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Growing through chaos in the Matsuyama map via subcritical flip bifurcation and bistability

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Abstract

Recent publications revisit the growth model proposed by Matsuyama ("Growing through cycles", Econometrica 1999), presenting new economic interpretations of the system as well as new results on its dynamics described by a one-dimensional piecewise smooth map (also called M-map). The goal of the present paper is to give the rigorous proof of some results which were remaining open, related to the dynamics of M-map. We prove that an *attracting* 2-cycle appears via border collision bifurcation, give the *explicit flip bifurcation value* at which this cycle looses stability, as well as the explicit coordinates of its points at the bifurcation value, proving that the flip bifurcation is always of *subcritical type*. We show that this leads to the existence of a region of *bistability* associated with an attracting 2-cycle coexisting with attracting 4-cyclic chaotic intervals. This means that the effects of the destabilization of the 2-cycle, related to a corridor stability, are catastrophic and irreversible. We give also the conditions related to the sharp transition to chaos, proving that the cascade of stable cycles of even periods cannot occur. The parameter region in which repelling cycles of odd period exist is further investigated, namely, we give an explicit boundary of this region and show its relation to the non existence of cycles of period three.

Keywords: Endogenous growth models, Matsuyama map, piecewise smooth map, subcritical flip bifurcation, border collision bifurcation, skew tent map as a normal form

1. Introduction

We consider the growth model proposed by Matsuyama in [13], also called M-map (for example, by Deng and Khan in [5]). The strength of the model is in its simple formulation which takes into account the two sources of economic growth,

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namely, via a process of capital accumulation and processes of technical change and innovation. The dynamics of the model by Matsuyama are represented by a unimodal map $x_{t+1} = \phi(x_t)$ consisting in an increasing function of the (normalized) capital accumulation given by $x_{t+1} = Gx_t^{1-\frac{1}{\sigma}}$ for $x_t \leq 1$ (related to the phase without innovation, also called Solow regime) and a decreasing function $x_{t+1} = Gx_t/(1+\theta(x_t-1))$ for $x_t \geq 1$ (representing a Romer regime of phase of innovation). Thus, the M-map is given by

$$x_{t+1} = \phi(x_t) = \begin{cases} f(x_t) = G{x_t}^{1-\frac{1}{\sigma}} & \text{if} \quad 0 < x_t < 1 \quad \text{(Solow regime)} \\ g(x_t) = \frac{Gx_t}{1+\theta(x_t-1)} & \text{if} \quad x_t > 1 \qquad \text{(Romer regime)} \end{cases} \tag{1}$$

where $G > 0$, $\sigma > 1$ and $\theta = (1 - \frac{1}{\sigma})^{(1-\sigma)}$. Here, x_t represents the ratio of accumulated capital to cumulative innovation. When the economy has a relatively small stock of capital (i.e., $x_t < 1$), the economy is in the Solow regime; there is no innovation and the economy can grow only through capital accumulation, which is subject to diminishing returns, with the share of capital in production, $(1-\frac{1}{\sigma})$ (which we shall denote by γ for convenience), being strictly less than one. When the economy has a relatively large stock of capital (i.e., $x_t > 1$), then the economy is in the Romer regime; there is active innovation. The parameter G is the growth potential of the economy. If $G < 1$, the economy will eventually stop growing, and remains in the Solow regime forever. If $G > 1$, there is a balanced growth path, along which the stock of capital and cumulative innovation continue to grow at the same rate, and for $\sigma < 2$ the ratio, x_t, remains constant in the Romer regime. However, the balanced growth path may be unstable, because innovations may arrive in waves, because the past innovations discourage innovations more than contemporary innovations. Matsuyama in [13] shows that the strength of this effect is captured by $\theta = (1-\frac{1}{\sigma})^{(1-\sigma)} > 1$, which is increasing in σ.

As we recall below, after the appearance of the M-map its dynamics have been considered in several papers, see e.g. Mitra [15], Mukherji [16], Gardini et al [7], Deng and Khan [5], and are still not completely understood. Moreover, besides the growth model as considered in the papers cited above, a new interpretation of a similar growth model has been recently proposed by Sunaga in [20], motivated by the understanding of why the development of the financial sector is not always promoting economic growth. To this goal, the author develops a discrete-time model in which entrepreneurs and financial intermediaries engage in their respective innovative activities, leading to a map having the same basic structure as that in [13], except it includes financial intermediaries' activities. The system is given by

$$k_{t+1} = \Phi(k_t) = \begin{cases} \overline{G}{k_t}^{1-\frac{1}{\sigma}} & \text{if} \quad 0 < k < \Omega \\ \frac{\overline{G}k_t\Omega^{1-\frac{1}{\sigma}}}{\Omega+\theta(k_t-\Omega)} & \text{if} \quad k > \Omega \end{cases} \tag{2}$$

(see [20]), where the new parameter Ω is related to financial activities. The system is studied as a model different from the M-map. However, the map in (2) is topologically conjugate to the M-map, so all the dynamics and bifurcations are qualitatively the same. In fact, via the following change of variable and redefinition of the parameter:

$$k = h(x) := \Omega x \ , \ G = \overline{G}\Omega^{-\frac{1}{\sigma}} \tag{3}$$

the M-map in (1) is obtained, that is: $\phi(x) = h^{-1} \circ \Phi \circ h(x)$. Clearly now both the state variable x ($x = h^{-1}(k) = k/\Omega$) and the parameter G depend on the financial parameter Ω.

Thus, stimulated from the various economic significance of the model, and the several open questions related to the properties of the M-map, we propose new results on the dynamics of this system. That is, the goal of this work is to give a detailed explanation of the bifurcation mechanisms occurring in the M-map which are not yet well clarified.

Recall that in terms of dynamical system theory, a bifurcation occurs when an infinitesimal change in the value of some parameter of a system causes a qualitative (topological) change in its dynamic behavior. The bifurcations occurring to a steady state, or fixed point, may be classified according to the qualitative changes in the local dynamics (i.e. in a neighborhood of the steady state). This is a general theory well developed for smooth systems. The flip bifurcation and pitchfork bifurcation are mainly known in their supercritical occurrence. However, both may occur in the subcritical form, which is of particular type, as described below for the flip bifurcation, occurring in the system here investigated, which also has the peculiarity to be not smooth.

The limitations of smooth dynamical systems was already evidenced by Baumol and Benhabib in [2], an early survey of chaotic dynamics in economics, and nowadays it occurs more and more often that an economic model is described by a nonsmooth system, for which different kinds of bifurcations may occur. In nonsmooth systems, when a kink point exists at which the function defining the system changes abruptly, bifurcations may occur called *border collision*. This happens when an invariant set, as a cycle of any period, has a point which merges with the kink point (the point $x = 1$ in our system), colliding with it. In such a case it is more difficult to classify the dynamic effect of the bifurcation. However, when the invariant set undergoing border collision is a $k-$cycle, then it is possible to use the skew tent map as a border collision normal form (as we shall recall in the following), applied to the $k-$th iterate of the system.

Also recall that in the theory of dynamical system, the set of initial conditions that converge to an attractor (that is, an attracting invariant set, such as an attracting steady state, an attracting period-2 cycle, a chaotic attractor, etc.) is called its basin of attraction. While the set of initial conditions converging to an invariant set which is not attracting (such as an unstable steady state, an unstable period-2 cycle, etc.) is called its stable set.

Coming back to the piecewise smooth system (1) of interest, recall that as shown in [13], the steady state x^* of the M-map is attracting in the Solow regime ($x < 1$) for $0 < G < 1$, at $G = 1$ it merges with the kink point $x = 1$ (so it undergoes a border collision bifurcation) and for $G > 1$ the steady state

x^* belongs to the Romer regime ($x > 1$). For $G > \theta - 1$ it is attracting and, decreasing G, it becomes unstable at $G = \theta - 1$. For $1 < G < \theta - 1$ there exist dynamics which are alternating between these two phases leading to a higher growth than that associated with the steady state.

Chaotic dynamics can occur. The condition of the first homoclinic bifurcation of the fixed point x^* in the Romer regime has been considered by Mitra in [15].

The flip bifurcation of the fixed point x^* in the Romer regime (at the bifurcation value $G = \theta - 1$) is not a standard one. This degenerate bifurcation has been investigated by Mukherji in [16], together with a sufficient condition for the stability of a 2-cycle, and a sequence of period doubling bifurcations was conjectured to exist.

The effect of the border collision bifurcation of the steady state x^* when (from attracting in the Solow regime) at $G = 1$ it merges with the kink point $x = 1$ and then, for $G > 1$, enters the Romer regime, has been shown in [7]. For $1 < \sigma < 2$ as G increases through 1 the steady state from globally attracting in the Solow regime becomes globally attracting in the Romer one. For $\sigma > 2$ as G increases through 1 the steady state in the Solow regime loses its stability and becomes unstable in the Romer regime. After this border collision bifurcation the dynamic behavior is characterized by one of the following kinds of asymptotic behaviors, depending on the value of the parameter σ: i) a stable cycle of period 2; ii) a robust $k-$cyclic chaotic attractor with $k = 4$ or $k = 2$ or $k = 1$.

Although the dynamics may be chaotic, Deng and Khan in [5] have shown the non existence of 3- and 5-cycles in the parameter range of interest, while it is given numerical evidence of the existence of a 7-cycle, which is thus the smallest possible period of a cycle of odd period occurring in the M-map.

Besides the results here summarized, the problems listed below are still open.

- $P1$ Is it true that with decreasing G an attracting cycle of period 2 always appears at the degenerate flip bifurcation of the fixed point x^* in the Romer regime (at $G = \theta - 1$)?

Indeed, numerical examples show the existence of a stable 2-cycle whose states are alternating between the Solow and Romer phases, but only examples and sufficient conditions have been given, so it is unclear if this is always the case. This is also a global bifurcation corresponding to the border collision bifurcation (with $x = 1$) of a 2-cycle and it may lead to a 2-cycle or to a 2^k-cyclic chaotic attractor for $k \geq 1$ (as described in [21]). We shall prove that for any $\sigma > 2$ an attracting 2-cycle necessarily appears crossing the bifurcation at $G = \theta - 1$. That is, either the unique steady state in the Romer regime is globally attracting, or fluctuations between the Solow and Romer regimes occur.

- $P2$ At which parameter values the stable 2-cycle becomes unstable? And which kind of bifurcation is it?

Indeed, the flip bifurcation of the 2-cycle of the M-map has never been determined so far, the attracting 2-cycle may become unstable via a border collisions bifurcation or via a smooth flip bifurcation, sub- or supercritical. In [16] besides a sufficient condition for the stability of the 2-cycle it is argued that when it becomes unstable, attracting cycles of period 2^n for $n > 1$ can exist. Differently, in [7] it is conjectured that when the 2-cycle is unstable, attracting cycles of period 2^n for $n > 1$ cannot exist, and that the asymptotic trajectories belong to 4-cyclic chaotic intervals. Here we give the explicit value of the parameter G at which a smooth flip bifurcation of the 2-cycle occurs, and the explicit expression of the periodic points of the 2-cycle. This also allows us to prove that it is of subcritical type.

*P*3 Is it possible the existence of an attracting 4-cycle? At which parameter values chaotic dynamics exist?

Proving that the 2-cycle undergoes a subcritical flip bifurcation, we also show that a repelling 4-cycle must appear (decreasing G) before such a flip bifurcation. The investigation of the possible mechanisms of the appearance of the 4-cycles leads to the conditions for which attracting cycles of period 2^n for $n > 1$ cannot exist. The occurrence of a subcritical flip bifurcation of the 2-cycle results in coexistence of two attracting sets, and we give the explicit conditions for which the 2-cycle and 4-cyclic chaotic intervals coexist. So, no stable 4-cycle exists, and chaotic dynamics appear when the 2-cycle is still attracting.

So, we show that as G decreases, stable fluctuations between the Solow and Romer regime always occur. These two-cyclical fluctuations become more complicated soon after the flip bifurcation of the 2-cycle. Since the 2-cycle looses stability via a subcritical flip bifurcation there is coexistence between an attracting 2-cycle and 4-cyclical chaotic fluctuations between the two regimes. That is, still when the 2-cycle is locally attracting, chaotic fluctuations occur, which are coexisting with the regular oscillations. The basins of the two attracting sets are separated by an unstable 4-cycle. This implies *corridor stability*, to use the terminology introduced by Leijonhufvud in [9] . That is, the 2-cycle is locally stable but globally unstable so that small shocks can be absorbed but not large ones. Recall that, as remarked in Benhabib and Miyao [3], see also in Matsuyama et al. [14], corridor stability is another implication of nonlinearity. Its occurrence evidences that only the local stability of a steady state or a cycle may be misleading in interpreting the dynamic behavior in a neighborhood of the invariant set. Moreover, as we describe later, when the 2-cycle loses its local stability via a subcritical flip, the effects are catastrophic and irreversible.

Other problems are related to the parameter ranges in which the system is (becomes) chaotic in a unique interval including the unstable steady state.

*P*4 In which range of the parameters is the fixed point x^* homoclinic?

Regarding the bounded chaotic dynamics which can exist involving the fixed point x^* in the Romer regime of the M-map, to our knowledge the first sufficient conditions were given by Mitra in [15]. It is shown that in a unimodal map, an invariant set in which the map is chaotic exists when $\phi^3(1) < x^*$ which corresponds to the existence of homoclinic orbits of the fixed point x^* (and x^*

is also called a snap-back repeller, after Marotto [12], see also [8]). In [7] it is given the exact value of σ at which this homoclinic bifurcation of the snap-back repeller, defined by $\phi^3(1) = x^*$, can occur. However, the equation is given by an implicit function, so that it is difficult to estimate when this first homoclinic bifurcation of the fixed point occurs. Here we give a strict estimate for that condition, proving that repelling cycles of odd period can occur only in a narrow region of the parameter plane of interest.

$P5$ Is the condition $\phi^3(1) > 1$ sufficient to state the non existence of a 3-cycle in the M-map?

In [5] it is shown that 3-cycles and 5-cycles cannot exist in the M-map. However, the nonexistence of 3-cycles was already argued in [13] although the argument, represented by the condition $\phi^3(1) > 1$, was not developed. Here we show that indeed for the M-map that condition is enough to prove the non existence of 3-cycles.

Summarizing, the content of the paper is as follows. In Section 2 we recall some results on the M-map and the parameter range of interest. In Section 3 we prove that the degenerate flip bifurcation of the fixed point x^* in the Romer regime occurring at $G = \theta - 1$ leads always to an attracting 2-cycle with periodic points x_L and x_R in the two regimes (Theorem 1). In Section 4 we prove (Theorem 2) that the smooth flip bifurcation of the 2-cycle is of subcritical type, giving the explicit bifurcation value $G = G_2^*$ (necessarily for any $\sigma > \sigma_4 \simeq 3.825$) as well as the explicit expression of the two periodic points x_L and x_R. The occurrence of a subcritical flip bifurcation of the 2-cycle implies the appearance of a repelling 4-cycle before the flip bifurcation. Such a repelling 4-cycle may be associated either with a smooth fold bifurcation or directly with the BCB which must exist and related to the implicit equation $\phi^4(1) = 1$, and thus to a range with bistability. In the same section we show, by using the skew tent map as a normal form, when the BCB of a 4-cycle leads to 4-cyclic chaotic intervals (Theorem 3) and give an explicit interval which includes the bifurcation value (Theorem 4). In Section 5 we prove a strict range, in the parameter plane (G, σ), for the occurrence of the first homoclinic bifurcation of the fixed point x^* showing that for any value of σ (and necessarily $\sigma > \sigma_1 \simeq 21.231$) the bifurcation occurs for G in the interval $(1, 1.15)$ (Theorem 5). In Section 6 we prove in Theorem 6 that due to the properties of the M-map, the condition $\phi^3(1) > 1$ (which always holds) leads to the non existence of 3-cycles. Section 7 concludes, while some details of the proofs are relegated to the Appendices.

2. Preliminaries

In this section we recall from [7] some results of the M-map and of the functions $f(x)$ and $g(x)$ defining it, which characterize the left and right regimes (with respect to the kink point $x = 1$, L side and R side, for short), respectively and, in particular, the result of the border collision bifurcation (BCB for short) occurring to the fixed point x^* when it crosses the kink point $x = 1$ (at $G = 1$).

The M-map is defined in (1), where $\theta = (1 - \frac{1}{\sigma})^{(1-\sigma)}$ and $\sigma > 1$. We also introduce $\gamma = (1 - \frac{1}{\sigma})$ for our convenience. We are interested in the values of the parameters belonging to the parameter plane (G, σ) for

$$\sigma > 2 \text{ (so } \gamma = (1-\frac{1}{\sigma}) \in (0.5, 1),\ \ \theta = \gamma^{(1-\sigma)} \in (2, e)) \text{ and } 1 < G \leq (\theta - 1) \quad (4)$$

Note that for $1 < \sigma \leq 2$ the fixed point of the map is always globally attracting (in $x > 0$), for $G < 1$ it belongs to the L side (Solow regime) while for $G > 1$ it belongs to the R side (Romer regime). Differently, for any $\sigma > 2$ the fixed point in the Romer regime (for $G > 1$) given by

$$x^* = 1 + \frac{(G-1)}{\theta}$$

is globally attracting only for $G > (\theta - 1)$. This can be immediately seen from the expression of the first derivatives of the map in the two regimes, and the kind of monotonicity in the two regimes. In fact, the involved functions are both monotone: $f(x)$ is increasing and concave (in $[0, 1]$):

$$f'(x) = G\gamma x^{-\frac{1}{\sigma}} > 0\ ,\ f''(x) = G\gamma(-\frac{1}{\sigma})x^{-\frac{1}{\sigma}-1} = -\frac{1}{\sigma x}f'(x) < 0$$

while $g(x)$ is decreasing and convex (in $[1, G]$):

$$g'(x) = -\frac{G(\theta-1)}{[1+\theta(x-1)]^2} < 0\ ,\ g''(x) = \frac{2\theta G(\theta-1)}{[1+\theta(x-1)]^3} > 0$$

The inverse functions are given by:

$$f^{-1}(y) = (\frac{y}{G})^{\frac{1}{\gamma}}, \quad g^{-1}(y) = \frac{y(\theta-1)}{y\theta - G}$$

We shall also use the following expressions:

$$f'(x) = \gamma x^{-1} f(x), \quad f''(x) = -\frac{1}{\sigma}x^{-1}f'(x) = -\frac{1}{\sigma}\gamma x^{-2} f(x) \quad (5)$$

$$\begin{aligned} g'(x) &= -\frac{(\theta-1)}{x[1+\theta(x-1)]}g(x) \quad (6) \\ g''(x) &= -\frac{2\theta}{[1+\theta(x-1)]}g'(x) = \frac{2\theta(\theta-1)}{x[1+\theta(x-1)]^2}g(x) \end{aligned}$$

It is immediate to see that for $G > 1$ the asymptotic dynamics of the map belong to the *absorbing interval* $[g(G), G]$ to which it is possible to restrict the analysis: due to the monotonicity of $f(x)$ each point in $(0, g(G))$ is mapped into the absorbing interval in a finite number of iterations, and this absorbing interval is invariant for map $\phi(x)$.

An example of the M-map when the fixed point belongs to the Romer regime is shown in Fig.1a.

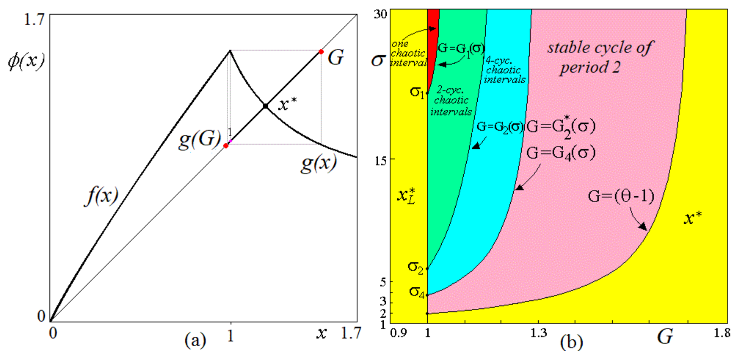

Figure 1: In (a) graph of the M-map and the absorbing interval $[g(G), G]$ for $\sigma = 10$ and $G = 1.5$. In (b) 2D bifurcation diagram in the plane (G, σ).

At a fixed value of σ, increasing the parameter G the stable fixed point in the Solow regime collides with the kink point $x = 1$ for $G = 1$.

This means that the economy neither grows nor stays stationary, it is into a divided state characterized, in formal terms, by a switching border. The economy is just in a critical point, from which it may develop to either a growth path (cyclical or chaotic) or to a stationary state, depending on the value of σ determined by the economic policy. For any $\sigma > 1$ the result of this border collision in the dynamic behavior is summarized in the following theorem, proved in [7], by using the skew tent map as a normal form (see also Appendix A).

Theorem (from [7]). *The border collision bifurcation of the fixed point $x^* = 1$ of the map ϕ given in (1), occurring at $G = 1$ for any $\sigma > 1$, gives rise to an*

- *attracting fixed point x^* in the Romer regime if* $1 < \sigma < 2$;
- *attracting cycle of period* 2 *if* $2 < \sigma < \sigma_4 \simeq 3.825$;
- *attracting* 4*-cyclical chaotic intervals if* $\sigma_4 < \sigma < \sigma_2 \simeq 6.123$;
- *attracting* 2*-cyclical chaotic intervals if* $\sigma_2 < \sigma < \sigma_1 \simeq 21.231$;
- *attracting chaotic interval if* $\sigma > \sigma_1$.

The normal form of the border collision bifurcation of a fixed point occurring in one-dimensional piecewise smooth maps (a detailed description of which can be found in [22]) is recalled in Appendix A, since it is used also in Section 3 to prove the outcome of the degenerate flip bifurcation of the fixed point x^*occurring at $G = (\theta - 1)$, as well as in Section 4 to prove the result of the BCB of a 4-cycle.

There are bifurcation curves issuing from the particular points of the straight line $G = 1$ in the 2D parameter plane (G, σ), whose equations are known in implicit or explicit form. The curve $G = (\theta - 1)$ is issuing from the point $(G, \sigma) = (1, 2)$, see Fig.1b. From the point $(G, \sigma) = (1, \sigma_4)$ the BCB curve of a 4-cycle $G = G_4(\sigma)$ of implicit equation $\phi^4(1) = 1$ is issuing. From $(G, \sigma) = (1, \sigma_2)$ the curve $G = G_2(\sigma)$ of the first homoclinic bifurcation of the 2-cycle is issuing, of implicit equation $\phi^4(1) = x_R$ where x_R is the periodic point of the repelling 2-cycle on the right side. Issuing from $(G, \sigma) = (1, \sigma_1)$ is the curve $G = G_1(\sigma)$

of the first homoclinic bifurcation of the fixed point x^*, of implicit equation $\phi^3(1) = x^*$.

The colors represent the results obtained numerically about the asymptotic behavior, and thus about the attracting set of the M-map. We can see the region between the curves $G = (\theta - 1)$ and $G = G_4(\sigma)$ related to an attracting 2-cycle. Indeed at this scale of the figure this is the result that can be numerically observed. However, as already remarked in the Introduction, the flip bifurcation of the 2-cycle is a new result leading to a curve of equation $G = G_2^*(\sigma)$ (given explicitly in Section 4), and also this curve is issuing from the point $(G, \sigma) = (1, \sigma_4)$ but at the scale of Fig.1b it cannot be distinguished from the other curve, we shall come back to this point in Section 4.

One more property, already proved by Matsuyama in [13] (and recalled in Appendix B), is that for any $G > 1$ each point of the absorbing interval $[g(G), G]$ on the left side, i.e. in $x < 1$, is mapped to the right side in one iteration. This follows from

Property 1 ([13])**.** *Let* $G > 1$*, then* $\phi^3(1) = f(g(G)) > 1$*.*

3. Degenerate bifurcation of x^* leading to an attracting 2-cycle

Consider the fixed point $x^* = 1 + \frac{(G-1)}{\theta}$ for $G > 1$. From $g'(x^*) = -\frac{(\theta-1)}{G}$ it is immediate to see that $-1 < g'(x^*) < 0$ for $G > (\theta - 1)$, and the fixed point is attracting, while $g'(x^*) < -1$ for $G < (\theta - 1)$, and the fixed point is repelling. As already noticed in [16], the flip bifurcation occurring at $G = \theta - 1$ is not a standard flip bifurcation (supercritical or subcritical). It is of degenerate type (as described in [21]). Recall that by the flip bifurcation theorem, to have a super- or subcritical flip bifurcation, the condition of non zero Schwarzian derivative must be satisfied (or, equivalently, the third derivative of the second iterate must not be zero). The Schwarzian derivative is zero iff the function is topologically conjugate to a linear or linear-fractional function (see e.g. [4]). The function $g(x)$ is linear-fractional, thus a standard flip bifurcation the fixed point x^* cannot occur. In fact, considering the branch of the second iterate $\phi^2(x)$ to which the fixed point belongs, given by

$$H(x) = g \circ g(x) = \frac{G^2 x}{x\theta(G - (\theta - 1)) + (\theta - 1)^2}$$

and defined in the interval $[1, g^{-1}(1)] = [1, \frac{\theta-1}{\theta-G}]$, for which clearly x^* is still a fixed point ($H(x^*) = x^*$), we can see that three fixed points are not allowed. As $g(x)$ also $H(x)$ is a branch of hyperbola, and the equation $H(x) = x$ for $G \neq (\theta - 1)$ is satisfied only by $x = 0$ (outside the region of interest) and $x = x^*$. Thus, the bifurcation occurring at $G = (\theta - 1)$ is not a standard one. Moreover, at the bifurcation value, substituting $G = (\theta - 1)$ in the expression of $H(x) = g^2(x)$ we can see that it reduces to the identity function, $H(x) = x$, which means that all the points of the interval $[1, G]$ are fixed, that is, all the points in this interval different from x^* are 2-periodic for map $\phi(x)$ (stable but not attracting).

Regarding the other points of the interval $(0, 1)$, at this bifurcation it is easy to see that each point is mapped into the absorbing interval $[g(G), G] = [1, G]$ in a finite number of iterations, thus becoming 2-periodic or fixed (such points are also called pre-periodic), and thus the iterates of $\phi^n(x)$ as reported in [5] are not only useless but can be misleading. In fact, what matters are the preimages of the kink point $x = 1$ on the left side, i.e. $x_{-n} = f^{-n}(1)$, for $n \geq 1$ (preimages which are accumulating to $x = 0$, repelling fixed point of $f(x)$). So any initial condition in the interval $[x_{-n}, x_{-(n-1)})$ (where $x_{-0} = 1$) is mapped into the absorbing interval in n application by $f(x)$, and becomes periodic, but only in the Romer regime.

What is now to be proved rigorously is which kind of dynamics of the M-map $\phi(x)$ occurs for $G < (\theta - 1)$, in the absorbing interval $[g(G), G]$, after the degenerate flip bifurcation, when the fixed point x^* is repelling. The degenerate flip bifurcation of the fixed point x^* described above may be considered as a global bifurcation (not a local one), since the prediction of what occurs after the bifurcation is not the result of a local analysis. This depends on the global properties of the map, since the 2-cycle at the border points of the invariant segment $[1, G]$ of $\phi(x)$ can also be interpreted as a 2-cycle undergoing a BCB. That is, considering this bifurcation as a BCB of a fixed point of $\phi^2(x)$ merging with $x = 1$ at the bifurcation value, we can use the skew tent map as a border collision normal form, obtaining the rigorous answer to what may happen (in the same way that the rigorous answer was given for the border collision bifurcation of the fixed point x^* at $G = 1$, as a function of σ).

Consider $\phi^2(x)$ at the bifurcation value $G = (\theta - 1)$. The point $x = 1$ is a local minimum of $\phi^2(x)$, and it is also a fixed point of $\phi^2(x)$, both for the function defined on the right side, $g \circ g(x)$ (which is the identity function, with slope equal to 1) and of the function $g \circ f(x)$ on the left side, since $g \circ f(1) = 1$. So the first derivatives of the function $\phi^2(x)$ in $x = 1$ are $\beta = \frac{d}{dx} g \circ f(x)|_{x=1} < 0$ on the left side and $\alpha = \frac{d}{dx} g \circ g(x)|_{x=1} = 1$ on the right side. Since in $x = 1$ we have a local minimum of $\phi^2(x)$, we use the skew tent map in the form reported in (23) in Appendix A, for which β and α represent the slopes on the left and right side, respectively.

By the chain rule, we have $\beta = \frac{d}{dx} g \circ f(x)|_{x=1} = g'(f(1))f'(1) = g'(G)f'(1)$ and by using $f'(1) = G\gamma$, $g'(G) = -\frac{G(\theta-1)}{[1+\theta(G-1)]^2}$ we get, at the bifurcation $G = (\theta - 1)$, $g'(G) = -\left[\frac{(\theta-1)}{1+\theta(G-1)}\right]^2 = -\frac{1}{G^2}$, thus

$$\beta = g'(G)f'(1) = -\frac{\gamma}{G} = -\frac{\gamma}{\theta - 1} \ .$$

Since for any admissible value of σ, $\sigma > 2$, we have $\theta \in (2, e)$, thus $0 < \frac{1}{\theta-1} < 1$ and $\gamma \in (0.5, 1)$, it follows that it is always $-1 < \beta < 0$. From the normal form theory (see Fig.10 in Appendix A at $\alpha = 1$) we can state that the bifurcation leads to a stable fixed point of $\phi^2(x)$, that is, an attracting 2-cycle of $\phi(x)$. So the degenerate flip bifurcation of the fixed point x^* leads always to the appearance of a stable 2-cycle with symbolic sequence LR, whose periodic points we denote by x_L and x_R. We have so proved the following

Theorem 1. *For any fixed value of σ, $\sigma > 2$, the degenerate flip bifurcation of the fixed point x^* of the M-map $\phi(x)$ occurring at $G = (\theta - 1)$ leads, with decreasing G, to an attracting 2-cycle with symbolic sequence LR.*

Notice that this is a peculiar property of the M-map. Considering for example the increasing function $x_{t+1} = Gx_t^{4\gamma}$ (in place of Gx_t^{γ}) for $x_t < 1$ (in (1)), it can be rigorously proved that no stable cycle of period 2^n for $n \geq 1$, can exist: When the fixed point in the Romer regime becomes unstable at the bifurcation occurring at $G = \theta - 1$, the asymptotic trajectories belong to cyclic chaotic intervals. In fact, the derivative on the left side in $x = 1$ becomes $4\gamma G$ which leads to $\beta = -\frac{4\gamma}{\theta-1} < -1$, so that, depending on the value of β, from Fig.10 in Appendix A at $\alpha = 1$, the bifurcation leads always to chaotic intervals, i.e. 2^n-cyclic chaotic intervals for $n \geq 1$ or a unique chaotic interval for the map $\phi^2(x)$, which means 2^n-cyclic chaotic intervals with $n \geq 1$ for $\phi(x)$.

4. Subcritical flip bifurcation of the 2-cycle and BCB of 4-cycles

As we have seen in the previous section, in the M-map a regime in the parameter space (G, σ) with an attracting 2-cycle always exists. Still unclear was the mechanisms through which the 2-cycle becomes unstable. The statements reported in [7] on the 2-cycle and 4-cycles are conjectures based on the graph of the map and its iterates. It is argued that also for the 2-cycle a degenerate flip bifurcations may occur, related to the BCB of a 4-cycle, and to the appearance of 4-cyclic chaotic intervals. It is shown in [21] that such a degenerate bifurcation can occur in maps which are locally topologically conjugate to a linear or linear-fractional map, and close to the bifurcation value the second iterate $\phi^2(x)$ and the fourth iterate $\phi^4(x)$ involved in the points of the 2-cycle have a graph which suggests that such a topological conjugacy may be possible. Given that the flip bifurcation value of the 2-cycle was not detected explicitly, it was difficult to verify that point. Here we succeed in determining the explicit value of the parameters at which the flip bifurcation of the 2-cycle occurs, and the explicit expression of the periodic points of the 2-cycle. This allows us to demonstrate that it is a smooth flip of subcritical type. Notice that this implies that a repelling 4-cycle must exist "before" the flip bifurcation of the attracting 2-cycle (i.e. when it is still attracting), and must disappear "after", leaving a repelling 2-cycle. In this section we also investigate the appearance of such repelling 4-cycle, as well as the occurrence of bistability. We give the condition leading to two coexisting attracting sets given by the 2-cycle and 4-cyclic chaotic intervals. This is related to the BCB occurring when the condition $\phi^4(1) = 1$ holds, and we give an explicit interval bounding this bifurcation value (denoted $G = G_4$ in Section 2).

4.1. Flip bifurcation of the 2-cycle

The local stability of the 2-cycle existing for $G < (\theta - 1)$ can be investigated evaluating the derivative $f'(x_L)g'(x_R)$. Considering the branches of the second iterate $\phi^2(x)$ for which the 2-cycle leads to x_L and x_R as fixed points, we can see that the decreasing functions are convex. Indeed, for the function

$$F_{LR}(x) = g \circ f(x) = \frac{G^2 x^\gamma}{1 + \theta(Gx^\gamma - 1)} \tag{7}$$

defined in the interval $[g(G), 1]$, whose fixed point corresponds to x_L, we have

$$F'_{LR}(x) = g'(f(x))f'(x) < 0 \text{ and } F''_{LR}(x) = g''(f(x))(f'(x))^2 + g'(f(x))f''(x) > 0.$$

Similarly it can also be shown that the other function $F_{RL}(x) = f \circ g(x)$, defined in the interval $[g^{-1}(1), 1]$, whose fixed point corresponds to x_R, is decreasing and convex.

Since the eigenvalue of the 2-cycle is negative, it may undergo a flip bifurcation, related to a 4-cycle. In Fig.2 besides the map $\phi(x)$ it is also shown the graph of the fourth iterate $\phi^4(x)$ for which x_L and x_R are fixed points (besides x^*), but now with positive slopes of the related branches of $\phi^4(x)$. In Fig.2a we can see that the 2-cycle is attracting, while in Fig.2b we can see that the 2-cycle is repelling, and also that a 4-cycle exists (moreover, from the slopes we can also say that it is repelling).

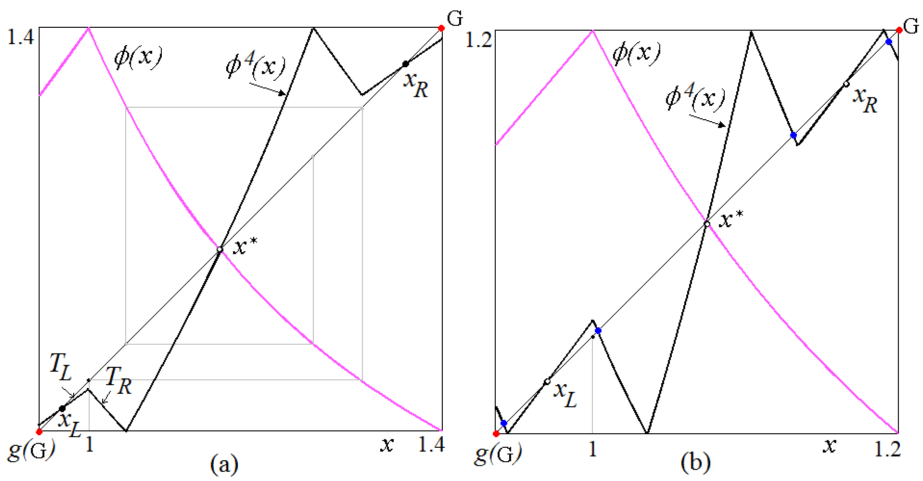

Figure 2: Graph of $\phi(x)$ and of $\phi^4(x)$ for $\sigma = 50$ and (a) $G = 1.4$; (b) $G = 1.2$.

In simple cases we can detect from the graph when the 2-cycle is attracting or repelling, because comparing with the diagonal we can see when the graph of $\phi^4(x)$ in the fixed points is crossing the diagonal from above to below (that is, with positive slope < 1) as in Fig.2a (thus showing an attracting 2-cycle) or *vice versa* from below to above the diagonal as in Fig.2b (that is, with positive slope > 1, thus showing a repelling 2-cycle).

For any fixed value of σ ($\sigma > \sigma_4$) decreasing G from $(\theta - 1)$ let us define as G_2^* the value of G at which the flip bifurcation of the 2-cycle occurs (i.e. at $G = G_2^*$ it is $F'_{LR}(x) = g'(x_R)f'(x_L) = -1$, as well as $F'_{RL}(x) = f'(x_L)g'(x_R) = -1$).

Recall that a standard smooth flip bifurcation of a fixed point $\overline{x}$ of a map $F(x)$ of supercritical type leads to an attracting 2-cycle close to the unstable fixed point, while a flip bifurcation of subcritical type is associated with a repelling 2-cycle close to the stable fixed point and merging with it at the flip bifurcation. At the bifurcation value only one fixed point exists, attracting when the bifurcation is supercritical, repelling when it is subcritical. It is also known that in order to see which kind of flip bifurcation occurs (supercritical or subcritical) we can consider the second iterate, $T(x) = F^2(x)$, for which the fixed point $\overline{x}$ has positive slope, and undergoes a pitchfork bifurcation (supercritical or subcritical), which can be investigated via the third derivative of the function $T(x)$ (see [19]). That is, if at the bifurcation value it holds that $T'''(x)|_{\overline{x}} > 0$ (resp. < 0) then the flip bifurcation is of subcritical (resp. supercritical) type, as qualitatively shown in Fig.3.

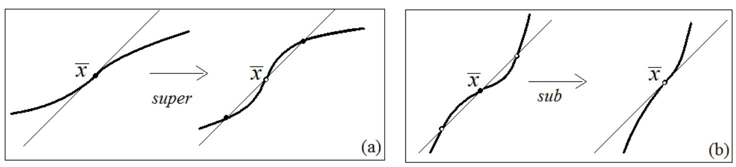

Figure 3: Qualitative representation of a pitchfork bifurcation of a fixed point $\overline{x}$ of a map $F(x)$, which is supercritical in (a) and subcritical in (b). For the flip bifurcation of a fixed point, it is related to the second iterate of the map.

Let us now characterize the flip bifurcation of the 2-cycle of the M-map with the following result:

Theorem 2. *For any fixed value of σ, $\sigma > \sigma_4$, the flip bifurcation of the 2-cycle occurs for $G = G_2^*$ where*

$$G_2^* = \left\{ (\theta - 1)\gamma^\gamma \left(\frac{1+\gamma}{\theta} \right)^{\frac{1}{\sigma}} \right\}^{\frac{1}{1+\gamma}} , \tag{8}$$

the 2-cycle is given by

$$x_R = \frac{(\theta - 1)(1+\gamma)}{\theta} \ , \ x_L = g(x_R) = \frac{G_2^*(1+\gamma)}{\theta\gamma} \tag{9}$$

and the flip bifurcation is of subcritical type.

Proof. Considering the point x_R of the 2-cycle, fixed point of the function $F_{RL}(x) = f \circ g(x)$, the bifurcation occurs when $F'_{RL}(x)|_{x_R} = -1$. Thus, at the bifurcation we have that the two following equations hold:

$$\left\{ \begin{array}{c} f \circ g(x_R) = x_R \\ f'(g(x_R))g'(x_R) = -1 \end{array} \right. \tag{10}$$

Since (by using (5) and (6))

$$f'(g(x))g'(x) = \frac{\gamma}{g(x)} f(g(x)) \frac{-(\theta-1)}{x[1+\theta(x-1)]} g(x) = f(g(x)) \frac{-\gamma(\theta-1)}{x[1+\theta(x-1)]}$$

from the second equation in (10) we have

$$f(g(x_R)) \frac{-\gamma(\theta-1)}{x_R[1+\theta(x_R-1)]} = -1$$

and considering the first equation in (10) (i.e. substituting x_R to $f \circ g(x_R)$) we get

$$\gamma(\theta-1) = 1 + \theta(x_R - 1)$$

that is, $x_R = \frac{(\theta-1)(1+\gamma)}{\theta}$, as reported in (9), and then its image x_L immediately follows.

Since the point x_R does not depend on the value of G, from the first equation in (10) we can obtain the bifurcation value of the parameter G. Considering the function $F_{RL}(x) = G\left[\frac{Gx}{1+\theta(x-1)}\right]^{\gamma}$ we have:

$$\begin{aligned} G\left[\frac{Gx_R}{1+\theta(x_R-1)}\right]^{\gamma} &= x_R \\ G\left[\frac{Gx_R}{\gamma(\theta-1)}\right]^{\gamma} &= x_R \\ G^{1+\gamma} &= x_R^{1-\gamma}\gamma^{\gamma}(\theta-1)^{\gamma} \end{aligned}$$

and substituting $x_R = \frac{(\theta-1)(1+\gamma)}{\theta}$:

$$G^{1+\gamma} = \frac{\gamma^{\gamma}(1+\gamma)^{(1-\gamma)}(\theta-1)}{\theta^{(1-\gamma)}}$$

from which the result (8) on the bifurcation value follows.

In order to show which kind of flip bifurcation occurs, we can consider the derivatives of the function $T_L(x) = F_{LR}^2(x) = g \circ f \circ g \circ f(x)$ (or equivalently of $F_{RL}^2(x) = f \circ g \circ f \circ g(x)$). In fact, as we have shown in the qualitative Fig.3 associated with a flip bifurcation, and recalled above, if at the bifurcation value it is $T_L'''(x)|_{x_L} > 0$ (resp. < 0) then the flip bifurcation is of subcritical (resp. supercritical) type. In Appendix C the computations are reported which show that $T_L'''(x)|_{x_L} > 0$ thus proving that always a subcritical flip bifurcation of the 2-cycle occurs.□

Let σ_2^* be the value of σ related to the point at which the curve $G = G_2^*(\sigma)$ intersects the vertical line $G = 1$ in the two-dimensional parameter plane. Then σ_2^* is the value of σ at which it holds

$$(\theta-1)\gamma^{\gamma}\left(\frac{1+\gamma}{\theta}\right)^{\frac{1}{\sigma}} = 1 \tag{11}$$

Since the bifurcation of the fixed point at $G = 1$ leads to a single value $\sigma_4 \simeq 3.825$ it must be $\sigma_4 = \sigma_2^*$ (that is, at $\sigma = \sigma_4$ it is $G_2^* = G_4 = 1$). Notice that assuming $G_2^* = 1$ the point x_L of the 2-cycle in (9) leads to $x_L = \frac{(1+\gamma)}{\theta\gamma}$, thus $x_R = f(x_L) = \left(\frac{1+\gamma}{\theta\gamma}\right)^\gamma$ so that, from the point x_R in (9), it must be

$$\left(\frac{1+\gamma}{\theta\gamma}\right)^\gamma = \frac{(\theta-1)(1+\gamma)}{\theta} \tag{12}$$

and after some algebraic steps, considering $\gamma = 1 - \frac{1}{\sigma}$, it can be seen that (11) holds iff (12) holds, confirming that $G_2^* = 1$. Considering that when the parameter G is equal to 1 the absorbing interval $[g(G), G]$ shrinks to the unique point $x = 1$, we have that the periodic points in (9) must also shrink to the same point, thus

$$\frac{(\theta-1)(1+\gamma)}{\theta} = 1\ ,\ \frac{1+\gamma}{\theta\gamma} = 1$$

which lead to $(\theta - 1)\gamma = 1$, that is:

$$(\gamma^{1-\sigma} - 1)\gamma = 1$$

and this equation is satisfied for $\sigma = \sigma_4$ (as in fact $\phi^4(1) = 1$ is also satisfied).

This proves that the curves of equation $G = G_2^*$ and $G = G_4$ are issuing from the same point $(1, \sigma_4)$ of the two-dimensional parameter plane (G, σ).

4.2. Appearance of 4-cycles

So the subcritical flip bifurcation of the 2-cycle at $G = G_2^*$ leads to the question of when the first 4-cycle of $\phi(x)$ appears since decreasing G from $(\theta-1)$ a repelling 4-cycle must exist before the value G_2^*.

Notice that decreasing G, as long as $\phi^4(G) < G$ (or equivalently $\phi^4(1) < 1$) the graph of ϕ^4 in the absorbing interval consists in only five branches, since only three preimages of $x = 1$ belong to the absorbing interval, giving the kink points of the function $\phi^4(x)$ besides $x = 1$ (i.e. $g^{-1}(1)$, $g^{-2}(1)$, $g^{-3}(1)$), as shown in Fig.2a. The rightmost branch of the function $\phi^4(x)$ (given by $F_{RL}^2(x) = f \circ g \circ f \circ g(x)$) as well as the leftmost branch (given by $T_L(x) = F_{LR}^2(x) = g \circ f \circ g \circ f(x)$), are the increasing branches related to the 2-cycle. When $\phi^4(G) = G$ occurs, which also corresponds to $\phi^4(1) = 1$, we have necessarily a 4-cycle undergoing a BCB. In fact, a 4-cycle exists, given by

$$\{1, G, g(G), f(g(G))\} \tag{13}$$

and it is at a border collision since one periodic point is $x = 1$. Decreasing G further, two more kink points of $\phi^4(x)$ (from the preimages of $x = 1$) exist inside the absorbing interval leading to two more branches of the function $\phi^4(x)$ in $[g(G), G]$, as can be seen in the example in Fig.2b. The new branches, $g^2 \circ f \circ g(x)$ on the rightmost side and $g^3 \circ f(x)$ on the leftmost side (see Fig.2b), are decreasing and necessarily both have a fixed point, leading to the existence of a 4-cycle of the M-map with symbolic sequence LR^3. Notice that in the case shown in Fig.2b only one 4-cycle exists since it is after the subcritical flip of the 2-cycle (thus one more repelling 4-cycle disappeared at the subcritical flip bifurcation).

So decreasing G we know that a 4-cycle exists at the border collision represented by the implicit equation $\phi^4(1) = 1$ (and we have already defined as $G = G_4$ the value of G at which this BCB occurs) and after, for $G < G_4$. Thus we are interested in the two branches of $\phi^4(1)$ on the left and right side of the kink point $x = 1$, given by

$$T_L(x) = g \circ f \circ g \circ f(x) \text{ and } T_R(x) = g \circ f \circ g \circ g(x) \tag{14}$$

and to the slopes of these functions evaluated in the kink point, say

$$\alpha = T_L'(x)|_{x=1} > 0 \text{ and } \beta = T_R'(x)|_{x=1} < 0 \tag{15}$$

The appearance of 4-cycles of map $\phi(x)$ may occur via smooth fold bifurcation or via fold BCB. Consider the monotone increasing branches of $\phi^4(x)$ related to the fixed points associated with the 2-cycle. We can fix the reasoning on the leftmost branch, $T_L(x)$, defined in the interval $[g(G), 1]$, as qualitatively shown in Fig.4.

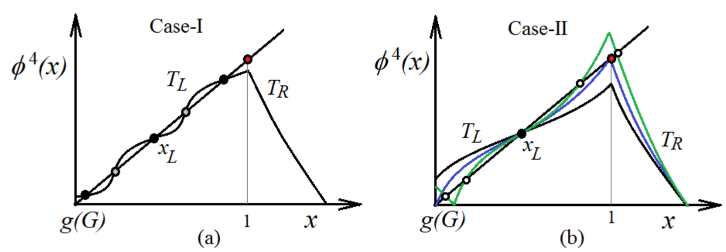

Figure 4: Qualitative scenario related to the leftmost branch of $\phi^4(x)$, the branch $T_L(x)$, of the bifurcation leading to the appearance of the first 4-cycle (a) via a smooth fold bifurcation (Case-I); (b) via a fold BCB (Case-II).

Case-I (smooth fold). If a pair of 4-cycles appears via smooth fold bifurcation, say at $G = G_4^* > G_2^*$, then at the occurrence of the border collision the 4-cycle merging with $x = 1$ (decreasing G at $G = G_4$) must be attracting, i.e. its eigenvalue α must be $\alpha < 1$, as qualitatively shown in Fig.4a. While the repelling 4-cycle is involved in the subcritical flip (and in this case the subcritical flip may occur for $G_2^* < G_4$ or $G_2^* > G_4$).

Case-II (fold BCB). Differently, the 4-cycles may appear via fold BCB, at $G = G_4 > G_2^*$ (leading to a pair of 4-cycles with symbolic sequence LR^3 and $LRLR$) as qualitatively shown in Fig.4b. In such a case, since on the left side the function $T_L(x)$ approaches the collision from below, the fixed point (and thus 4-cycle) appearing on the left side of $x = 1$ must be necessarily repelling, that is, it must necessarily be $\alpha \geq 1$ at the bifurcation and $\alpha > 1$ after.

Recall that in both cases the result of the BCB at $G = G_4$ depends on the value of both derivatives of the function ϕ^4 in $x = 1$, given in (15), $\alpha(G_4, \sigma)$ on the left side of $x = 1$ and $\beta(G_4, \sigma)$ on the right side. In particular, the stability/instability of the fixed point in $x > 1$ existing after the bifurcation (4-cycle for ϕ with symbolic sequence LR^3) depends on the branch of ϕ^4 on the right side of $x = 1$, the function $T_R(x)$ in (14).

From Theorem 2 we have the explicit bifurcation value $G = G_2^*$, and in Fig.5 we have plotted the values of $\phi^4(1)$ at $G = G_2^*$ as a function of σ. This value is always very close to 1 but higher than 1, which means that at $G = G_2^*$ it is $\phi^4(1) > 1$, that is, the BCB has already occurred, leading to

$$G_2^* < G_4$$

and this whichever is the bifurcation leading to the appearance of the first pair of 4-cycles (i.e. in Case-I via smooth fold or Case-II via fold BCB).

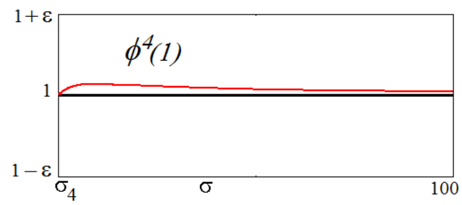

Figure 5: Values of $\phi^4(1)$ at $G = G_2^*$ as a function of σ up to 100, showing that it is always $G_2^* < G_4$, $\varepsilon = 10^{-4}$.

Let us now characterize the result of the border collision occurring at $G = G_4$. As recalled above, for any $\sigma > \sigma_4$, when $\phi^4(1) = 1$ occurs there exists a colliding 4-cycle, given in (13), i.e. $\{1, G, g(G), f(g(G))\}$. The point $x = 1$ is a fixed point of the two different components of $\phi^4(x)$, at the two sides of $x = 1$, that is, both of the function $T_L(x)$ and $T_R(x)$ in (14). The first derivatives $\alpha(G_4) = T_L'(x)|_{x=1} > 0$ and $\beta(G_4) = T_R'(x)|_{x=1} < 0$ are those characterizing the effect of the bifurcation, via the skew tent map as normal form, and since $x = 1$ is a local maximum, we use the skew tent map in the form reported in (22) in Appendix A. In the following Property we give the explicit expressions of these derivatives.

Property 2. *For any fixed value of σ, $\sigma > \sigma_4$, consider the border collision bifurcation of the 4-cycle $\{1, G, g(G), f(g(G))\}$ occurring at $G = G_4$, then*

$$\alpha(G_4) = T_L'(x)|_{x=1} = \frac{(\theta - 1)\gamma^2(\theta - G_4)}{G_4[1 + \theta(G_4 - 1)]} \tag{16}$$

$$\beta(G_4) = T_R'(x)|_{x=1} = -\alpha(G_4)\frac{\theta - 1}{\gamma} \tag{17}$$

So we can now detect the result of the BCB of the 4-cycle, showing the conditions for which it leads directly to 4-cyclic chaotic intervals of the map, so that the border collision leads to the sudden appearance of all the harmonics related to the 2-cycle, since it proves that all the cycles with even periods 2^n for $n > 1$ exist after the bifurcation and are unstable.

Theorem 3. *For any fixed value of σ, $\sigma > \sigma_4$, consider the border collision bifurcation of the 4-cycle (13) occurring at $G = G_4$ and the derivatives in (16) and (17).*

If Case-II occurs ($\alpha(G_4) \geq 1$), then the bifurcation leads to 4-cyclic chaotic intervals of the M-map $\phi(x)$, and for $G \in (G_2^*, G_4)$ there is coexistence of an attracting 2-cycle and attracting 4-cyclic chaotic intervals of the M-map $\phi(x)$;

If Case-I occurs ($\alpha(G_4) < 1$) and $\beta(G_4) < -\frac{1+\sqrt{1+4\alpha(G_4)}}{2\alpha(G_4)}$, then the bifurcation leads to 4-cyclic chaotic intervals of the M-map $\phi(x)$;

If Case-I occurs ($\alpha(G_4) < 1$) and $\alpha(G_4)(\frac{\theta-1}{\gamma})^{1/2} > 1$, then the bifurcation leads to $4 \cdot 2^n$-cyclic chaotic intervals of the M-map $\phi(x)$ for some $n \geq 0$.

Proof. Recall that if the border collision bifurcation occurs with a cycle born by fold BCB (Case-II) then it is $\alpha(G_4) \geq 1$, while if it occurs with a cycle born by smooth fold (Case-I) then it is $\alpha(G_4) < 1$. Then the value of β determines the result. From the skew tent map as a normal form (see Appendix A with a local maximum as reported in (22)) we have that given the value of α, if $\beta < h_1(\alpha) = -\frac{1+\sqrt{1+4\alpha}}{2\alpha}$ (from (27) in Appendix A) then the result for map $\phi^4(x)$ is a unique chaotic interval, which means 4-cyclic chaotic intervals for map $\phi(x)$. Also we have that given the value of α, if $\beta < -\frac{1}{\alpha}$ (from (26) in Appendix A) then the result for map $\phi^4(x)$ is 2^n-cyclic chaotic intervals for some $n \geq 0$, which for map $\phi(x)$ means $4 \cdot 2^n$-cyclic chaotic intervals.

Proof for Case-II. Let us first show that the result is true for $\alpha(G_4) = 1$ and then for $\alpha(G_4) > 1$.

For $\alpha(G_4) = 1$ if $\beta(G_4) < h_1(1) = -\frac{1+\sqrt{5}}{2} \cong -1.618$ then there is a unique chaotic interval for $\phi^4(x)$ (4-cyclic chaotic intervals of map $\phi(x)$). In the M-map, for $\alpha(G_4) = 1$ this inequality is satisfied. In fact, this leads to $\beta(G_4) = -\frac{\theta-1}{\gamma} < -1$, which is a decreasing function of σ, and for $\sigma > \sigma_4$ it takes values in the interval $(-1.7183, -1.83329)$. This comes as follows: for $\sigma = \sigma_4 \cong 3.825$ it is $\frac{\theta-1}{\gamma} = 1.83329$ and for $\sigma \to \infty$, then $\gamma \to 1$ while $\theta \to e$ thus $\beta \to -(e-1) \cong -1.7183$.

For $\alpha(G_4) > 1$: since we have seen that for $\alpha(G_4) = 1$ in the M-map the inequality $\beta(G_4) < h_1(\alpha) = -\frac{1+\sqrt{1+4\alpha}}{2\alpha}$ is satisfied, then also for $\alpha(G_4) > 1$ a fortiori it is satisfied. In fact, it is $h_1(\alpha) = -\frac{1+\sqrt{1+4\alpha}}{2\alpha} > h_1(1)$, while $\beta(G_4) = -\alpha(G_4)\frac{\theta-1}{\gamma} < -\frac{\theta-1}{\gamma} < h_1(1)$.

Since for $\alpha(G_4) \geq 1$ the repelling 4-cycle involved in the subcritical flip at G_2^* appears at the BCB, with $G_2^* < G_4$, it follows that for values of G in the interval (G_2^*, G_4) two attracting sets coexist (a 2-cycle and 4 cyclic chaotic intervals).

Proof for Case-I. To end the possible cases we have to consider also the case in which $\alpha(G_4) < 1$ (if it may occur).

If $\alpha(G_4) < 1$ and $\beta(G_4) < h_1(\alpha(G_4)) = -\frac{1+\sqrt{1+4\alpha(G_4)}}{2\alpha(G_4)}$ (from (27) in Appendix A) then the bifurcation leads to 4-cyclic chaotic intervals of map $\phi(x)$.

If $\alpha(G_4) < 1$ and $\beta(G_4) < -\frac{1}{\alpha(G_4)}$ (from (26) in Appendix A) that is, if $\alpha(G_4)(\frac{\theta-1}{\gamma})^{1/2} > 1$[1], then the bifurcation leads to $4 \cdot 2^n$-cyclic chaotic intervals of map $\phi(x)$ for some $n \geq 0$. □

Numerically, for each fixed value of $\sigma > \sigma_4$ we have always observed Case-II, $\alpha(G_4) > 1$, $G_2^* < G_4$, and the two values differing for less than 10^{-4}. In Fig.1b the two curves $G = G_2^*$ and $G = G_4$ are not distinguishable from each other.

A detailed investigation at $\sigma = 5$ shows that indeed the scenario of the qualitative Fig.4b occurs, that is Case-II with $G_2^* < G_4$ and coexistence, as illustrated in Fig.6.

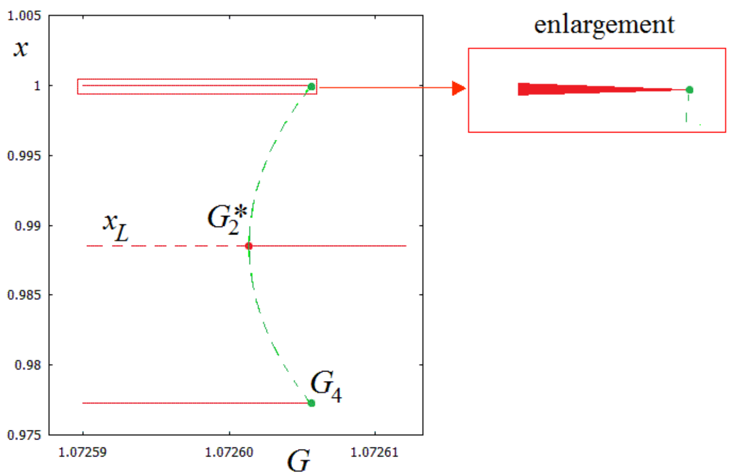

Figure 6: A part of the one-dimensional bifurcation diagram at $\sigma = 5$, as a function of G, enlarged in a neighborhood of point x_L of the 2-cycle. At $G = G_4$ a fold BCB occurs, leading to 4-cyclic chaotic intervals. In green the repelling 4-cycle is shown, which merges with x_L at the subcritical flip bifurcation.

Decreasing G, first a fold BCB occurs at $G = G_4$ leading to a pair of 4-cycles (both unstable), and the result of the BCB is the appearance of a new attracting set made up of 4-cyclic chaotic intervals, so that there is coexistence of two attractors: the 2-cycle and 4-cyclic chaotic intervals. One repelling 4-cycle (with symbolic sequence LR^3) is inside the chaotic intervals, the other one (with symbolic sequence $LRLR$, in green in Fig.6) approaches the 2-cycle leading to the subcritical flip bifurcation at $G = G_2^*$ after which only the 4-cyclic chaotic intervals are left as attracting set.

So, in Case-II of particular interest is the small range between the fold BCB at which the pair of unstable 4-cycles appears and the subcritical flip of the

[1]Recall that in this range it is $\frac{\theta-1}{\gamma} \in (-1.7183, -1.83329)$.

2-cycle. In fact, here the locally stable 2-cycle coexists with the attracting set made up of 4-cyclical intervals. And the basin of attraction of the 2-cycle is bounded by the unstable 4-cycle external to the chaotic intervals, so that the 2-cycle possesses the corridor stability a la Leijonhufvud [9]: i.e., it is stable and self-correcting when small shocks occur, but unstable against large shocks. Furthermore, when decreasing G the 2-cycle loses stability via the subcritical flip, the effects are both catastrophic and irreversible. They are catastrophic because the economy, initially located in a stable oscillation of period 2, soon becomes oscillating at different values, far away from the previous values, and the fluctuations are no longer predictable. Furthermore, these effects are irreversible in the sense that reversing the parameter, increasing G to the original value and restoring the stability of the 2-cycle, the states persist in the attracting 4-cyclic chaotic set for a while.

Indeed we think that what occurs is always Case-II with $G_2^* < G_4$ and related interval of coexistence, although occurring in a very narrow interval of values for G. Notice that even if we cannot have the explicit expression of the BCB value $G = G_4$ we can have an explicit upper bound (or lower bound) of it, given in the following theorem.

Theorem 4. *For any fixed value of σ, $\sigma > \sigma_4$, if $\alpha(G_4) \geq 1$ (Case-II) then $G_4 \in (G_2^*, G^*]$ while if $\alpha(G_4) < 1$ (Case-I) then $G_2^* < G^* < G_4$ where G^* is given by*

$$G^* = \frac{1}{2}\left\{\frac{(\theta-1)(1-\gamma^2)}{\theta} + \left[\frac{(\theta-1)^2(1-\gamma^2)^2}{\theta^2} + 4(\theta-1)\gamma^2\right]^{\frac{1}{2}}\right\}. \qquad (18)$$

Proof. The value $\alpha(G_4)$ has been computed in (16), so let us investigate when it is higher or smaller than 1. It is easy to see that

$$\alpha(G_4) = \frac{(\theta-1)\gamma^2(\theta-G_4)}{G_4[1+\theta(G_4-1)]} < 1 \text{ iff } (G_4)^2 - G_4\frac{(\theta-1)(1-\gamma^2)}{\theta} - (\theta-1)\gamma^2 > 0 \qquad (19)$$

For positive G_4 then $\alpha(G_4) < 1$ holds only for $G_4 > G^*$ where G^* is the positive solution of

$$(G^*)^2 - G^*\frac{(\theta-1)(1-\gamma^2)}{\theta} - (\theta-1)\gamma^2 = 0 \qquad (20)$$

leading to the value given in (18). Considering the function $\alpha(G) = \frac{(\theta-1)\gamma^2(\theta-G)}{G[1+\theta(G-1)]}$ it is easy to see that $\frac{d}{dG}\alpha(G) < 0$. Thus, when the 4-cycle appears via smooth fold (Case-I with $\alpha(G_4) < 1$), then it must be $G^* < G_4$; while when the 4-cycle appears via fold BCB (Case-II with $\alpha(G_4) \geq 1$) then it must be $G_4 \leq G^*$.□

Numerically the values of G^*, G_4 and G_2^* differ for less than 10^{-4} and we have always observed $G_2^* < G_4 \leq G^*$ (i.e. Case-II). In Fig.7 in black we have plotted the curve $G = G_2^*(\sigma)$ (from (8)) as a function of σ up to 100 while in red we have plotted the curve $G = G^*(\sigma)$ (from (18)) and the two curves are not distinguishable, while in the enlargement the two curves can be clearly distinguished, and the curve $G = G_4$ is in between, as well as the bistability region.

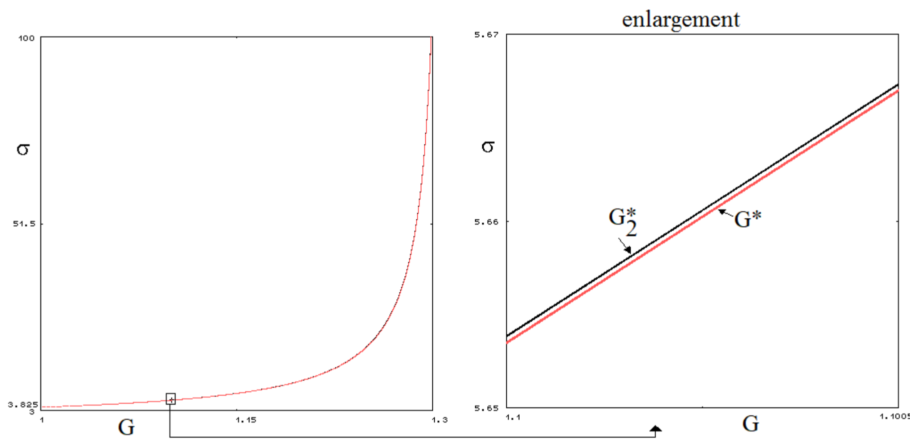

Figure 7: In the (G, σ)-parameter plane it is shown in black the curve $G = G_2^*(\sigma)$ given in (8), and in red the curve $G = G^*(\sigma)$ given in (18), which are better visible in the enlargement.

5. Chaos and first homoclinic bifurcation of the fixed point x^*

The relevance of the fixed point to be homoclinic is associated with the rigorous proof of chaos. That is, if x^* is homoclinic then we can state that the M-map is chaotic. By chaotic map we mean that an invariant set Λ exists on which the restriction of $\phi(x)$ is topologically conjugate to the shift map on two symbols, also called Devaney chaos [6] or Li-Yorke chaos [10]. Notice that this result occurs, both for smooth and piecewise smooth systems, whenever a cycle (of any period) is homoclinic, not necessarily the fixed point (for the proof we refer to [8]). In the previous section we have seen how the M-map becomes chaotic abruptly at a BCB, with a sharp transition from the existence of only a few (a finite number) of cycles to infinitely many repelling cycles and chaos. In fact, the result of the BCB at $G = G_4$ described above allows us to precisely state when chaos first appear in the M-map, which may also coexist with an attracting 2-cycle. The following result holds, as a consequence of Theorem 3:

Corollary 1. *For any fixed value of* σ, $\sigma > \sigma_4 \simeq 3.825$, *if* $\alpha(G_4) \geq 1$ *or* $\alpha(G_4) < 1$ *and* $\alpha(G_4)(\frac{\theta-1}{\gamma})^{1/2} > 1$, *then the M-map is chaotic for* $G \in (1, G_4)$.

Indeed, when the assumptions are satisfied there are homoclinic cycles, and cyclic chaotic intervals exist.

However, since it is known that in unimodal maps cycles of odd period can exist only when the fixed point is homoclinic, it is relevant to determine the parameter range in which this occurs. The goal of this section is to get an upper boundary for the value of G, that we have denoted G_1, at which the first homoclinic bifurcation of the fixed point $x^* = 1 + \frac{(G-1)}{\theta}$ occurs, decreasing G from $(\theta - 1)$. Recall that for any fixed $\sigma > \sigma_1$, decreasing G from $(\theta - 1)$ (or, better, $G < G_2(\sigma)$), G_1 is solution for $G > 1$ of the equation $\phi^3(1) = x^*$. For the M-map it holds $\phi^3(1) > 1$ (Property 1 in Section 2, see also Appendix B), as well as the following property, proved in Appendix B:

Property 3. *For the M-map in the range $G > 1$ the function $\phi^3(1) = f(g(G))$ is increasing with respect to G.*

To get an estimate of G_1 let us consider the solutions of the equation $f(g(G)) = x^*$, that is:

$$G[\frac{G^2}{1+\theta(G-1)}]^\gamma = 1 + \frac{(G-1)}{\theta}. \quad (21)$$

Clearly $G = 1$ is a solution, and the function $f(g(G))$ is increasing with respect to G (Property 3). The graph of the function in the right side of (21) is a straight line with slope $\frac{1}{\theta}$. For $G > 1$, we have that as long as $2 < \sigma \leq \sigma_1$ it is always $f(g(G)) > x^*(G)$, as shown in Fig.8a, so that the homoclinic bifurcation of x^* cannot occur, while for $\sigma > \sigma_1$ the two increasing functions have one more intersection point, as shown in Fig.8b. We prove that the new intersection point is always smaller that 1.15.

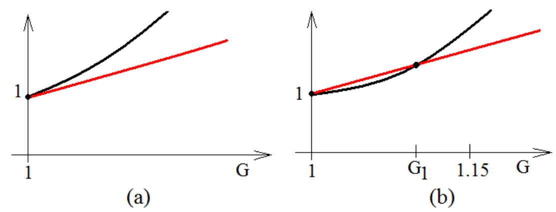

Figure 8: Qualitative graphs of the left and right hand side functions in (21) for (a) $2 < \sigma \leq \sigma_1$ and (b) $\sigma > \sigma_1$.

Remark. The value σ_1 must also correspond to the value at which the slope of the function $f(g(G))$ in $G = 1$ is equal to $\frac{1}{\theta}$ (i.e. that slope must be larger, equal and smaller than $\frac{1}{\theta}$ for $\sigma < \sigma_1$, $\sigma = \sigma_1$ and $\sigma > \sigma_1$, respectively). That is, σ_1 is also solution of $2\gamma + 1 - \gamma\theta = \frac{1}{\theta}$ i.e. of

$$1 - \gamma(\theta - 2) = \frac{1}{\theta}$$

where $\gamma = (1 - \frac{1}{\sigma})$ and $\theta = (1 - \frac{1}{\sigma})^{(1-\sigma)}$, for $\sigma > 2$.

Theorem 5. *For any fixed σ, $\sigma > \sigma_1 \simeq 21.231$, the value G_1 at which the first homoclinic bifurcation of the fixed point x^* occurs decreasing G from $(\theta - 1)$ satisfies $1 < G_1 < 1.15$.*

Proof. As remarked above, G_1 is solution for $G > 1$ of the equation $\phi^3(1) = x^*$ and for $\sigma > \sigma_1$ considering a value of G in a right neighborhood of $G = 1$ it is $f(g(G)) < x^*(G)$. We show that for $G = 1.15$ it is $f(g(1.15)) > x^*(1.15)$ which implies that the root G_1 must exist in that interval.

For $\sigma > \sigma_1 \simeq 21.231$ it is $1 > \gamma > \gamma_1 = (1 - \frac{1}{\sigma_1}) \simeq 0.95238$, $2.72 > e > \theta > \theta_1 = (1 - \frac{1}{\sigma_1})^{(1-\sigma_1)} \simeq 2.65329$. In the right side of (21) we have $x^*(1.15) = 1 + \frac{0.15}{\theta}$ which for $\theta_1 < \theta < 2.72$ takes values in the interval $[1.055147, 1.0565335]$. Since

$g(G) = \frac{G^2}{1+\theta(G-1)} < 1$ and $\gamma < 1$ so that $f(g(G) = Gg(G)^\gamma > Gg(G)$, in the left side of (21) we have:

$$f(g(1.15)) = 1.15[\frac{1.3225}{1+\theta(0.15)}]^\gamma$$
$$> 1.15[\frac{1.3225}{1+\theta(0.15)}] = \frac{1.520875}{1+\theta(0.15)}$$

and $\frac{1.520875}{1+\theta(0.15)}$ takes values in the interval $[1.08, 1.0878984]$. □

6. Non existence of cycles of period three

In this section we give the proof which follows from the properties of the M-map. Consider the rightmost branch of the function $\phi^3(x)$, given by $Q(x) = g \circ f \circ g(x)$ (see an example in Fig.9). In order to have a smooth fold bifurcation (leading to a pair of 3-cycles of the M-map) $Q(x)$ must be increasing and concave, $Q'(x) > 0$ and $Q''(x) < 0$, while we prove that in the region of interest it is convex.

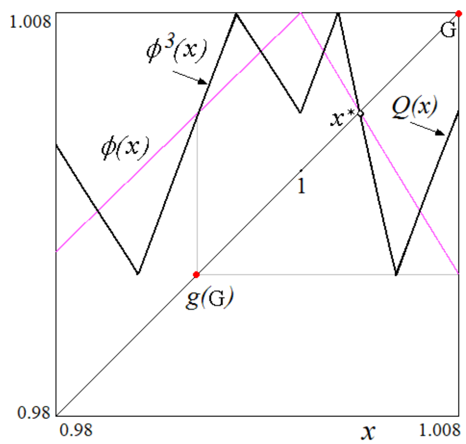

Figure 9: Graph of the M-map $\phi(x)$ and its third iterate $\phi^3(x)$ at $\sigma = 30$, $G = 1.008$.

Property 4. *For any value of* (G, σ) *in the range* $(1, G_1) \times (\sigma_1, +\infty)$ *the function* $Q(x) = g \circ f \circ g(x)$ *defined in the interval* $[g^{-1}(1), G] = [\frac{\theta-1}{\theta-G}, G]$ *is increasing and convex.*

The proof is given in Appendix B.

Theorem 6. *Let* $\sigma > 2$ *and* $1 < G < (\theta - 1)$. *The M-map cannot have a 3-cycle.*

Proof. Recall that a unimodal continuous map as $\phi(x)$ can have a cycle of odd period only when the fixed point x^* is homoclinic, thus we have to consider $\sigma > \sigma_1$ and $1 < G < G_1$. Since it is $\phi^3(1) > 1$ (Property 1) we have that any point $x < 1$ is mapped by $f(x)$ to the region $x > 1$, thus a smooth fold bifurcation may occur only for cycles with symbolic sequence LRR. This requires the rightmost branch of the function $\phi^3(x)$ defined by $Q(x) = g \circ f \circ g(x)$ to be increasing and concave. But from Property 4, it is convex, thus a smooth fold cannot occur. The only possibility is via a fold BCB, which requires $\phi^3(1) = 1$, as well as $\phi^3(1) < 1$ which cannot occur.$\square$

7. Conclusions

In this work we have reconsidered the M-map, the growth model proposed by Matsuyama in [13], which recently raised to the novel interest and new interpretations of its dynamic behaviors. We have given the rigorous proof of some open problems. We have proved in Theorem 1 (by using the skew tent map as a normal form) that in the whole parameter range of interest the degenerate flip bifurcation of the fixed point in the Romer regime leads to a unique attracting 2-cycle, with one periodic point in the Solow regime and one in the Romer regime. The flip bifurcation of such a 2-cycle has been detected in explicit form, showing that it is always of subcritical type (Theorem 2), and thus a bistability regime is expected to exist. In Theorems 3 and 4 we have given the conditions for which the result of the border collision related to 4-cycles leads to cyclic chaotic intervals, showing that a cascade of stable cycles of even periods cannot occur. The parameter range in which repelling cycles of odd period exist has been further investigated, giving a suitable explicit boundary (Theorem 5), as well as its relation to the non existence of cycles of period three in Theorem 6. Although many results have been rigorously proved, still there are some conjectures: the appearance of the first 4-cycles may occur via smooth fold bifurcation (called Case-I) or via fold BCB (called Case-II), but we have numerical evidence that in the M-map Case-II occurs.

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Appendix A (Border collision normal form)

The use of the skew tent map $\psi : x \mapsto \psi(x)$ defined by the function

$$\psi(x) = \begin{cases} \alpha x + 1, & x \leq 0, \\ \beta x + 1, & x \geq 0. \end{cases} \tag{22}$$

as a normal form of the border collision bifurcation of a fixed point occurring in one dimensional piecewise smooth maps is known since the early works by Nusse and Yorke ([17], [18]). A survey and detailed description of its use can be found in [22]. This is possible since the dynamic behavior of map ψ depending on the two parameters α and β is now well known (see [11], [1], [22] and references therein). All the possible kinds of border collision bifurcation of the fixed point x^* are classified according to the partition of the (α, β)-parameter plane into subregions in which the same qualitative dynamics take place. We summarize these results in Fig.10 in the region of interest.

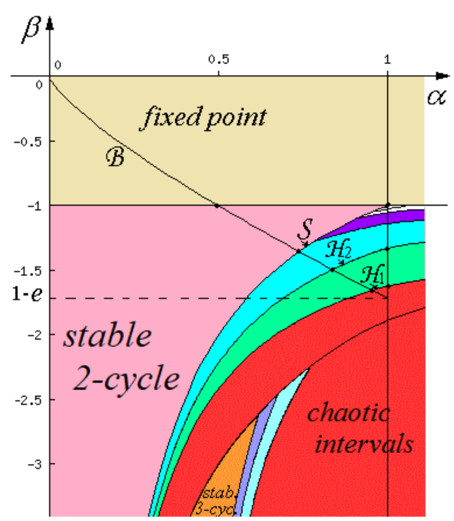

Figure 10: Bifurcation structure of the (α, β)-parameter plane of the skew tent map $\psi(x)$.

The explicit equations of the bifurcation curves characterizing the different regions represented in Fig.10 can be found in [22].

The case considered here for map ψ in (22) is used for a map with a maximum in the kink point, and the region of interest is for $0 < \alpha < 1$, $\beta < 0$. However, the case with a minimum is topologically conjugate, thus Fig.10 also represents the bifurcations in the case of a minimum considering the slopes as follows:

$$\psi(x) = \begin{cases} \beta x - 1, & x \leq 0 \\ \alpha x - 1, & x \geq 0 \end{cases} \tag{23}$$

When a fixed point of a one-dimensional piecewise smooth unimodal map collides with the kink point, the two slopes of the functions existing on the two sides of the kink point determine the result of the BCB.

Considering from [7] the case of the BCB of the fixed point $x^* = 1$, at the bifurcation it is $G = 1$, the slopes of the functions on the two sides of the kink point to which the fixed point is colliding are given by

$$\alpha = f'(x)|_{x=1} = \left(1 - \frac{1}{\sigma}\right) \in (0,1), \quad \beta = g'(x)|_{x=1} = (1-\theta) < 0 \tag{24}$$

From $1 < \theta < e$ we have that $1 - e < \beta < 0$, thus, the region which interests us in the (α, β)-parameter plane is $0 < \alpha < 1$, $1 - e < \beta < 0$ (see the rectangle in Fig.10). Substituting first $\theta = (1 - \frac{1}{\sigma})^{1-\sigma}$ and then $\sigma = 1/(1-\alpha)$ into (24) we get the expression of the border collision curve of the fixed point $x^* = 1$ in terms of the parameters α and β, which is denoted $\mathcal{B}$,

$$\mathcal{B}: \quad \beta = 1 - \alpha^{\alpha/(\alpha-1)} \tag{25}$$

and it is also shown in Fig.10. It can be noticed that the curve $\mathcal{B}$ in the region of interest does not intersect the curve related to 3-cycles of the skew tent map, as expected, due to Property 1, $\phi^3(1) > 1$, for the M-map.

The curves denoted $\mathcal{S}$, $\mathcal{H}_2$ and $\mathcal{H}_1$ in Fig.10 are given by

$$\mathcal{S}: \quad \alpha\beta = -1, \text{ i.e. } \quad \beta = -\frac{1}{\alpha} \tag{26}$$

$$\mathcal{H}_2: \quad \alpha^2\beta^3 + \alpha - \beta = 0, \text{i.e.} \quad \alpha = \frac{-1 - \sqrt{1 + 4\beta^4}}{2\beta^3}$$

$$\mathcal{H}_1: \quad \alpha\beta^2 - \alpha + \beta = 0, \text{i.e.} \quad \beta = h_1(\alpha) := \frac{-1 + \sqrt{1 + 4\alpha^2}}{2\alpha} \tag{27}$$

The curves $\mathcal{S}$ and $\mathcal{H}_1$ are used also in Theorem 3 of Section 4.

Appendix B (Proofs of Properties 1, 2, 3, 4)

Proof of Property 1. $\phi^3(1) = f(g(G)) > 1$ (from [13]).

Considering $f(g(G)) = G[\frac{G^2}{1+\theta(G-1)}]^\gamma$ we have the following inequalities:

$$G[\frac{G^2}{1+\theta(G-1)}]^\gamma > 1$$
$$\frac{G^2}{1+\theta(G-1)} > \frac{1}{G^{1/\gamma}}$$
$$G^{2+1/\gamma} > 1+\theta(G-1)$$
$$G^{2+1/\gamma} - 1 - \theta(G-1) > 0$$

Let $h(G) = G^{2+1/\gamma} - 1 - \theta(G-1)$, then $h(G)$ is increasing and convex. In fact, from $h(1) = 0$, $h'(G) = (2+1/\gamma)G^{1+1/\gamma} - \theta$, $h'(1) = (2+1/\gamma) - \theta > 3 - e$ we have $h(G) > 0$ for any $G > 1$.□

Proof of Property 2. *For any fixed value of* σ, $\sigma > \sigma_4$, *decreasing* G, *consider the border collision bifurcation of the 4-cycle* $\{1, G, g(G), f(g(G))\}$ *in (13) occurring at* $G = G_4$, *then*

$$\alpha(G_4) = T_L'(x)|_{x=1} = \frac{(\theta-1)\gamma^2(\theta-G_4)}{G_4[1+\theta(G_4-1)]} \tag{28}$$

$$\beta(G_4) = T_R'(x)|_{x=1} = -\alpha(G_4)\frac{\theta-1}{\gamma} \tag{29}$$

Proof. To show this we consider the fixed point of $\phi^4(x)$ colliding with $x = 1$, and the slopes of the functions on the left and right side of $x = 1$. The derivative of the function $T_L(x) = g \circ f \circ g \circ f(x)$ is detailed in Appendix C, leading to:

$$T_L'(x) = \frac{[T_L(x)]^2}{xf(x)}\left[\frac{(\theta-1)\gamma}{G}\right]^2 \frac{g \circ f(x)}{f \circ g \circ f(x)}$$

Evaluating this derivative at $x = 1$ and for $G = G_4$ we have $T_L(1) = 1$, $f(1) = G_4$, $g \circ f(1) = g(G_4)$, $f \circ g \circ f(x) = f(g(G_4))$ so that we get

$$T_L'(1) = \frac{1}{G_4}\left[\frac{(\theta-1)\gamma}{G_4}\right]^2 \frac{g(G_4)}{f \circ g(G_4)}$$

Then considering $g(G_4) = \frac{G_4^2}{[1+\theta(G_4-1)]}$, and $f(g(G_4)) = g^{-1}(1) = \frac{\theta-1}{\theta-G_4}$ we have

$$\alpha = T_L'(1) = \left[\frac{(\theta-1)\gamma}{G_4}\right]^2 \frac{{G_4}^2}{G_4[1+\theta(G_4-1)]}\frac{\theta-G_4}{\theta-1}$$

that is

$$\alpha(G_4) = \frac{(\theta-1)\gamma^2(\theta-G_4)}{G_4[1+\theta(G_4-1)]} \tag{30}$$

We can repeat the computations with the function $T_R(x) = g \circ f \circ g \circ g(x)$, and its first derivative, leading to $\beta = T_R'(x)|_{x=1}$. However, comparing the first derivatives we can immediately notice that $T'_R(x)$ evaluated in $x = 1$ differs from $T'_L(x)$ evaluated in $x = 1$ only by one factor. In fact, it is

$$\begin{aligned} T'_L(x) &= g'(f \circ g \circ f(x)) \cdot f'(g \circ f(x)) \cdot g'(f(x)) \cdot f'(x) \\ T'_L(1) &= g'(f \circ g \circ f(1)) \cdot f'(g \circ f(1)) \cdot g'(f(1)) \cdot f'(1) \\ \alpha &= g'(f \circ g(G)) \cdot f'(g(G)) \cdot g'(G) \cdot f'(1) \end{aligned}$$

while

$$\begin{aligned} T'_R(x) &= g'(f \circ g \circ g(x)) \cdot f'(g \circ g(x)) \cdot g'(g(x)) \cdot g'(x) \\ T'_R(1) &= g'(f \circ g \circ g(1)) \cdot f'(g \circ g(1)) \cdot g'(g(1) \cdot g'(1) \\ \beta &= g'(f \circ g(G)) \cdot f'(g(G)) \cdot g'(G) \cdot g'(1) \end{aligned}$$

so that we have

$$\begin{aligned} \beta(G_4) &= \alpha(G_4)\frac{g'(1)}{f'(1)} = -\alpha(G_4)\frac{\theta-1}{\gamma} \\ &= -\frac{(\theta-1)^2\gamma(\theta-G_4)}{G_4[1+\theta(G_4-1)]}.\square \end{aligned}$$

Proof of Property 3. *For the M-map in the range $G > 1$ the function $\phi^3(1) = f(g(G))$ is increasing with respect to G.*

From $f(g(G)) = G[\frac{G^2}{1+\theta(G-1)}]^\gamma = \frac{G^{2\gamma+1}}{[1+\theta(G-1)]^\gamma}$, the derivative with respect to G, $D = \frac{d}{dG}f(g(G))$, is given by

$$\begin{aligned} D &= \frac{1}{[1+\theta(G-1)]^{2\gamma}}\left\{(2\gamma+1)G^{2\gamma}[1+\theta(G-1)]^\gamma - G^{2\gamma+1}\gamma\theta[1+\theta(G-1)]^{\gamma-1}\right\} \\ &= [\frac{G^2}{1+\theta(G-1)}]^\gamma[(2\gamma+1) - \frac{G\gamma\theta}{1+\theta(G-1)}] \end{aligned}$$

so that it is $D > 0$ when

$$\begin{aligned} 2\gamma+1 &> \frac{G\gamma\theta}{1+\theta(G-1)} \\ 2+\frac{1}{\gamma} &> \frac{G\theta}{1+\theta(G-1)} \end{aligned}$$

which is true, since it is $\gamma < 1$ so that the left side is $2 + \frac{1}{\gamma} > 3$ while the right side is $\frac{G\theta}{1+\theta(G-1)} < 3$. To see this last statement consider

$$\begin{aligned} \frac{G\theta}{1+\theta(G-1)} &< 3 \\ G\theta &< 3 + 3G\theta - 3\theta \\ G &> \frac{3}{2}\frac{\theta-1}{\theta} \end{aligned}$$

which holds. In fact, it is $\frac{3}{2}\frac{\theta-1}{\theta} < 1$ iff $\theta < 3$ which is true since $\theta \in (2, e)$, while on the left side it is $G > 1$.□

Proof of Property 4. *For any value of (G, σ) in the range $(1, G_1) \times (\sigma_1, +\infty)$ the function $Q(x) = g \circ f \circ g(x)$ defined in the interval $[g^{-1}(1), G] = [\frac{\theta-1}{\theta-G}, G]$ is increasing and convex.*

For the first derivative of $Q(x) = g \circ f \circ g(x)$, taking into account the derivatives given in (5) and in (6), we have

$$Q'(x) = g'(f(g(x)))f'(g(x))g'(x) > 0$$

thus $Q(x)$ is increasing. Recall that $Q(x)$ is defined in the interval $[g^{-1}(1), G] = [\frac{\theta-1}{\theta-G}, G]$ so that for any x it is (to be used below) $Q(x) > Q(\frac{\theta-1}{\theta-G}) = g(G) = \frac{G^2}{1+\theta(G-1)}$.

Considering the second derivative of $Q(x)$ we have:

$$\begin{aligned}
Q''(x) &= g''(f(g(x)))[f'(g(x))g'(x)]^2 + g'(f(g(x)))f''(g(x))[g'(x)]^2 + g'(f(g(x)))f'(g(x))g''(x) \\
&= \frac{-2\theta}{[1+\theta(f(g(x))-1)]} g'(f(g(x)))[f'(g(x))g'(x)]^2 + \\
&\qquad g'(f(g(x)))[\frac{-1}{\sigma g(x)} f'(g(x))][g'(x)]^2 + g'(f(g(x)))f'(g(x))[\frac{-2\theta}{1+\theta(x-1)} g'(x)] \\
&= Q'(x)\left\{\frac{-2\theta}{[1+\theta(f(g(x))-1)]}[f'(g(x))g'(x)] + \frac{-g'(x)}{\sigma g(x)} + \frac{-2\theta}{1+\theta(x-1)}\right\} \\
&= Q'(x)\left\{\frac{-2\theta f(g(x))}{[1+\theta(f(g(x))-1)]}[\gamma\frac{g'(x)}{g(x)}] + \frac{-g'(x)}{\sigma g(x)} + \frac{-2\theta}{1+\theta(x-1)}\right\} \\
&= Q'(x)\left\{\frac{-g'(x)}{g(x)}[\frac{2\theta\gamma}{G} g(f(g(x))) + \frac{1}{\sigma}] - \frac{2\theta}{1+\theta(x-1)}\right\} \\
&= Q'(x)\left\{\frac{(\theta-1)}{x[1+\theta(x-1)]}[\frac{2\theta\gamma}{G} g(f(g(x))) + \frac{1}{\sigma}] - \frac{2\theta}{1+\theta(x-1)}\right\} \\
&= \frac{Q'(x)2\theta}{1+\theta(x-1)}\left\{\frac{(\theta-1)}{x}[\frac{\gamma}{G} g(f(g(x))) + \frac{1}{\sigma 2\theta}] - 1\right\} \\
&> \frac{Q'(x)2\theta}{1+\theta(x-1)}\left\{\frac{(\theta-1)}{x}[\frac{\gamma}{G}\frac{G^2}{1+\theta(G-1)} + \frac{1}{\sigma 2\theta}] - 1\right\} \\
&> \frac{Q'(x)(\theta-1)2\theta}{1+\theta(x-1)}\left\{\frac{1}{x}[\frac{\gamma G}{1+\theta(G-1)} + \frac{1}{\sigma 2\theta}] - \frac{1}{(\theta-1)}\right\}
\end{aligned}$$

Since $1 < x < G$ it is $\frac{1}{x} > \frac{1}{G}$ so that

$$\begin{aligned}
Q''(x) &> \frac{Q'(x)(\theta-1)2\theta}{1+\theta(x-1)}\left\{\frac{\gamma}{[1+\theta(G-1)]} + \frac{1}{G\sigma 2\theta} - \frac{1}{(\theta-1)}\right\} \\
&= \frac{Q'(x)(\theta-1)2\theta}{1+\theta(x-1)}\left\{\frac{\gamma(\theta-1) - 1 - \theta(G-1)}{[1+\theta(G-1)](\theta-1)} + \frac{1}{G\sigma 2\theta}\right\} \\
&= \frac{Q'(x)(\theta-1)2\theta}{1+\theta(x-1)}\left\{\frac{(1+\gamma)(\theta-1) - \theta G}{[1+\theta(G-1)](\theta-1)} + \frac{1}{G\sigma 2\theta}\right\} > 0
\end{aligned}$$

The last inequality follows from

$$(1+\gamma)(\theta-1) > \theta G$$

In fact, considering that for $\sigma > \sigma_1 \simeq 21.231$ it is $1 > \gamma > \gamma_1 = (1-\frac{1}{\sigma_1}) \simeq 0.95238$ we have $(1+\gamma) > (1+\gamma_1) \simeq 1.95238$ and from $2.72 > e > \theta > \theta_1 = (1-\frac{1}{\sigma_1})^{(1-\sigma_1)} \simeq 2.65329$ we have $(\theta-1) > (\theta_1-1) \simeq 1.65329$ so that on the left side it is $(1+\gamma)(\theta-1) > 3.22785$, while (by using Theorem *) θG is bounded from above by the value $(2.72)(1.15) = 3.128$.□

Appendix C: Derivatives of the function $T_L(x)$

We consider the function $T_L(x) = g \circ f \circ g \circ f(x)$ defined in the interval $[g(G), 1]$ and its derivatives, in particular evaluated at the flip bifurcation value of the 2-cycle.

The first derivative is given by

$$\begin{aligned}
T_L'(x) &= g'(f \circ g \circ f(x)) \cdot f'(g \circ f(x)) \cdot g'(f(x)) \cdot f'(x) \\
&= \frac{-(\theta-1)}{f \circ g \circ f(x)[1+\theta(f \circ g \circ f(x)-1)]} T_L(x) \cdot \frac{\gamma}{g \circ f(x)} f \circ g \circ f(x) \cdot g'(f(x)) \cdot f'(x) \\
&= [T_L(x)]^2 \frac{-(\theta-1)}{f \circ g \circ f(x) G} \cdot \frac{\gamma}{g \circ f(x)} \cdot \frac{-(\theta-1)}{f(x)[1+\theta(f(x)-1)]} \cdot g \circ f(x) \cdot f'(x) \\
&= [T_L(x)]^2 \frac{(\theta-1)^2\gamma}{f \circ g \circ f(x) G} \cdot \frac{1}{[1+\theta(f(x)-1)]} \cdot \frac{f'(x)}{f(x)} \\
&= [T_L(x)]^2 \frac{(\theta-1)^2\gamma}{f \circ g \circ f(x) G^2 f(x)} \cdot \frac{G f(x)}{[1+\theta(f(x)-1)]} \cdot \frac{\gamma}{x} \\
&= [T_L(x)]^2 \frac{(\theta-1)^2\gamma}{f \circ g \circ f(x) G^2 f(x)} \cdot g \circ f(x) \cdot \frac{\gamma}{x} \\
&= \frac{[T_L(x)]^2}{x f(x)} \left[\frac{(\theta-1)\gamma}{G}\right]^2 \frac{g \circ f(x)}{f \circ g \circ f(x)}
\end{aligned}$$

The first derivative evaluated at the fixed point x_L of the 2-cycle, considering that $T_L(x_L) = x_L$, $f(x_L) = x_R$, $g \circ f(x_L) = x_L$, $f \circ g \circ f(x_L) = x_R$ leads to

$$\begin{aligned}
T_L'(x_L) &= \frac{[T_L(x_L)]^2}{x_L f(x_L)} \left[\frac{(\theta-1)\gamma}{G}\right]^2 \frac{g \circ f(x_L)}{f \circ g \circ f(x_L)} \\
&= \left[\frac{x_L}{x_R} \frac{(\theta-1)\gamma}{G}\right]^2
\end{aligned}$$

and computed at the flip bifurcation value $G = G_2^*$ with $x_R = \frac{(\theta-1)(1+\gamma)}{\theta}$, $x_L = g(x_R) = \frac{G_2^*(1+\gamma)}{\theta\gamma}$ we have, as expected, $T_L'(x_L) = 1$.

Second derivative of the function $T_L(x) = g \circ f \circ g \circ f(x)$. Considering

$$T_L'(x) = \frac{[T_L(x)]^2}{x f(x)} \left[\frac{(\theta-1)\gamma}{G}\right]^2 \frac{g \circ f(x)}{f \circ g \circ f(x)}$$

then

$$T_L''(x) = \frac{2T_L T_L' x f(x) - [T_L]^2(f(x) + x f'(x))}{[x f(x)]^2} \cdot \frac{g \circ f(x)}{f \circ g \circ f(x)} + \frac{[T_L]^2}{x f(x)} \cdot \frac{N}{[f \circ g \circ f(x)]^2}$$
$$= \frac{T_L}{x f(x) f \circ g \circ f(x)} \left\{ \frac{2T_L' x f(x) - T_L(f(x) + x f'(x))}{x f(x)} \cdot g \circ f + \frac{T_L}{f \circ g \circ f} N \right\}$$

where

$$N = g'(f(x)))f'(x) f \circ g \circ f(x) - g \circ f(x) f'(g \circ f(x)) g'(f(x)) f'(x)$$
$$= g'(f(x)))f'(x) f \circ g \circ f(x) - g \circ f(x) \frac{\gamma}{g \circ f(x)} f \circ g \circ f(x) g'(f(x)) f'(x)$$
$$= f \circ g \circ f(x)(1-\gamma) g'(f(x))) f'(x)$$

We have

$$T_L''(x) = \frac{T_L}{x f(x) f \circ g \circ f(x)} \left\{ \frac{2T_L' x f(x) - T_L(f(x) + x f'(x))}{x f(x)} \cdot g \circ f(x) + T_L(1-\gamma) g'(f(x)) f'(x) \right\}$$
$$= \frac{T_L(x) g \circ f(x)}{x f(x) f \circ g \circ f(x)} \left\{ \frac{2T_L' x f(x) - T_L(f(x) + x f'(x))}{x f(x)} + T_L(1-\gamma) \frac{g'(f(x)) f'(x)}{g \circ f(x)} \right\}$$

and

$$\frac{2T_L' x f(x) - T_L(f(x) + x f'(x))}{x f(x)} = \frac{2T_L' x f(x) - T_L(f(x) + \gamma f(x))}{x f(x)}$$
$$= \frac{2T_L' x - T_L(1+\gamma)}{x}$$
$$= T_L \left[\frac{2T_L'}{T_L} - \frac{(1+\gamma)}{x} \right]$$
$$= T_L \left[2 \frac{T_L(x)}{x f(x)} \left[\frac{(\theta-1)\gamma}{G} \right]^2 \frac{g \circ f(x)}{f \circ g \circ f(x)} - \frac{(1+\gamma)}{x} \right]$$
$$= \frac{T_L}{x} \left[2 \frac{T_L(x)}{f(x)} \left[\frac{(\theta-1)\gamma}{G} \right]^2 \frac{g \circ f(x)}{f \circ g \circ f(x)} - (1+\gamma) \right]$$

so that

$$T_L''(x) = \frac{T_L(x) g \circ f(x)}{x f(x) f \circ g \circ f(x)} \left\{ \frac{T_L}{x} \left[2 \frac{T_L(x)}{f(x)} \left[\frac{(\theta-1)\gamma}{G} \right]^2 \frac{g \circ f(x)}{f \circ g \circ f(x)} - (1+\gamma) \right] + T_L(1-\gamma) \frac{g'(f(x)) f'(x)}{g \circ f(x)} \right\}$$
$$= \frac{[T_L(x)]^2 g \circ f(x)}{x^2 f(x) f \circ g \circ f(x)} \left\{ 2 \frac{T_L(x)}{f(x)} \left[\frac{(\theta-1)\gamma}{G} \right]^2 \frac{g \circ f(x)}{f \circ g \circ f(x)} - (1+\gamma) - (1-\gamma) \frac{[-g'(f(x))] f'(x) x}{g \circ f(x)} \right\}$$

we can notice that at the fixed point x_L it is

$$
\begin{aligned}
T_L''(x_L) &= \frac{x_L}{[x_R]^2}\left\{2\frac{T_L(x_L)}{f(x_L)}\left[\frac{(\theta-1)\gamma}{G}\right]^2\frac{g\circ f(x_L)}{f\circ g\circ f(x_L)}-(1+\gamma)-(1-\gamma)\frac{[-g'(f(x_L))]f'(x_L)x_L}{g\circ f(x_L)}\right\} \\
&= \frac{x_L}{[x_R]^2}\left\{2\left[\frac{x_L}{x_R}\frac{(\theta-1)\gamma}{G}\right]^2-(1+\gamma)-(1-\gamma)[-g'(f(x_L))]f'(x_L)\right\} \\
&= \frac{x_L}{[x_R]^2}\left\{2\left[\frac{x_L}{x_R}\frac{(\theta-1)\gamma}{G}\right]^2-(1+\gamma)-(1-\gamma)\frac{(\theta-1)\gamma g\circ f(x_L)}{x[1+\theta(f(x_L)-1)]}\right\} \\
&= \frac{x_L}{[x_R]^2}\left\{2\left[\frac{x_L}{x_R}\frac{(\theta-1)\gamma}{G}\right]^2-(1+\gamma)-(1-\gamma)\frac{(\theta-1)\gamma}{[1+\theta(x_R-1)]}\right\}
\end{aligned}
$$

at the flip bifurcation value we obtain, as expected,

$$
T_L''(x_L) = \frac{x_L}{[x_R]^2}\left\{(1-\gamma)-(1-\gamma)\right\} = 0
$$

Third derivative. Considering

$$
\begin{aligned}
T_L''(x) &= \frac{[T_L(x)]^2 g\circ f(x)}{x^2 f(x) f\circ g\circ f(x)}\left\{2\frac{T_L(x)}{f(x)}\left[\frac{(\theta-1)\gamma}{G}\right]^2\frac{g\circ f(x)}{f\circ g\circ f(x)}-(1+\gamma)+(1-\gamma)\frac{g'(f(x))f'(x)x}{g\circ f(x)}\right\} \\
&= \frac{T_L'(x)}{x}\left[\frac{G}{(\theta-1)\gamma}\right]^2\left\{2x\frac{T_L'(x)}{T_L(x)}-(1+\gamma)+(1-\gamma)\frac{g'(f(x))f'(x)x}{g\circ f(x)}\right\} \\
&= \left[\frac{G}{(\theta-1)\gamma}\right]^2\left\{2\frac{[T_L'(x)]^2}{T_L(x)}-\frac{(1+\gamma)T_L'(x)}{x}+T_L'(x)(1-\gamma)\frac{g'(f(x))f'(x)}{g\circ f(x)}\right\}
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{1}{\left[\frac{G}{(\theta-1)\gamma}\right]^2}T_L'''(x) = &\frac{2}{[T_L(x)]^2}\left[2T_L'(x)T_L''(x)T_L(x)-[T_L'(x)]^3\right]-\frac{(1+\gamma)}{x^2}[xT_L''(x)-T_L'(x)]+ \\
&+T_L''(x)(1-\gamma)\frac{g'(f(x))f'(x)}{g\circ f(x)}+T_L'(x)(1-\gamma)D\left[\frac{g'(f(x))f'(x)}{g\circ f(x)}\right]
\end{aligned}
$$

evaluating in x_L we know that $T_L(x_L)=x_L$, $T_L'(x_L)=1$ and $T_L''(x_L)=0$, so that we have

$$
\begin{aligned}
\frac{1}{\left[\frac{G}{(\theta-1)\gamma}\right]^2}T_L'''(x)|_{x_L} &= \left\{\frac{-2}{[x_L]^2}+\frac{(1+\gamma)}{[x_L]^2}+(1-\gamma)\frac{d}{dx}\left[\frac{g'(f(x))f'(x)}{g\circ f(x)}\right]\right\} \\
&= \left\{\frac{1}{[x_L]^2}\frac{-1}{\sigma}+(1-\gamma)\frac{d}{dx}\left[\frac{g'(f(x))f'(x)}{g\circ f(x)}\right]\right\} \\
&= \frac{1}{\sigma}\left\{-\frac{1}{[x_L]^2}+\frac{d}{dx}\left[\frac{g'(f(x))f'(x)}{g\circ f(x)}\right]\right\}
\end{aligned}
$$

where

$$\frac{d}{dx}\left[\frac{g'(f(x))f'(x)}{g\circ f(x)}\right] = \frac{1}{[g\circ f(x)]^2}\left\{g''(f(x))[f'(x)]^2 g\circ f(x) + g'(f(x))f''(x)g\circ f(x) - [g'(f(x))f'(x)]^2\right\}$$
$$= \frac{1}{[g\circ f(x)]^2}\left\{-\frac{2\theta g\circ f(x)}{[1+\theta(f(x_L)-1)]}g'(f(x))[f'(x)]^2 + g'(f(x))\frac{-g\circ f(x)}{\sigma x}f'(x)\right.$$
$$\left.-[g'(f(x))f'(x)]^2\right\}$$

and at the bifurcation value (clearly at the bifurcation we consider $G = G_2^*$)

$$g'(f(x_L))f'(x_L) = g'(x_R)f'(x_L) = -1$$

so that

$$\frac{d}{dx}\left[\frac{g'(f(x))f'(x)}{g\circ f(x)}\right]|_{x_L} = \frac{1}{[x_L]^2}\left\{-\frac{2\theta g\circ f(x_L)}{[1+\theta(f(x_L)-1)]}g'(f(x_L))[f'(x_L)]^2\right.$$
$$\left.+g'(f(x_L))\frac{-g\circ f(x_L)}{\sigma x_L}f'(x_L) - [g'(f(x_L))f'(x_L)]^2\right\}$$
$$= \frac{1}{[x_L]^2}\left\{\frac{2\theta g\circ f(x_L)f'(x_L)}{[1+\theta(f(x_L)-1)]} + \frac{g\circ f(x_L)}{\sigma x_L} - 1\right\}$$
$$= \frac{1}{[x_L]^2}\left\{\frac{2\theta x_L f'(x_L)}{\gamma(\theta-1)} + \frac{x_L}{\sigma x_L} - 1\right\}$$
$$= \frac{1}{[x_L]^2}\left\{\frac{2\theta}{\gamma(\theta-1)}\frac{x_L G\gamma}{x_L^{1/\sigma}} + \frac{1}{\sigma} - 1\right\}$$
$$= \frac{1}{[x_L]^2}\left\{\frac{2\theta x_R}{(\theta-1)} + \frac{1}{\sigma} - 1\right\}$$
$$= \frac{1}{[x_L]^2}\left\{2(1+\gamma) + \frac{1}{\sigma} - 1\right\}$$
$$= \frac{1}{[x_L]^2}\left\{1 + 2\gamma + \frac{1}{\sigma}\right\}$$

leading to

$$\frac{1}{\left[\frac{G}{(\theta-1)\gamma}\right]^2}T_L'''(x)|_{x_L} = \frac{1}{\sigma}\left\{-\frac{1}{[x_L]^2} + \frac{d}{dx}\left[\frac{g'(f(x))f'(x)}{g\circ f(x)}\right]\right\}$$
$$= \frac{1}{\sigma}\left\{-\frac{1}{[x_L]^2} + \frac{1}{[x_L]^2}[1 + 2\gamma + \frac{1}{\sigma}]\right\}$$
$$= \frac{1}{\sigma}\frac{1}{[x_L]^2}\left\{2\gamma + \frac{1}{\sigma}\right\} > 0$$