the original 1/13-cycle in Fig. 2. This tongue is located between the $2/18 = 1/9$, and the $2/20 = 1/10$ tongues in Fig. 2, which indeed agrees with the aforementioned rule that between the rotation numbers 1/9 and 1/10 there exists also the rotation number $(1 + 1)/(9 + 10) = 2/19$. As the fundamental resonance the 9-period and 10-period tongues from Fig. 2 in Fig. 3 are dislocated to the places of the 12-period and the 14-period tongues, it is quite natural that the 2/19 we just studied takes the place of the previous 13-period tongue.

9. Proof of the rule for change of rotation numbers

In this section we describe the dynamics associated with the piecewise linear map F given in (12) and (13) at the bifurcation value $a = 1$ for which we have a *centre-like* situation. The map F is defined by the linear map F_1 (given by the matrix M) for $Y \geqslant 0$, and by the linear map F_2 (given by the matrix R) for $Y < 0$ (see (14)). Note that the matrix R does not depend on the parameters, being a rotation with the rotation number 1/6.

Recall briefly what was already written about the geometric shape of an invariant curve on which a trajectory of F is located. It is convenient to denote by NW, SW and SE, respectively, the northwest, southwest and southeast quadrants of the phase plane. Then for any initial point $(Y_0, Z_0) \in \mathbb{R}^2 \setminus SW$, its trajectory under the map F belongs to a closed invariant curve made up of arcs of the three ellipses: The first arc belongs to $\mathbb{R}^2 \setminus SW$ being a portion of an invariant ellipse $\mathscr{E}$ of the linear map F_1 through the point (Y_0, Z_0); The other two arcs belong to SW being the two successive images by F_2 of the arc $\mathscr{E} \cap NW$. The equations for the corresponding ellipses in the case $(Y_0, Z_0) = (0, -1)$ are given in (20), (23) and (24). Several examples of the invariant curves for the different values of c are shown in Figs. 6, 8 and one more example is given below (see the dashed curve in Fig. 9). The phase plane is filled with such invariant curves related to different initial conditions. As we prove in this section the trajectories on an invariant curve are either periodic or quasiperiodic, depending on the value of the parameter c.

Let us recall some properties associated with a *centre*, which hold for the linear map F_1. If the value of the parameter c is given as in (15), i.e., $c = c_{m/n}$, then the map F_1 is defined by the matrix $M_{m/n}$ (see (16)). In such a case any initial point of the phase plane is periodic with rotation number m/n, i.e., any point is n-periodic and the whole orbit is obtained with m turns around the fixed point O. The orbit belongs to an invariant curve which is an ellipse $\mathscr{E}$ with centre in O. Note that for $c > 0$ we have $m/n < 1/6$. Due to $\det M = 1$ the rotation is in counterclockwise direction.

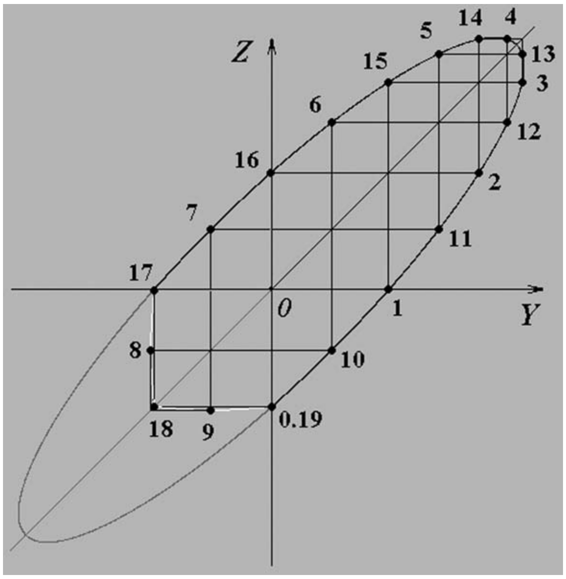

Fig. 8. The 2/13-cycle of the piecewise linear map F.

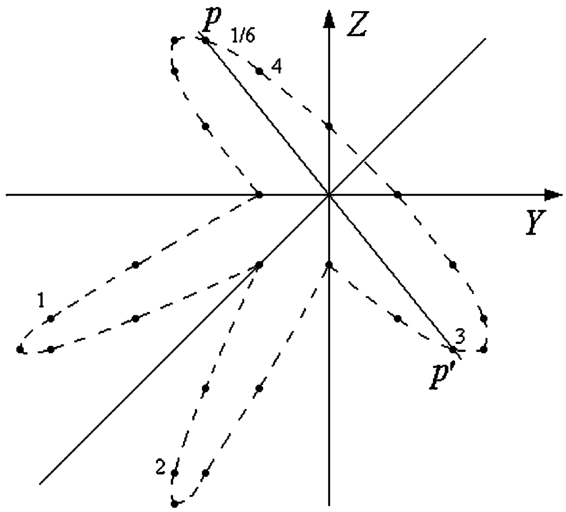

Fig. 9. An example of the periodic orbit of the map F with the rotation number $m'/n' = 6/25$.

The number n/m can be interpreted as the fraction of the total amount n of periodic points which are visited in one full turn around the fixed point. This ratio is an integer number only for $m = 1$, while it is a fraction in the general case: $n/m = k + l/m$, where k is an integer and $0 \leqslant l < m$.

Let us also comment on the position of the periodic points. If n is *even* then each point $p = (Y_0, Z_0)$ belongs to a periodic orbit denoted $o(p)$ which is symmetric with respect to the origin, i.e., the point $p' = (-Y_0, -Z_0)$ belongs to the same periodic orbit. While if n is *odd* then each point p belongs to a periodic orbit $o(p)$ which is not symmetric, and the symmetric point p' belongs to another orbit denoted $o(p')$ which is symmetric to $o(p)$.

We shall prove that for $c = c_{m/n}$ also the piecewise linear map F is associated with a rotation, but with a different rotation number m'/n' (as we have seen in the examples of the previous section). The result is stated in the following:

Proposition 1. *Let $a = 1$ and $c = c_{m/n}$ given in (15). Then the fixed point of the piecewise linear map F is centre-like with rotation number m'/n' where*:

$m' = m$, $n' = (6m + n)/2$ for n even;
$m' = 2m$, $n' = (6m + n)$ for n odd.

The proof consists of two parts. First we assume that the map F has a periodic orbit and show that the rotation number is $m'/n' = 2m/(6m + n)$. In the second part we prove that any initial point of the phase plane is periodic.

So, assume that the map F has a periodic orbit $o(p)$. We can obtain the new rotation number m'/n' by computing the fraction n'/m' as a function of the fractions related to the rotation numbers 1/6 (related to the matrix R) and m/n (related to the matrix $M_{m/n}$). To do this let us consider, without loss of generality, an initial point p of the orbit $o(p)$ belonging to the *NW* quadrant. Then the matrix R is applied to p three times before the trajectory enters the *SE* quadrant, where the matrix $M_{m/n}$ is applied. As $R^3 p = p'$, after 3 steps the map F has already done half round, while the second half is associated only with the matrix $M_{m/n}$, so that the number of steps in the second half of the round is $n/2m$. Summing up we have

$$\frac{n'}{m'} = 3 + \frac{n}{2m} = \frac{6m + n}{2m}, \tag{31}$$

and the new rotation number is

$$\frac{m'}{n'} = \frac{m}{3m + n/2} \quad \text{for } n \text{ even}, \tag{32}$$

$$\frac{m'}{n'} = \frac{2m}{6m + n} \quad \text{for } n \text{ odd}. \tag{33}$$

To prove the second part of the assertion, we consider an initial point $p \in NW$ without loss of generality, given that a point $(0, Z)$ is mapped by F into $(-Z, 0)$, and the arc of the invariant ellipse $\mathscr{E}$ through p in that quadrant and its images give a whole invariant curve. The map F_2 applies to p three times, and we get the symmetric point $p' \in SE$, to which the linear map F_1 applies. Clearly the point p' belongs to the same periodic orbit $o(p)$ if n is even (so that $o(p') = o(p)$), or it belongs to the symmetric periodic orbit $o(p')$ if n is odd. In any case, after a finite number of iterations by F_1, the iterated point moves on the periodic orbit $o(p')$ and reaches the NW quadrant, where again the map F_2 applies three times, and we get the symmetric point, which is necessarily a periodic point of the same orbit if n is even, or of the symmetric one if n is odd. And so on. It is clear that on the invariant ellipse associated with the matrix $M_{m/n}$ only the points of the periodic orbits $o(p)$ and $o(p')$ are visited, which are finite in number, and thus after a given suitable integer number of iterations the trajectory visits the initial point, which is thus periodic.

In fact, let us consider the case *n even*. We know that p belongs to an orbit $o(p)$ associated with the map $M_{m/n}$ which includes also the symmetric point p'. This orbit also includes the $n/2$ points of $o(p)$ belonging to the region $Y > 0$, and the m points in the NW quadrant, between a point $(0, Z)$ and its image by F, together with their images by the map F_2. Thus we have in total $n/2 + 3m$ points which necessarily belong to the orbit of p and no other points can exist. Being this number exactly n', it is periodic.

Similarly we reason when n is *odd*. We know that p belongs to an orbit $o(p)$ associated with the map F_1, and the orbit of F through p also involves the symmetric point p' belonging to the symmetric orbit $o(p')$. The orbit of F through p in $Y > 0$ involves only points of $o(p)$ and of $o(p')$ which are in total n ($\lfloor n/2 \rfloor$ points of $o(p)$ plus $\lfloor n/2 \rfloor + 1$ points of $o(p')$, or vice versa). There are also the $2m$ points of $o(p)$ and of $o(p')$ in the NW quadrant, between a point $(0, Z)$ and its image by F, together with their images by application of the map F_2. Thus we have a total of $n + 6m$ points which necessarily belong to the orbit of p and no other points can exist. Being this number exactly n', the trajectory is periodic.

As an example, let $c = c_{3/7}$, so that the rotation number for the map F_1 is $m/n = 3/7$. In Fig. 9 we present an orbit of the map F at $c = c_{3/7}$, which is a cycle with rotation number $m'/n' = 6/25$. The initial point p maps to the point p' in 3 steps, $n/2m = 1 + 1/6$, so that $n'/m' = 3 + 1 + 1/6 = 25/6$ and $m'/n' = 6/25$.

Remark. It is worth to note that the *summation rule*, which holds for the rotation numbers in the general case with smooth functions, also holds for the piecewise linear case. That is, if m_1/n_1 and m_2/n_2 are two rotation numbers associated with the parameters c_1 and c_2 at $a = 1$, then also the rotation number $(m_1 + m_2)/(n_1 + n_2)$ exists in between. The same rule works also for m_1'/n_1' and m_2'/n_2': the rotation number $(m_1' + m_2')/(n_1' + n_2')$ exists in between.

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Appendix A. Dynamics of the nonlinear Hicksian model

The purpose of this Appendix is to describe briefly the global dynamics of the map (9) in the phase space, in accordance with the two-dimensional bifurcation diagram shown in Fig. 2. To proceed let us rewrite the difference equation (9) as a two-dimensional piecewise linear map $H : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$H:\quad (Y, Z) \mapsto \begin{cases} F_1(Y, Z), Z \leqslant Y + D/a; \\ F_2(Y, Z), Z > Y + D/a; \end{cases}$$

where

$$F_1:\quad \begin{pmatrix} Y \\ Z \end{pmatrix} \mapsto \begin{pmatrix} (c + a)Y - aZ \\ Y \end{pmatrix},$$

$$F_2:\quad \begin{pmatrix} Y \\ Z \end{pmatrix} \mapsto \begin{pmatrix} cY - D \\ Y \end{pmatrix}.$$

Here a, c and D are real parameters: $0 < c < 1$, $a > 0$, $D > 0$. So, the map H is given by two linear maps F_1 and F_2 defined, respectively, below and above the critical line

$$LC_{-1} = \{(Y, Z) : Z = Y + D/a\}.$$

The map H has a unique fixed point $(Y^*, Z^*) = (0,0)$ which is the fixed point of the map F_1 (while the fixed point of the map F_2, $(Y, Z) = (D/(c-1), D/(c-1))$, does not belong to the region of definition of F_2, so that it is not a fixed point for the map H).

Recall that using the eigenvalues $\lambda_{1,2}$ of the Jacobian matrix of the map F_1, given in (3), we get that the fixed point (Y^*, Z^*) is a node for $(c+a)^2 > 4a$ (attracting for $a < 1$, repelling for $a > 1$), while it is a focus for $(c+a)^2 < 4a$ (attracting for $a < 1$, repelling for $a > 1$).

At $a = 1$ the fixed point loses stability with the pair of complex-conjugate eigenvalues crossing the unit circle, so, a center bifurcation occurs which can be considered as an analog of the Neimark–Sacker bifurcation for smooth maps: Namely, similar to the smooth case, for the piecewise linear map H the result of this bifurcation is an attracting invariant closed curve homeomorphic to a circle. It appears not in the neighbourhood of the fixed point, but far from it (its position is defined by the critical line LC_{-1}). On this curve the map H is topologically conjugate to a piecewise linear orientation preserving circle homeomorphism, thus, H is reduced to a rotation with some rational or irrational rotation number (see [6]). In case of a rational rotation number m/n the map H has an attracting cycle of period n, and we can see the corresponding n-periodicity region in the (a,c)-bifurcation plane shown in Fig. 2. An irrational rotation number corresponds to a quasiperiodic orbit of the map H. After some additional considerations we give examples of the phase portrait of the map H in the case of a periodic orbit for $a = 1$ and $a > 1$.

First note that the map F_2 has a rather simple dynamics: As the eigenvalues of the Jacobian matrix of the map F_2 are $\mu_1 = c$, $\mu_2 = 0$, where $0 < c < 1$, any initial point of the half-plane above LC_{-1} is mapped by F_2 in one step into a point of the straight line

$$LC_0 = H(LC_{-1}) = \{(Y, Z) : Z = (Y + D)/c\}.$$

Following iterations by F_2 give points on this straight line approaching the fixed point of F_2 (which is below LC_{-1}), until the trajectory enters the half-plane below LC_{-1} where the map F_1 is applied. Then, at $a > 1$, as long as the fixed point is a focus, the trajectory spirals away from it and in a finite number of iterations the trajectory enters the region above LC_{-1} where the map F_2 is applied. Then F_2 acts as a return mechanism for the orbit to the region below LC_{-1}. In such a way we can get either a periodic, or a quasiperiodic orbit.

What does the phase portrait of the map H exactly look like at the bifurcation $a = 1$? It is easy to show that at $a = 1$ the fixed point is locally a center with the rotation number m/n if the value of the parameter c is $c = 2\cos(2\pi m/n) - 1$, obtained from $\mathrm{Re}\,\lambda_{1,2} = \cos(2\pi m/n)$. Similarly as in [2], it can be shown that in this case in the phase space of the map H there exists an invariant polygon $\mathscr{P}$ made up by n segments of the critical lines LC_i, where $LC_i = H(LC_{i-1})$, $i = 0, 1, \ldots, n-1$. Any point of the inner part of the polygon is periodic with the rotation number m/n, while any point outside $\mathscr{P}$ is mapped in a finite number of steps into a point of the boundary of $\mathscr{P}$ which is also periodic. Fig. 10 shows an example of the polygon $\mathscr{P}$ at $a = 1$, $c = c_{1/8} = \sqrt{2} - 1$, $D = 100$, made up by 8 segments of the critical lines, and corresponding to a center bifurcation with the rotation number 1/8.

In the case of an irrational rotation number there exists a closed invariant region $\mathscr{Q}$ bounded by an invariant ellipse $\mathscr{E}$ of the map F_1, tangent to LC_i, $i = 0, 1, \ldots$ The region $\mathscr{Q}$ is filled with other ellipses, such that any inner point of $\mathscr{Q}$ belongs to a quasiperiodic orbit dense in the corresponding invariant ellipse of F_1. Any point outside $\mathscr{Q}$ is mapped

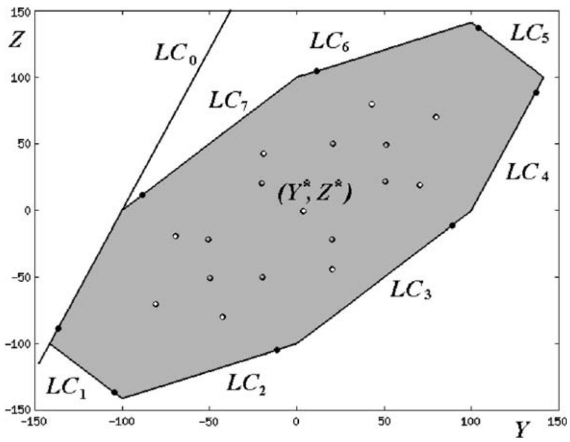

Fig. 10. The invariant polygon $\mathscr{P}$ of the map H at $a = 1$, $c = c_{1/8} = \sqrt{2} - 1$, $D = 100$.

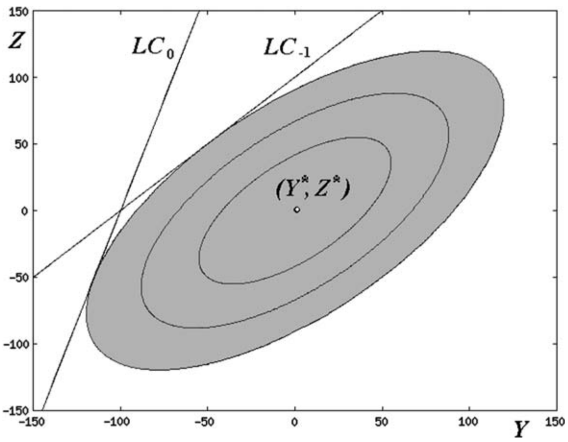

Fig. 11. The invariant region of the map H at $a = 1$, $c = 0.3$, $D = 100$.

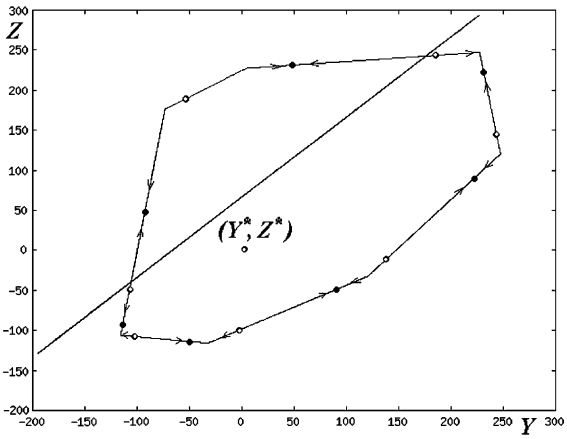

Fig. 12. The attracting closed invariant curve of the map H at $a = 1.5$, $c = 0.15$, $D = 100$.

in finite number of steps into a point of $\mathscr{E}$ which is also a point of a quasiperiodic orbit (Fig. 11 shows the periodic orbit of high period at $a = 1$, $c = 0.3$, $D = 100$, which gives an idea of what the quasiperiodic orbit looks like).

Let now $a > 1$ and let the (a, c)-parameter point belong to the m/n-periodicity region. Then the map H has two m/n-cycles, one attracting and one saddle, so that the corresponding closed invariant curve in the phase plane is formed by the stable set of the saddle cycle approaching the points of the attracting cycle. This stable set is also made up by segments of corresponding LC_i, $i \geqslant 0$. Fig. 12 shows an example of the attracting and saddle 1/7-cycles of the map H, together with the stable manifold of the saddle cycle, which forms an attracting closed invariant curve, at $a = 1.5$, $c = 0.15$, $D = 100$.

Appendix B. Dynamics of the modified Hicksian model

It this appendix, we show that the modified Hicksian model (11) has dynamics qualitatively similar to those of the model (9). With the introduction of the additional constraint (10), there exists one more critical line LC'_{-1} in the (Y, Z) phase plane given by $Y = 0$, besides the critical line LC_{-1} given by $Z = Y + D/a$. So, the phase plane becomes separated into four regions R_i, $i = 1, \ldots, 4$, where four different linear maps F_i are applied. Thus, we consider the piecewise linear continuous map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F_1:\quad \begin{pmatrix} Y \\ Z \end{pmatrix} \mapsto \begin{pmatrix} (c+a)Y - aZ \\ Y \end{pmatrix} \quad \text{for } (Y,Z) \in R_1 = \{(Y,Z) : Y \geqslant 0, Z \leqslant Y + D/a\},$$

$$F_2:\quad \begin{pmatrix} Y \\ Z \end{pmatrix} \mapsto \begin{pmatrix} a(Y - Z) \\ Y \end{pmatrix} \quad \text{for } (Y,Z) \in R_2 = \{(Y,Z) : Y \leqslant 0, Z \leqslant Y + D/a\},$$

$$F_3:\quad \begin{pmatrix} Y \\ Z \end{pmatrix} \mapsto \begin{pmatrix} cY - D \\ Y \end{pmatrix} \quad \text{for } (Y,Z) \in R_3 = \{(Y,Z) : Y \geqslant 0, Z > Y + D/a\},$$

$$F_4:\quad \begin{pmatrix} Y \\ Z \end{pmatrix} \mapsto \begin{pmatrix} -D \\ Y \end{pmatrix} \quad \text{for } (Y,Z) \in R_4 = \{(Y,Z) : Y \leqslant 0, Z > Y + D/a\}.$$

The map F has a unique fixed point $(Y^*, Z^*) = (0,0)$ which is the fixed point for both maps F_1 and F_2, and belongs to LC'_{-1}. (The fixed point $(D/(c-1), D/(c-1))$ of the map F_3 is out the region R_3, and neither the fixed point $(-D,-D)$ of the map F_4 does belong to R_4, so these points are not fixed points of F.)

It can be easily shown that also for the modified map F the stability condition of (Y^*, Z^*) is $a < 1$: The eigenvalues of the map F_2 are $\lambda'_{1,2} = (a \pm \sqrt{a^2 - 4a})/2$ being complex-conjugate for $a < 4$, with $|\lambda'_{1,2}| < 1$ for $a < 1$. Thus, considering the map F_2 alone, for $a < 1$ its fixed point is an attracting focus, and it becomes a center for $a = 1$ with rotation number $1/6$ ($\operatorname{Re}\lambda'_{1,2} = 1/2 = \cos 2\pi/6$).

Recall that considering the map F_1 alone, we find that the fixed point (Y^*, Z^*) is a focus for $(c+a)^2 < 4a$, being attracting for $a < 1$ and repelling for $a > 1$. At $a = 1$ it is a center with either a rational or an irrational rotation number. If $\operatorname{Re}\lambda_{1,2} = \cos(2\pi m/n)$, i.e., if $c = 2\cos(2\pi m/n) - 1$, then the fixed point is a center with the rational rotation number m/n.

At $c = 0$ the maps F_1 and F_2 are identical, representing rotation with the rotation number $1/6$. In general, at $a = 1$, $c = 2\cos(2\pi m/n) - 1$, we say that locally (in a neighbourhood of (Y^*, Z^*), specified below) the map F is *center-like*. The corresponding rotation number m'/n' is equal to $2m/(n + 6m)$, which for n even becomes $m/(n/2 + 3m)$ (see Proposition 1).

Let $a > 1$. Then as long as (Y^*, Z^*) is a focus, any initial point from its neighbourhood spirals away from (Y^*, Z^*) under the maps F_1 and F_2, until the trajectory enters the region R_3. The eigenvalues of F_3 are $\mu_1 = c$, $\mu_2 = 0$, thus any point of R_3 is mapped by F_3 in one step into a point of the critical line $LC_0 = F_1(LC_{-1}) = \{(Y,Z): Z = (Y + D)/c, Z \geqslant 0\}$. Then iterations by F_3 give points belonging to LC_0, approaching the fixed point of F_3 (which is in R_2), until necessarily the trajectory enters the region R_4. Any point of R_4 is mapped by F_4 in two steps into the point $(-D,-D) \in R_2$. So, the trajectory again begins to rotate first under F_2, then under F_1, and so on. In such a way we get either periodic or quasiperiodic trajectory.

Let $a = 1$ and $c = 2\cos(2\pi m/n) - 1$. In a similar way as in [2], it can be shown that in the phase space of the map F we obtain an invariant polygon $\mathscr{P}$ such that any point of the inner region bounded by the polygon is periodic with rotation the number m'/n', while any point outside $\mathscr{P}$ is mapped in a finite number of steps into a point of $\mathscr{P}$ which is also periodic with rotation number m'/n'. The polygon $\mathscr{P}$ is made up by n' segments of the critical lines LC_i, $LC_i = F(LC_{i-1})$, $i = 0, 1, \ldots, n' - 1$. Note that

$$LC_0 = \{(Y,Z) : Z = (Y + D)/c, Z \geqslant 0\} \cup \{(Y,Z) : Y = -D, Z < 0\}.$$

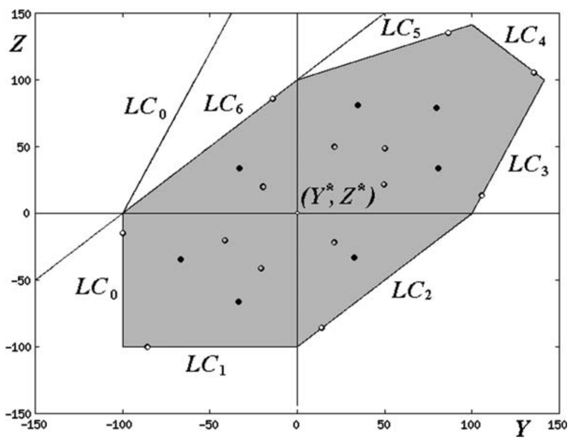

Fig. 13. The invariant polygon $\mathscr{P}$ of the map F at $a = 1$, $c = c_{1/8} = \sqrt{2} - 1$, $D = 100$.

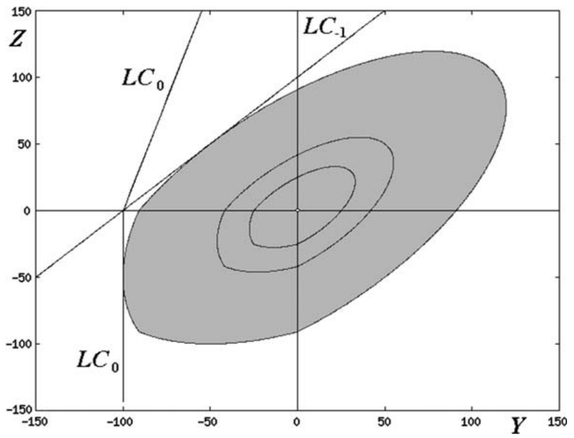

Fig. 14. The invariant region of the map F at $a = 1$, $c = 0.3$, $D = 100$.

Fig. 13 presents an example of the polygon $\mathscr{P}$ at $a = 1$, $c = \sqrt{2} - 1$, $D = 100$, made up by 7 segments of the critical lines and corresponding to a center-like bifurcation with the rotation number $m'/n' = 1/7$.

In the case of an irrational rotation number, there exists a closed invariant region $\mathscr{Q}$ bounded by an ellipse-like closed curve $\mathscr{C}$ made up by a portion of the corresponding invariant ellipse $\mathscr{E}$ of the map F_1, and portions of its two images by F_2. The closed curve $\mathscr{C}$ is tangent to LC_i, $i \geqslant 0$. The region $\mathscr{Q}$ is filled with other ellipse-like curves, such that any inner point of $\mathscr{Q}$ belongs to a quasiperiodic orbit dense in the corresponding ellipse-like curve, while any point outside $\mathscr{Q}$ is mapped in finite number of steps into a point of $\mathscr{C}$ which is also a point of the quasiperiodic orbit. Fig. 14 shows some quasiperiodic orbits (or rather periodic orbits of high period, due to numerical precision), and an invariant closed region at $a = 1$, $c = 0.3$, $D = 100$.

For $a > 1$ and the (a, c)-parameter point belonging to some m'/n'-periodicity region (see Fig. 3) the map F has an attracting and a saddle m'/n'-cycle. The unstable set of the saddle cycle, approaching the points of the attracting cycle, forms the attracting invariant closed curve in the phase plane. This unstable set is also made up of segments of LC_i for $i \geqslant 0$.

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