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Expectation-Stock Dynamics in Multi-Agent Fisheries

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Abstract. In this paper we consider a game-theoretic dynamic model describing the exploitation of a renewable resource. Our model is based on a Cournot oligopoly game where n profit-maximizing players harvest fish and sell their catch on m markets. We assume that the players do not know the law governing the reproduction of the resource. Instead they use an adaptive updating scheme to forecast the future fish stock. We analyze the resulting dynamical system which describes how the fish population and the forecasts (expectations) of the players evolve over time. We provide results on the existence and local stability of steady states. We consider the set of initial conditions which give non-negative trajectories converging to an equilibrium and illustrate how this set can be characterized. We show how such sets may change as some structural parameters of our model are varied and how these changes can be explained. This paper extends existing results in the literature by showing that they also hold in our two-dimensional framework. Moreover, by using analytical and numerical methods, we provide some new results on global dynamics which show that such sets of initial conditions can have complicated topological structures, a situation which may be particularly troublesome for policymakers.

Keywords: game theory, resource modelling, adaptive expectations, dynamical systems, stability

1. Introduction

The dynamics of a fish stock results from two effects. First, a fish population is a renewable resource, that is, it is able to reproduce itself over time. Second, the fish stock is reduced by the activities of fishermen, who extract fish and sell their catch on markets. In fishery economics, dynamic models are used to capture these effects and to study the evolution of the resource stock, to derive optimal harvesting policies and to give recommendations to policy makers which regulatory measures are suitable in order to e.g. avoid overfishing. Game-theoretic models are used frequently in fishery economics to take strategic interactions among the fishermen into account. Levhari and Mirman (1982) were among the first who studied optimal harvesting policies in a duopoly framework with infinite

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horizon; see also Clark (1990). A repeated game setting for fishery management is used e.g. by Hannesson (1995) and for game-theoretic insights into the management of transboundary fisheries, see McKelvey (1997). Mesterton-Gibbons (1993) provides a survey of game-theoretic modeling in resource economics. More recently, multi-market, multi-agent models have been introduced by Szidarovszky and Okuguchi (1998, 2000). They study memoryless harvesting strategies—i.e. they assume that players are myopic and only consider the short-run success of their policies—in an n-player, n-market oligopoly game of international commercial fishing. The authors provide existence and stability results for the non-cooperative and the full cooperation case, where in the former players maximize their own profits without taking the effects on overall profits into account and in the latter each player's goal is to maximize joint profits. Szidarovszky and Okuguchi (2000) compare the relative stability of the two situations by considering the extensions of the basins of attractions, i.e. the set of initial conditions converging to the corresponding sustainable equilibrium.

In the present paper we take the Szidarovszky-Okuguchi-model as a starting point, but consider a discrete-time version and relax the assumption that fishermen are perfectly informed about the fish stock (see also Bischi and Kopel (2002)). The assumption that at any point in time fishermen are able to accurately predict the future fish stock prevailing in the sea is very common in fishery economics. However, since the environmental system where fishermen operate is very complex and highly nonlinear (see e.g. Rosser (2001, 2002)), this is a very strong requirement.[1] In our model, at the time when the fishermen determine their optimal harvest for the next period, they use an imperfect estimate of the fish stock which is derived from past data. As soon as new information about the fish stock becomes available, this estimate is updated. The crucial point is that the introduction of an adaptive process of this kind increases the dimension of the (discrete-time) dynamical system from one to two. As a result, it makes the study of the stability properties of the equilibria and, in particular, a characterization of the basins of attraction much more difficult. Hence, in order to answer questions related to the extension and the topological structure of the set of initial conditions which generate acceptable or feasible time paths we introduce and use a global dynamic approach based on a combination of analytical and numerical methods.

An investigation of the extension and the shape of the basins of the sustainable equilibria becomes crucial in order to shed some light on the question of conservation or extinction of the resource. Surprisingly, there are only limited attempts in the literature to address this important issue, which is probably due to the high complexity of the models. Sethi and Somanathan (1996) study the use of a common property resource in an evolutionary game-theoretic framework and analyze the size of the basins of attraction of the stable states and the changes the basins undergo as structural parameters are varied. Mäler (2000) considers the dynamical behavior of ecosystems in lakes. He studies the size of the basins of the two stable steady states (an eutrophic and an oligotrophic state) in order to get insights into the resilience of the system. In intertemporal optimization models on resource management global dynamics also play an important role. The optimal long term policy will be often path-dependent: for initial conditions smaller than a threshold (often referred to as a Skiba point) the optimal policy converges to the smaller steady state, whereas for initial conditions larger than this threshold the optimal policy converges to the larger one (see also Mäler, (2000)).[2]

Summarizing, our paper makes two contributions to the literature on dynamic games. First, we extend the multi-agent multi-market model of Szidarovszky and Okuguchi by allowing imperfect stock information. Second, we use non-standard tools for analyzing the global dynamics of dynamic games in discrete time. Although here we consider resource management, our approach has been quite helpful in various applications, namely in duopoly games with adaptive expectations (Bischi and Kopel, 2001), evolutionary games of market competition with spillovers (Bischi, Dawid, and Kopel, 2003a, b) and game-theoretic models of rent-seeking contests (Bischi, Gardini, and Kopel, 2000; Bischi, Kopel, and Naimzada, 2001).

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we study existence of equilibria and in Section 4 the local stability of these equilibria is analyzed. Section 5 focuses on the global dynamics. Throughout our analysis we compare the case where players act non-cooperatively with the case where players fully cooperate. We end the paper with a discussion and conclusions in Section 6.

2. The model

There are n players and m markets, where $n, m > 1$. The n players harvest fish and each player sells the fish on the m markets.[3] The inverse demand functions for the markets $i = 1, 2, \dots, m$ are given by

$$p_i = a_i - b_i(x_{1i} + x_{2i} + \cdots + x_{ni}), \tag{1}$$

where $x_{ki}(t)$ denotes the amount of fish harvested by player k and sold in market i at time period t. Let $X(t)$ be the total fish biomass at time t in the common sea and

$$h_k(t) = x_{k1}(t) + x_{k2}(t) + \cdots + x_{km}(t) \tag{2}$$

be the amount of fish *harvested* (*and sold*) by player k at time t. Each player's harvesting costs depend on the harvest rate h_k and, additionally, on the total fish stock X (this assumption captures the fact that it is easier and less expensive to catch fish, if the fish population is large). The cost function of player k is given by

$$C_k = c_k + \gamma_k h_k^2 \big/ X, \tag{3}$$

which satisfies the common assumptions that costs are decreasing in the fish stock and increasing in harvest.[4] Let $s_i(t) = x_{1i}(t) + x_{2i}(t) + \cdots + x_{ni}(t)$ be the amount of fish *supplied* (*and sold*) in market i at time period t. We assume that the total fish harvested by the players equals the total fish supplied in the markets, i.e.

$$H(t) := h_1(t) + h_2(t) + \cdots + h_n(t) = s_1(t) + s_2(t) + \cdots + s_m(t). \tag{4}$$

Following Szidarovszky and Okuguchi (1998, 2000) we assume that in the absence of harvesting the stock of the fish population is driven by the discrete-time logistic equation[5]

$$X(t+1) = X(t)\,(1 + \alpha - \beta X(t))\,. \tag{5}$$

The parameter α is referred to as the intrinsic growth rate. For any $\alpha > 0$ the system without harvesting has two fixed points

$$X_0^* = 0 \quad \text{and} \quad X_1^* = \frac{\alpha}{\beta}. \tag{6}$$

The fixed point $X_0^* = 0$ represents the extinction of the species and the second steady state, $K = \alpha/\beta$, is called the "carrying capacity". The equilibrium point $X_0^* = 0$ is unstable for each $\alpha > 0$, and the positive equilibrium X_1^* of (5) is stable for $0 < \alpha < 2$. For $2 < \alpha < 3$, even if X_1^* is unstable, a bounded positive attractor exists around it, characterized by oscillatory dynamics. For each $0 < \alpha < 3$ the basin is given by the interval[6]

$$\mathcal{B} = \left(0, \frac{1+\alpha}{\beta}\right).$$

Hence, if the intrinsic growth rate is not too large, the unharvested fish population might fluctuate, but never becomes extinct as long as the initial fish stock is in the interval $\mathcal{B}$. Only if the initial fish stock is taken out of the interval $\mathcal{B}$, the trajectory would take on negative values, which can be regarded as extinction of the fish population in finite time (see e.g. Clarke (1990), p. 13).

As mentioned above, it is usually assumed that fishermen have perfect stock information. In this case the dynamics of the *harvested* fish stock is governed by a one-dimensional system

$$X(t+1) = X(t)\,(1 + \alpha - \beta X(t)) - H(X(t)) =: F(X(t)) \tag{7}$$

where $H(X(t))$ is the total harvest in which notation we show the dependence of the optimal harvest rate on the fish stock. However, we find the assumption that fishermen know the relation (5) governing the reproduction of the fish population as very strong. Hence, here we assume that the players try to predict next period's fish stock based on past observations of the fish stock level. To keep the model tractable, we consider a situation where all players have homogeneous expectations with respect to the future fish stock, which might be due to the fact that they are engaged in the same business and share a common experience of working in this industry and, furthermore, have access to the same kind of environmental information. The common expectation of the fishermen is denoted by $X^e(t+1)$ and stands for the level of the fish stock predicted for period $t+1$. We assume that this prediction is formed by the following adaptive scheme

$$X^e(t+1) = \lambda X(t) + (1-\lambda)\, X^e(t) \tag{8}$$

with $0 \leq \lambda \leq 1$, that is as a weighted average of the previous estimate and the observed actual fish stock. The parameter λ can be interpreted as a measure of the inertia of the fishermen. Re-written as

$$X^e(t+1) = X^e(t) + \lambda(X(t) - X^e(t))$$

it becomes clear that players revise their previous forecasts of the fish stock in proportion to the difference between actual fish stock and the previously predicted fish stock level. Moreover, it is well-known that the forecasts $X^e(t+1)$ can be written as a weighted sum of all past observations of the fish stock, with higher weights given to more recent observations (e.g. Nerlove (1958)).

Given that players have imperfect stock information and predict the fish stock one period ahead using (8), the evolution of the fish population and the predicted values of the fish stock can now be described by a two-dimensional dynamical system:

$$\begin{aligned} X(t+1) &= X(t)(1+\alpha-\beta X(t)) - H(X^e(t)) \\ X^e(t+1) &= \lambda X(t) + (1-\lambda)X^e(t). \end{aligned} \tag{9}$$

The first equation determines the dynamics of the resource which is subject to the harvesting activities of the players. Total harvest results from the individual harvests of the n players, see (4), who determine their individual quantities on the basis of the predicted level of the fish population. The second dynamic equation describes the prediction updating. Observe that for $\lambda = 1$ we get $X^e(t+1) = X(t)$ for each t. This special case can be interpreted as a situation where fishermen have "naive" expectations in the sense that they believe that from one period to the next the fish stock will not change.[7]

Note that any non-negative steady state of system (9) (if existing) has to fulfil $X^e = X$. This corresponds to the equilibrium condition for the case (7) of perfect stock information. Consequently, the equilibria in the case of perfect stock information and for the two-dimensional model of imperfect stock information are the same.[8] The point $X_0^* = 0$ is always an equilibrium, since clearly we must have $H(0) = 0$. Moreover, a positive X is an equilibrium if and only if

$$\alpha - \beta X = \frac{H}{X} =: g(X). \tag{10}$$

Accordingly, the steady state fish stocks are given by the positive intersections of the linear function $\alpha - \beta X$ and the graph of the function $g(X)$.

In the next section we derive expressions for the total harvesting quantity $H(X^e(t))$ when players behave noncooperatively (each maximizing its own profit) and when they fully cooperate (each player maximizes the sum of the profits of all players). Using the properties of the corresponding functions $g(X)$, we also give results on the existence of long run steady states in the two cases.

3. Existence of equilibria

3.1. The non-cooperative case

Each player determines its memoryless harvesting strategy such that its expected profit is maximized, without taking into account any effect on the total profit. That is, the players select harvest rates as determined by the non-cooperative Nash equilibrium. The expected profit of player k in period t is

$$\pi_k^e(t) = \sum_{i=1}^{m}[a_i - b_i(x_{1i} + x_{2i} + \cdots + x_{ni})]x_{ki} - c_k - \gamma_k \frac{h_k^2(t)}{X^e(t)}. \tag{11}$$

Observe that in this setup the number of players who are actively harvesting in a Nash equilibrium is in general endogenously determined since the (expected) fish stock level influences the harvesting costs of each player. If players differ with respect to their cost parameters c_k and γ_k, then depending on the level of the (expected) fish stock harvesting might be profitable for some players, whereas other players exit the market. Although, a model without interrelated markets has been recently analyzed by Szidarovszky, Okuguchi, and Kopel (2003), a similar analysis of the more general model where interrelated markets are considered seems impossible. Hence, here we focus on situations where all players are actively harvesting independent of the level of the expected fish stock. In Appendix 1 we show that if the value of the cost parameters γ_k of the players are sufficiently similar and fixed costs sufficiently small, then $\pi_k^e > 0$. Furthermore, h_k is always positive; a sufficient condition for $s_i > 0$ is that $a_i \equiv a$ for all i; and, if we additionally assume that $\gamma_k \equiv \gamma$ for all k, then $x_{ki} > 0$. In summary, if we consider the symmetric model where players face the same costs ($\gamma_k \equiv \gamma$) and assume that $a_i \equiv a$, we can be assured that our problem has a feasible positive solution with all players being active in all markets.[9] Although we will derive the total harvest for the general case, we have to keep in mind that the values of the parameters have to be chosen appropriately in order to fulfill the sufficient conditions given above.

The first order conditions for player k are:

$$\frac{\partial \pi_k^e}{\partial x_{ki}} = a_i - b_i(x_{1i} + x_{2i} + \cdots + x_{ni}) - b_i x_{ki} - 2\gamma_k \frac{h_k(t)}{X^e(t)} = 0 \quad i = 1, \ldots, m$$

from which

$$x_{ki} = \frac{a_i}{b_i} - (x_{1i} + x_{2i} + \cdots + x_{ni}) - 2\frac{\gamma_k}{b_i}\frac{h_k}{X^e} \quad i = 1, \ldots, m \tag{12}$$

follows. We focus on the total amount of harvest by player k since it is the total harvest of all players which determines the dynamics of the fish stock. We add the equations above for all $i = 1, \dots, m$ to obtain

$$h_k = A - (h_1 + h_2 + \cdots + h_n) - 2\frac{B\gamma_k}{X^e} h_k \tag{13}$$

where $A = \sum_{i=1}^{m} (a_i/b_i)$ and $B = \sum_{i=1}^{m} (1/b_i)$. This relation can also be written as

$$h_k = \frac{A}{1 + \frac{2B\gamma_k}{X^e}} - \frac{H}{1 + \frac{2B\gamma_k}{X^e}},$$

which, after addition over $k = 1, 2, \dots, n$, gives

$$H = A \sum_{k=1}^{n} \frac{1}{1 + \frac{2B\gamma_k}{X^e}} - H \sum_{k=1}^{n} \frac{1}{1 + \frac{2B\gamma_k}{X^e}} .$$

By defining

$$f(X^e) := \sum_{k=1}^{n} \frac{1}{1 + \frac{2B\gamma_k}{X^e}}$$

we obtain the optimal total harvesting quantity of all players in the Nash equilibrium:

$$H(X^e) = A \frac{f(X^e)}{1 + f(X^e)}. \tag{14}$$

From this expression, we can derive the relative harvest

$$g(X^e) = \frac{H(X^e)}{X^e}. \tag{15}$$

Straightforward, although tedious, calculations show that total harvest $H(X^e)$ is strictly increasing and strictly concave in the expected fish stock,

$$\partial H/\partial X^e > 0 \quad \text{and} \quad \partial^2 H/\partial X^{e2} < 0.$$

Moreover, $H(0) = 0$, $\lim_{X^e \to \infty} H(X^e) = An/(n + 1)$, and $\lim_{X^e \to \infty} H'(X^e) = 0$. The relative harvest (15) is strictly decreasing and strictly convex in X^e (see Szidarovszky and Okuguchi (1998)), i.e.

$$\partial g/\partial X^e < 0 \quad \text{and} \quad \partial^2 g/\partial X^{e2} > 0,$$

and $\lim_{X^e \to \infty} g(X^e) = 0$, $\lim_{X^e \to \infty} g'(X^e) = 0$. Since the total harvesting quantity is given by (14), the dynamical system (9) can be written as

$$\begin{aligned} X(t + 1) &= X(t)(1 + \alpha - \beta X(t)) - \frac{Af(X^e(t))}{1 + f(X^e(t))} \\ X^e(t + 1) &= \lambda X(t) + (1 - \lambda)X^e(t) \end{aligned} \tag{16}$$

where the parameters γ_k, α, β, A and B are positive and $0 \leq \lambda \leq 1$. Notice that $X = 0$ is always an equilibrium, and the positive equilibria are obtained by the positive solutions of equation

$$\alpha - \beta X = g(X) = \frac{Af(X)}{(1 + f(X))X}$$

with $C := \sum_{i=1}^{n} (1/\gamma_k)$. Clearly, $g(0) = AC/2B$ and $g'(0) = -A(\sum_{i=1}^{n}(1/\gamma_k^2) + C^2)/4B^2$.

3.2. *The full cooperation case*

We now assume that each player determines its harvesting activity such that the *joint profit of all players* is maximized. That is, player k's harvesting quantity x_{ki} is chosen such that $\pi^e = \sum_{k=1}^{n} \pi_k^e$, is maximized, where π_k^e is given in (11). It is easy to see that the solution obtained under this assumption of full cooperation is the same as for a situation where a sole owner determines the total harvest and delegates the individual harvesting quantities to the n players such that total costs are minimized. As in the non-cooperative case, an industry equilibrium may be determined endogenously, since due to their costs some players might not find it worthwhile to be active or might pull out from certain markets. In Appendix 1 we show that, in contrast to the non-cooperative case, π_k^e is always positive if fixed costs are sufficiently small. Therefore, industry profits are positive. Furthermore, again h_k is always positive. A sufficient condition for $s_i > 0$ is that $a_i \equiv a$ for all i and there exist $x_{ki} > 0$ as long as $s_i > 0$. In summary, if we consider the case where $a_i \equiv a$ for all i, we can be assured that our problem has a feasible positive solution with all n players being active. Note that in contrast to the non-cooperative case, no restrictions with respect to the players' cost values are needed in order to ensure this existence.

The first-order conditions for each player k are

$$\frac{\partial \pi^e}{\partial x_{ki}} = a_i - 2b_i(x_{1i} + x_{2i} + \cdots + x_{ni}) - 2\gamma_k \frac{h_k(t)}{X^e(t)} = 0 \quad i = 1, \ldots, m.$$

We rewrite these conditions as

$$\frac{a_i}{b_i} - 2(x_{1i} + x_{2i} + \cdots + x_{ni}) - 2\gamma_k \frac{h_k(t)}{b_i X^e(t)} = 0 \quad i = 1, \ldots, m. \tag{17}$$

Adding over all markets i yields

$$A - 2H - 2\gamma_k \frac{Bh_k(t)}{X^e(t)} = 0,$$

where we use the same definitions of A and B as before. Rewriting this condition as

$$\frac{A-2H}{2B\gamma_k} = \frac{h_k}{X^e}, \tag{18}$$

and summing over all players gives the expression for the total harvest if players fully cooperate

$$H^V(X^e) = \frac{ACX^e}{2(CX^e + B)}, \tag{19}$$

where as before $C := \sum_{k=1}^{n} \frac{1}{\gamma_k}$. The relative harvest in this case is given by

$$g^V(X^e) = \frac{H^V}{X^e} = \frac{AC}{2(CX^e + B)}.$$

It is easy to see that, as in the non cooperative case, H^V is strictly increasing and strictly concave. Furthermore, $H^V(0) = 0$, $\lim_{X^e\to\infty} H^V(X^e) = A/2$, and $\lim_{X^e\to\infty} H^{V\prime}(X^e) = 0$, and g^V is strictly decreasing and strictly convex in X^e. In addition, $\lim_{X^e\to\infty} g^V(X^e) = \lim_{X^e\to\infty} g^{V\prime}(X^e) = 0$.

The dynamic equations obtained for the full cooperation case under imperfect stock information are now given by

$$\begin{aligned} X(t+1) &= X(t)(1+\alpha-\beta X(t)) - \frac{ACX^e}{2(CX^e+B)} \\ X^e(t+1) &= (1-\lambda)X^e(t) + \lambda X(t) \end{aligned} \tag{20}$$

and the equilibrium condition (10) is

$$\alpha - \beta X = g^V(X) = \frac{AC}{2(CX+B)}.$$

Clearly, $g^V(0) = AC/2B = g(0)$, but $g^{V\prime}(0) = -AC^2/2B^2 < g'(0)$.

3.3. Equilibrium analysis

We can state the following result on the existence and number of steady states in both non-cooperative and cooperative cases, which can be derived directly from the properties of the function g and from figure 1.

Proposition 3.1. *Let* $H(X)$ be the total harvest (14) or (19) and let $g(X) = H(X)/X$ denote the relative harvest. Then the following holds.

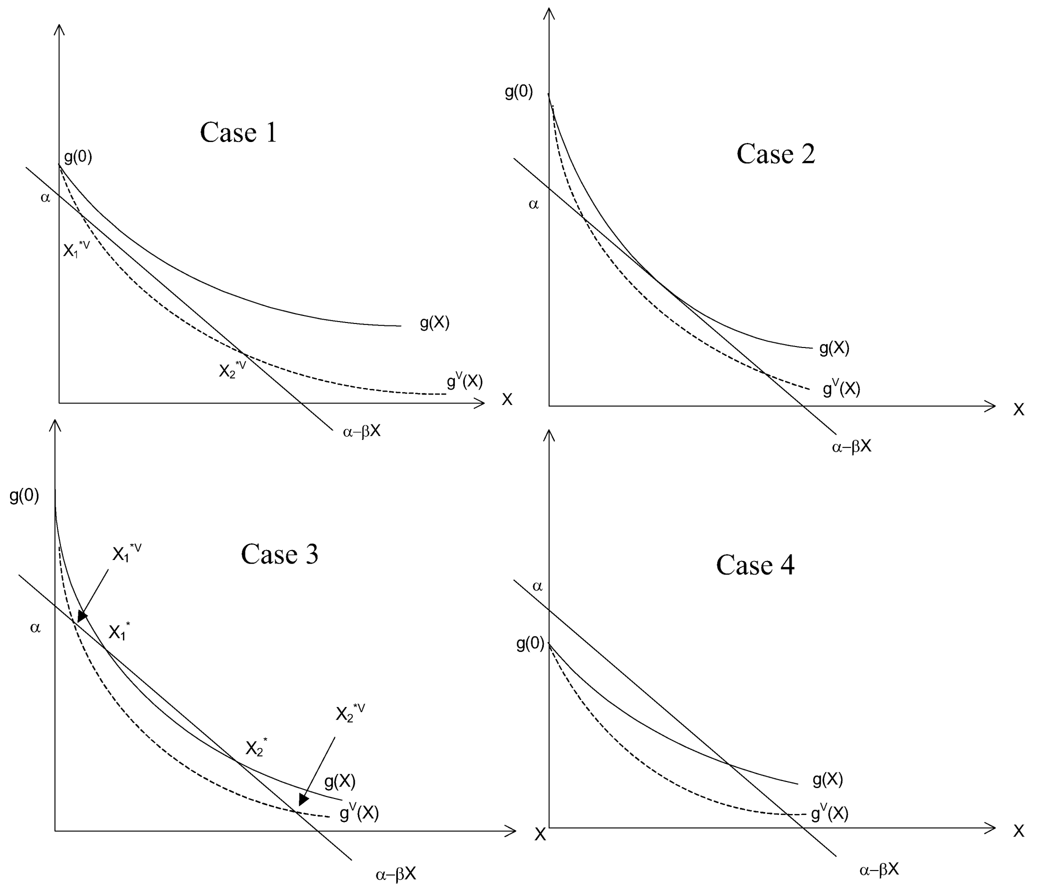

Figure 1. Graphical representation of the equilibrium condition $\alpha - \beta X = g(X)$, where $g(X)$ and $g^V(X)$ represent the relative harvests under the assumptions of no cooperation (solid line) and full cooperation (dashed line) respectively. Four different situations can be distinguished resulting in a different number of equilibria.

(i) Assume first that $g(0) > \alpha$ and $g'(0) < -\beta$. Then, there is a unique $\bar{X}$ such that $g'(\bar{X}) = -\beta$.
(iA) If $g(\bar{X}) > \alpha - \beta\bar{X}$, then no positive equilibrium exists.
(iB) If $g(\bar{X}) = \alpha - \beta\bar{X}$, then there is a unique positive equilibrium.
(iC) If $g(\bar{X}) < \alpha - \beta\bar{X}$, then there are two positive equilibria.

(ii) Assume next that $g(0) > \alpha$ and $g'(0) \geq -\beta$. Then no positive equilibrium exists.

(iii) Assume that $g(0) = \alpha$.
(iiiA) If $g'(0) < -\beta$, then there is a unique positive equilibrium.
(iiiB) If $g'(0) \geq -\beta$, then there is no positive equilibrium.

(iv) Assume finally that $g(0) < \alpha$. Then there is a unique positive equilibrium.

Notice that (iiiB) is the borderline case of (ii), (iiiA) is a borderline case of (iv), and (ii) and (iA) can be treated in the same way. So we have four basic cases (see figure 1). In case 1, no positive equilibrium exists. Case 2 is characterized by the existence of a unique positive equilibrium. In case 3, two positive equilibria X_1^* and X_2^* exist, where $X_1^* < X_2^*$. In case 4, again only 1 unique equilibrium exists.

We can compare the total harvest rates of all players in the non-cooperative and cooperative cases in the following way.

Lemma 3.2. For a given expected fish stock X^e and a given number of players n, we have

$$H^V(X^e) < H(X^e),$$

i.e. the total harvest in the full cooperation case (19) is smaller than in the non-cooperative case (14).

Proof. The inequality $H^V(X^e) < H(X^e)$ is equivalent to

$$f(X^e(t)) > \frac{CX^e}{CX^e + 2B}.$$

Let $z_k = \frac{X^e}{2B\gamma_k}$. Then $f(X^e) = \sum_{k=1}^n \frac{1}{1+\frac{1}{z_k}} = \sum_{k=1}^n \frac{z_k}{1+z_k}$. The right hand side of the inequality can be also expressed in terms of z_k as $\frac{CX^e}{CX^e+2B} = \frac{1}{1+2B/CX^e} = \frac{1}{1+1/\sum_{k=1}^n z_k} = \frac{\sum_{k=1}^n z_k}{\sum_{k=1}^n z_k+1}$. Since, for $k = 1, 2, \ldots, n$ we have $\frac{z_k}{1+z_k} > \frac{z_k}{1+\sum_{k=1}^n z_k}$, adding these inequalities for all k proves the claim. □

Note, however, that this result is valid only for a given expected fish stock. In general, the positive long-run equilibrium fish stock levels for the full cooperation case and the non-cooperative case differ. Therefore, this result does not tell us how these steady states and the corresponding total harvests in these steady states relate to each other.[10]

In comparing the two cases we have the following additional result.

- In case 1, if there is no positive equilibrium for the non-cooperative case, full cooperation may result in the emergence of one or two (positive) equilibria, X_1^{*V} and X_2^{*V} (figure 1(a)).
- In case 2, full cooperation will result in the appearance of two positive equilibria, X_1^{*V} and X_2^{*V}, where the original positive equilibrium (for the non-cooperative case) is always between the two new equilibria (for the full cooperation case) (figure 1(b)).
- In case 3, with full cooperation two equilibria still exist. The change occurs with respect to the location of the equilibria: the smaller equilibrium decreases and the larger one increases, i.e. $X_1^{*V} < X_1^*$ and $X_2^{*V} > X_2^*$ (figure 1(c)).

- Considering case 4 we notice that with full cooperation the unique positive equilibrium increases with respect to the noncooperative game, i.e. $X^{*V} > X^*$ (figure 1(d)).

Intuitively, our results show that for the same expected fish stock, agents acting cooperatively harvest less than if they act in a noncooperative way. The aggregate behavior eventually leads to a higher sustainable fish stock in the long run. Therefore, cooperation leads in this sense to conservation of the resource. These insights are in line with earlier results for game-theoretic models of fisheries (e.g. Clark, 1990; Levhari and Mirman, 1982).

4. Local stability of equilibria

The two-dimensional dynamical system in (9) can be represented as an iterated point mapping

$$T : \begin{cases} X(t+1) = X(t)(1+\alpha-\beta X(t)) - H(X^e(t)) \\ X^e(t+1) = \lambda X(t) + (1-\lambda)X^e(t). \end{cases} \tag{21}$$

Each time the map T is applied, a point of the plane (X, X^e) is moved to another point, which represents the state of the system at the next time step. A trajectory of the system

$$\tau(X(0), X^e(0)) = \{(X(t), X^e(t)) = T^t(X(0), X^e(0)), t \geq 0\}$$

is generated by T starting from an initial condition $(X(0), X^e(0))$. The projection of points of a trajectory on the horizontal axis gives the time evolution of the fish stock, the projection on the vertical axis gives the time evolution of players' forecasts. Time periods at which the trajectory is close to the diagonal $X^e = X$ correspond to periods at which expectations are quite accurate, whereas points far away from the diagonal represent situations where players overestimate (points above the 45-degree line) or underestimate (points below the 45-degree line) the actual fish stock.

For the non-cooperative and cooperative case, the total harvest H is given by (14) or (19). To derive conditions for the *local stability* of the equilibria, we have to analyze the eigenvalues of the Jacobian matrix

$$DT(X, X^e) = \begin{bmatrix} 1+\alpha-2\beta X & -H'(X^e) \\ \lambda & 1-\lambda \end{bmatrix},$$

evaluated at the equilibrium (or fixed point) under consideration. At a given fixed point (X^*, X^*), the characteristic equation becomes $P(z) = z^2 - Tr^*z + Det^* = 0$, where

$$Tr^* = 2+\alpha-2\beta X^* - \lambda \quad \text{and} \quad Det^* = (1-\lambda)(1+\alpha-2\beta X^*) + \lambda H'(X^*).$$

A sufficient condition for the stability is given by the following system of inequalities

$$P(1) = 1 - Tr^* + Det^* > 0\,; \quad P(-1) = 1 + Tr^* + Det^* > 0\,; \quad Det^* < 1 \tag{22}$$

which provide necessary and sufficient conditions for the two eigenvalues to be inside the unit circle of the complex plane (see e.g. Gumowski and Mira (1980) or Medio and Lines (2001)).

As an example consider the equilibrium $O = (0, 0)$. The first condition in (22) gives $H'(0) > \alpha$. It is easy to see that this condition coincides with the stability condition of the one-dimensional model (7) where agents have perfect stock information (see also Bischi and Kopel (2002)). The second condition becomes $4 + (2 - \lambda)\alpha + \lambda H'(0) - 2\lambda > 0$, which is always satisfied, since $0 \le \lambda \le 1$. The third condition gives the extra stability condition

$$H'(0) < (\lambda - \alpha\,(1 - \lambda))\,/\lambda.$$

So, we can conclude that the range of local asymptotic stability of the extinction equilibrium, defined by

$$\alpha < H'(0) < 1 - \alpha\frac{1-\lambda}{\lambda}$$

is non-empty only if $\alpha < \lambda$. It is again easy to see that the range of stability of the extinction equilibrium is smaller under the adaptive forecasting scheme (8) than in the case of perfect stock information (Bischi and Kopel, 2002). However, from this insight we must not conclude that extinction is less probable if agents adapt their beliefs using such an adaptive scheme. In fact, the conditions above only concern the *asymptotic* stability of the extinction equilibrium, whereas extinction may occur in finite time. The reason for this is that trajectories may exit the positive quadrant in finite time, although eventually converging towards an equilibrium. As an example consider the situation where the fixed point (0, 0) is an unstable focus (complex eigenvalues with modulus greater than one). Trajectories can be described as spiralling around (0, 0) with increasing amplitude. Of course, this implies that the time paths would involve negative and hence unfeasible values of the fish stock. In such a case, in fishery economics this matter is solved by saying that extinction occurs in finite time (see Clark (1990) and Bischi and Kopel (2002) for a more detailed discussion on this point).

A rigorous analytical study of the conditions for the stability of the positive equilibria, when they exist according to Proposition 3.1, is not easy, because in the general case we do not have analytical expressions of their coordinates. However, we know that if a positive equilibrium (X^*, X^*) of the model (9) exists, then X^* is also an equilibrium of the model with perfect stock information (7). Starting from this observation, we investigate the influence of the degree of inertia in revising expectations (which is represented by the parameter λ) on the stability of the equilibrium (X^*, X^*) given that a positive equilibrium X^* is stable under the assumption of perfect stock information. Although the coordinates of a steady state of (9) are independent of the parameter λ, its stability is influenced by λ according to the following Proposition.

Proposition 4.1. Let X^* be a positive steady state which is stable under the dynamics with perfect stock information (7) with H defined in (14) or (19).

(i) If $H'(X^*) < 1$ and $\alpha - 2\beta X^* < 0$, then (X^*, X^*) is a stable steady state of (9) for each $\lambda \in [0, 1]$.
(ii) If $H'(X^*) > 1$ and $\alpha - 2\beta X^* < 0$, then (X^*, X^*) is a stable steady state of (9) for $\lambda \in \left[0, \bar{\lambda}\right]$, where

$$\bar{\lambda} = \frac{\alpha - 2\beta X^*}{1 + \alpha - 2\beta X^* - H'(X^*)} \tag{23}$$

and it loses stability through a Neimark-Hopf bifurcation as λ is increased across the bifurcation value $\bar{\lambda} \in (0, 1)$.
(iii) If $H'(X^*) < 1$ and $\alpha - 2\beta X^* > 0$, then (X^*, X^*) is a stable steady state of (9) for $\lambda \in [\bar{\lambda}, 1]$, where $\bar{\lambda} \in (0, 1)$ is given by (23), and it loses stability through a Neimark-Hopf bifurcation as λ is decreased across the bifurcation value $\bar{\lambda}$.
(iv) If $H'(X^*) > 1$ and $\alpha - 2\beta X^* > 0$, then (X^*, X^*) is unstable for each $\lambda \in [0, 1]$ under the dynamics of (9).

Proof. Stability under perfect stock information means that $|F'(X^*)| < 1$, i.e. $H'(X^*) - 2 < \alpha - 2\beta X^* < H'(X^*)$. Under this assumption it is easy to show that the first and second stability conditions (22) are always satisfied. In fact, $P(1) = \lambda(2\beta X^* - \alpha + H'(X^*)) > 0$, and $P(-1) = (2 - \lambda)(\alpha - 2\beta X^* + 2) + \lambda H'(X^*) > 2H'(X^*) > 0$ for each $\lambda \in [0, 1]$. The third stability condition can be written as $Det^*(\lambda) < 1$, where $Det^*(\lambda)$ is a linear function of λ such that $Det^*(0) = (1 + \alpha - 2\beta X^*)$ and $Det^*(1) = H'(X^*)$. So, if the assumptions in (i) hold, then $Det^*(\lambda) < 1$ for each $\lambda \in [0, 1]$, if the assumptions in (iv) hold, then $Det^*(\lambda) > 1$ for each $\lambda \in [0, 1]$, whereas under the assumptions in (ii) and (iii) the graph of $Det^*(\lambda)$ crosses the value $Det^*(\lambda) = 1$ at $\lambda = \bar{\lambda} \in (0, 1)$, with positive or negative slope respectively. The conditions $P(1) > 0$, $P(-1) > 0$ and $Det^*(\bar{\lambda}) = 1$ ensure that for $\lambda = \bar{\lambda}$ the two eigenvalues of $DT(X^*, X^*)$ are complex conjugate with unitary modulus, i.e. they are on the unit circle of the complex plane, and are crossing it with $\frac{dDet^*(\bar{\lambda})}{d\lambda} \neq 0$. These are the conditions for the occurrence of a Neimark-Hopf bifurcation at $\lambda = \bar{\lambda}$. □

Our numerical investigations suggest that the Neimark-Hopf bifurcation is subcritical. That is, a repelling invariant closed curve exists around the stable equilibrium when the parameter λ is close to the bifurcation value (23) and this curve constitutes the boundary of the immediate basin of the stable equilibrium. This basin shrinks as λ approaches the bifurcation value $\bar{\lambda}$, at which the equilibrium becomes unstable.

Propositions 3.1 and 4.1 are valid in both cases, non-cooperative and cooperative. However, important differences in the quantitative effects are caused by the fact that total harvest is lower in the cooperative case. As argued above, when a unique equilibrium exists in both cases, we have $X^{*V} > X^*$; when two equilibria exist, then for the larger one (which is the only one that can be stable under perfect stock information) the relation $X_2^{*V} > X_2^*$ holds. Now start from a given equilibrium X^* which is stable in the perfect stock information case with harvesting H. Denote by $X^{*V} > X^*$ the corresponding stable equilibrium in the perfect stock information case with cooperative harvesting H^V. Then, the inequalities $H^{V\prime}(X^{*V}) < H'(X^*)$ and $\alpha - 2\beta X^{*V} < \alpha - 2\beta X^*$ hold and, consequently, the stability range as the parameter λ is varied is larger in the case of full cooperation.

As far as stability and the stability extent of the equilibria are concerned, the main differences between the two cases, non-cooperation and full cooperation, can be better illustrated by numerical studies which are guided by the analytic and qualitative insights presented above.

5. Global dynamics

Now we turn to an analysis of the *global* dynamics of our fishery model for various choices of the parameters. Our goal in this section is to characterize the set of initial conditions which (i) lead to time paths involving only non-negative values of the fish stock and (ii) converge to a positive steady state. Following Bischi et al. (2000) we will refer to this set as the *feasible set*. If negative values of the fish stock are obtained, we consider the fish populations as extinct (in finite time). Hence, the feasible set is (in most cases considered here) a subset of the basin of attraction in a mathematical sense. As numerical experiments will show, the feasible set may have complicated topological structure. Indeed, it may be formed by several non-connected portions or it may be a connected set with "holes" inside, where such "holes" represent portions of the basins of other attractors.[11] These phenomena are related to the fact that the two-dimensional dynamical system in discrete time which governs the adaptive process is obtained through the iteration of a noninvertible (or many-to-one) map.[12] The global bifurcations that cause qualitative changes of the structure of the feasible set will be explained by using the concept of critical curves. A brief introduction into this concept is provided in Appendix 2. The interested reader is also referred to Mira et al. (1996), Puu (2000); see Bischi, Gardini, and Kopel (2000), Bischi and Kopel (2001, 2002, 2003), Bischi, Dawid, and Kopel (2003), for recent applications in economics.

In the numerical examples described in this section, the values of the "biological" parameters are[13]

$$\alpha = 3, \quad \beta = 1. \tag{24}$$

Moreover, for both cases—cooperative and non-cooperative—situations with two players and two markets, i.e. $n = m = 2$, and no fixed costs, $c_k = 0$, are considered. To make sure that quantities and profits are positive, we choose values for the parameters such that the sufficient conditions provided in Appendix 1 are fulfilled ($\gamma_1 = \gamma_2 = \gamma$ and $a_1 = a_2 = a$).

5.1. *The non-cooperative case*

We first study the global dynamics of the non-cooperative case with $a_1 = a_2 = 4.8$, $b_1 = b_2 = 2$, $\gamma_1 = \gamma_2 = 1.1$, and $\lambda = 0.6$. These parameter values yield $A = 4.8$, $B = 1$, $C = 1.8182$. We have $AC/(2B) > \alpha$, and there are two positive equilibria for the competitive harvesting case with perfect stock information, $X_1^* = 0.6 < X_2^* = 1.67$, where X_1^* is unstable and X_2^* is stable. The corresponding steady states of the two-dimensional model with imperfect stock information, (X_1^*, X_1^*) and (X_2^*, X_2^*), are a saddle point and a stable focus respectively. The feasible set of the stable equilibrium (X_2^*, X_2^*), represented by the white region in figure 2(a), is rather large. Moreover, the equilibrium is far away from the boundaries. This can be taken as an indication that in such a situation even exogenous shocks will not lead to disaster, i.e. the stable equilibrium is robust with respect to noise (see Mäler, 2000). The grey region in figure 2(a) represents the set of initial conditions which result in extinction of the fish population (recall that this means extinction in the long run or extinction in finite time). The boundary that separates the two sets is formed by the stable manifold of the saddle point (X_1^*, X_1^*).

We will study the impact of changes in the cost parameters γ_1 and γ_2 on the extent of the feasible set and its structure. The influence of changes in the cost parameters is of significant interest for policy makers, since the costs of harvesting can be changed by such methods as restricting the length of the fishing season, setting total catch limitations, and regulating the type of fishing gear used. Furthermore, we will also investigate the effect of variations in λ, which measures the inertia of the fishermen to revise their forecasts as new information becomes available. The influence of changes in λ is interesting from a behavioral point of view: if fishermen put a higher weight on the most recently observed fish stock, how does this change the feasible set of the stable equilibrium?

Intuitively speaking, high harvesting costs should prevent over-fishing and lead to more conservation. Indeed, this is the situation shown in figure 2(a). The situation changes drastically, however, for decreasing cost values. This is shown in figure 2(b), obtained with $\gamma_1 = \gamma_2 = 0.9$. In this case, the size of the feasible set (which here coincides with the basin) of the positive equilibrium is small, and it will reduce even further for decreasing values of the cost parameters, until the equilibrium will become unstable and every initial condition will lead to extinction. So, as the intuition suggests, as it becomes cheaper to harvest fish, extinction of the fish stock becomes increasingly more likely.

To investigate the role of the inertia of fishermen to revise their forecasts, we use the same set of parameters as in figure 2(b), and consider the stability results stated in Proposition 4.1. In this case we have $H'(X_2^*) = 0.8 < 1$ and $\alpha - 2\beta X_2^* = 0.2 > 0$. So, we expect that the positive equilibrium will lose stability via a Neimark-Hopf bifurcation, if

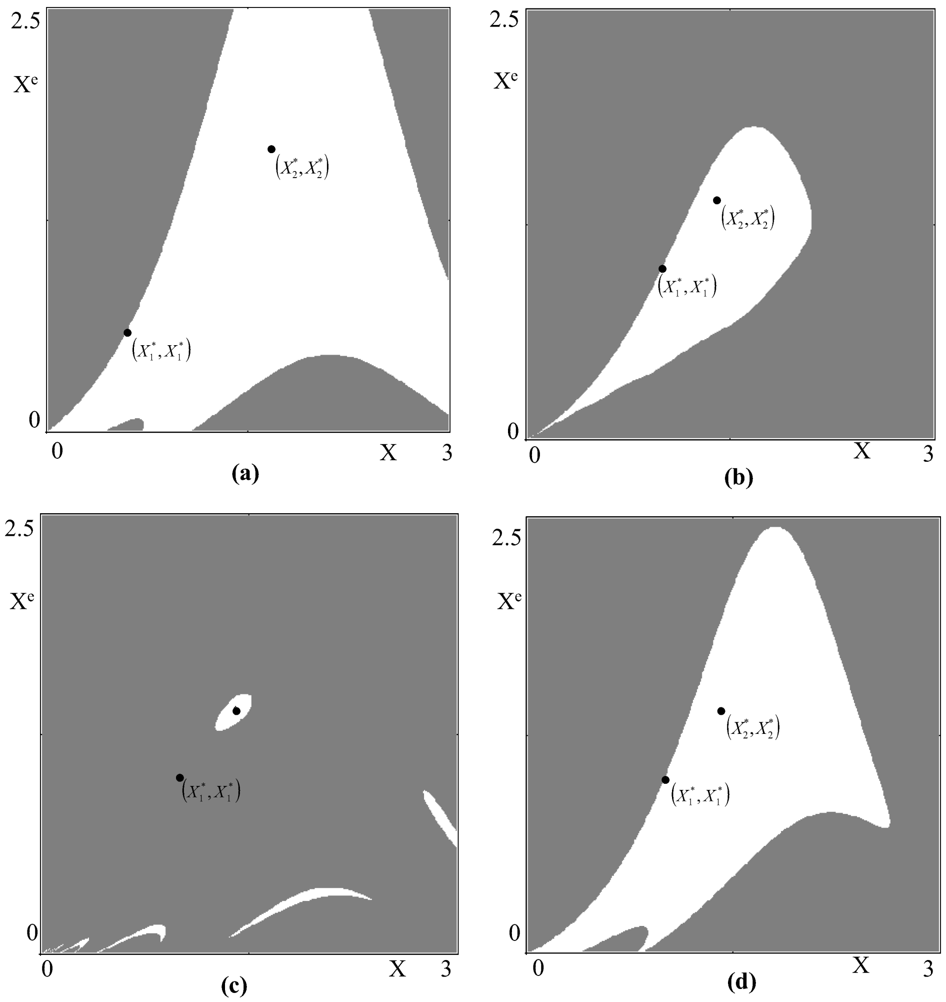

Figure 2. The non-cooperative case. The figures show the set of points (white region) that generate positive trajectories converging to the equilibrium (X_2^*, X_2^*). All points in the grey region generate trajectories that converge to the (0, 0) equilibrium (i.e. asymptotic extinction) and points that generate trajectories leading to extinction in finite time. (a) $a_1 = a_2 = 4.8$, $b_1 = b_2 = 2$, $\gamma_1 = \gamma_2 = 1.1$, $\lambda = 0.6$. (b) same as (a), but $\gamma_1 = \gamma_2 = 0.9$. (c) same as (a), but $\lambda = 0.3$. (d) same as (b), but $\lambda = 0.9$.

λ becomes lower than $\bar{\lambda} = 0.278$. Indeed, for $\lambda = 0.3$ the feasible set (which coincides with the basin) of the positive equilibrium is so small that the term "stability" has lost any practical meaning (see figure 2(c)). It can be noticed that now the basin is formed by several non connected portions: the "immediate basin", i.e. the portion of the basin that includes the stable equilibrium, is bounded by a repelling closed invariant curve of circular shape, thus suggesting that a subcritical Neimark-Hopf bifurcation is going to occur if λ is further decreased. The other portions are preimages of the immediate basin. Of course, in such a situation an increase in λ, i.e. less inertia in revising expectations, causes an enlargement of the basin or feasible set. This is shown in figure 2(d), which is obtained with the same values of the cost parameters as figure 2(b) and 2(c) and $\lambda = 0.9$. This seems to suggest that less inertia in revising expectations yields more robustness of the stable equilibrium.

Consequently, it seems that there are two possible ways to achieve more stability in terms of the extent of the feasible set of the equilibrium. A policy maker may increase the costs of harvesting the resource and may prevent over-fishing of the resource. If it is more expensive for fishermen to harvest the resource, total harvesting activity is reduced. As a result, conservation of the resource is achieved from a larger set of initial conditions for fish stock and predictions. A different route to higher stability seemingly is to make the fishermen believe that the use of the most recent observation of the fish stock gives a better prediction of future fish stocks than relying on past observations.

However, the second route does not always work, as suggested by Proposition 4.1. For example, let us consider the situation shown in figure 3(a), which is obtained for $a_1 = a_2 = 5.5$, $b_1 = b_2 = 1$, $\gamma_1 = \gamma_2 = 2.58$ and $\lambda = 0.3$. These values yield $A = 11$, $B = 2$, $C = 0.775$. In this case $AC/(2B) < \alpha$, so we have only one positive equilibrium at $X^* = 1.522$. Since $H'(X^*) = 1.02$ and $\alpha - 2\beta X^* = -0.0443$, the case (ii) of Proposition 4.1 implies that stability is *lost* for increasing values of the parameter λ at the bifurcation value $\bar{\lambda} = 0.64$. In fact, figure 3(b) shows the result obtained for $\lambda = 0.6$, which indicates that a subcritical Neimark-Hopf bifurcation at which stability

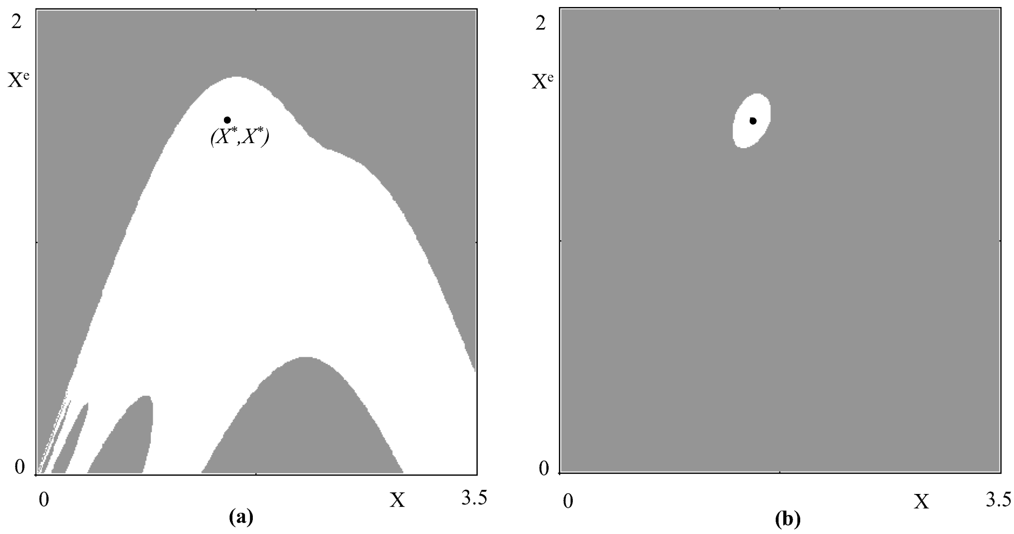

Figure 3. The non-cooperative case. The values of the parameter are $a_1 = a_2 = 5.5$, $b_1 = b_2 = 1$, $\gamma_1 = \gamma_2 = 2.58$. (a) $\lambda = 0.3$; (b) $\lambda = 0.6$.

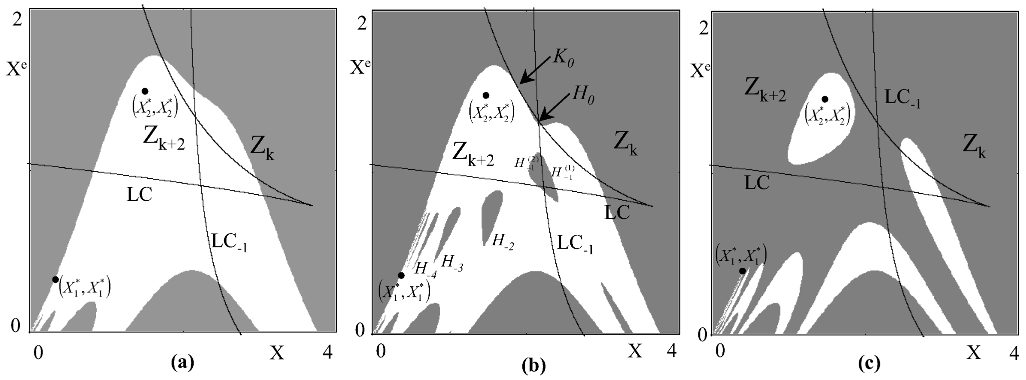

Figure 4. The non-cooperative case. The values of the parameters are $a_1 = a_2 = 6,\ b_1 = b_2 = 2, \lambda = 0.3$. (a) $\gamma_1 = \gamma_2 = 1.75, \lambda = 0.3$. (b) $\gamma_1 = \gamma_2 = 1.725$. (c) $\gamma_1 = \gamma_2 = 1.71$. In the figures the critical curves LC_{-1} (locus of vanishing Jacobian) and $LC = T(LC_{-1})$ are depicted. Branches of LC constitute the boundaries that separate regions with different number of preimages, denoted by Z_k and Z_{k+2}.

is lost will occur. So, in this situation stability (as well as a larger feasible set) is ensured for *higher* inertia in revising expectations, i.e. for lower values of λ.

Another interesting situation is obtained for $a_1 = a_2 = 6,\ b_1 = b_2 = 2, \lambda = 0.3$, $\gamma_1 = \gamma_2 = 1.75$ and $\lambda = 0.3$, which yields $A = 6$, $B = 1$, $C = 1.1428$ (figure 4(a)). In this case two positive equilibria exist for harvesting under perfect stock information, the stable one being $X_2^* = 1.5$. Since $H'(X_2^*) = 0.65$ and $\alpha - 2\beta X_2^* = 0$, case (i) of Proposition 4.1 ensures that the positive equilibrium is stable for each $\lambda = [0, 1]$, i.e. the equilibrium is not destabilized as the predictions move closer to so-called "naive" forecasts ($\lambda = 1$).

However, surprising and unexpected effects are obtained for decreasing cost values. As we will demonstrate, the observed changes in the feasible set can be explained by global bifurcations due to contacts between critical curves and the feasible sets' boundaries (see Mira et al. (1996), or Bischi and Kopel (2001), for more details). We refer the reader to Appendix 2 where we give a brief introduction into the concepts used for our analysis. In figure 4(a) the critical curves LC_{-1} and LC are shown. The region inside the two branches of LC (joining at a cusp point) is characterized by points that have two more preimages than points located outside of this region. Hence, in figure 4, we denote these two regions by Z_{k+2} and Z_k. The critical curve LC can now be used to understand the qualitative changes in the topological structure of the feasible sets shown in figure 4 in terms of contacts between $LC = T(LC_{-1})$ and the sets' boundaries (global bifurcations). Starting from a situation presented in figure 4(a), for decreasing cost values the feasible set of the stable equilibrium shrinks. In figure 4(b), obtained for $\gamma_1 = \gamma_2 = 1.725$, we show the situation right after the boundary of the feasible set had a contact with LC. A small portion of the grey set (the "extinction set")—denoted by H_0—entered the region Z_{k+2} and two new rank-1 preimages of H_0 are created, which are located at opposite sides with respect to LC_{-1} and joining along it (this is due to the fact that LC_{-1} is the locus of merging preimages of the points of LC). These newly created two preimages, denoted by $H_{-1}^{(1)}$ and $H_{-1}^{(2)}$ in figure 4(b), constitute a "hole" inside the feasible set of the positive stable equilibrium. This hole belongs to the grey set, and hence initial conditions in this set lead to extinction of the fish population.[14] The rank-1 preimages of the points which belong to this "main hole" form another "hole", denoted by H_{-2}, whose points are mapped into H_0 after two iterations. This iterative procedure gives rise to a sequence of holes nested inside the feasible set of the positive equilibrium. Observe that this global bifurcation drastically changes the topological structure of the feasible set from a simply connected set to a multiply connected set (compare figures 4(a) and (b)). However, if cost values are decreased further, another global bifurcation can be detected. In this case it is due to the contact between the portion of the boundary of the feasible set denoted by K_0 in figure 4(b). The effect on the feasible set is shown in figure 4(c), obtained after a decrease of γ_1 and γ_2 to 1.71. Now the feasible set is formed by the immediate basin (bounded by a repelling invariant closed curve) and several non connected portions which are preimages of the immediate basin.

Generally speaking, if we denote by $\mathcal{B}_0$ the immediate basin, then the total basin can be expressed as

$$\mathcal{B}(A) = \bigcup_{n=0}^{\infty} T^{-n}(\mathcal{B}_0(A))$$

where $T^{-n}(x)$ represents the set of all the rank-n preimages of x, i.e. the set of points which are mapped in x after n iterations of the map T. This suggests an intuitive explanation of the mechanism that gives rise to non-connected sets or basins. If a map is noninvertible, it maps distinct points into the same point. Put differently, we can say it folds the phase plane. Equivalently, the backward iteration of a noninvertible map *repeatedly unfolds* the phase space. This implies that the basins (or feasible sets) may be non-connected, i.e. formed by several disjoint portions (see e.g. Mira et al., 1996; Abraham, Gardini, and Mira, 1997; Bischi, Gardini, and Kopel, 2000; Bischi and Kopel, 2001; Agliari, Bishi, and Gardini, 2002). Note that such a complexity of the basin (multiply connected or non-connected) can only arise in discrete dynamical systems generated by the iteration of a noninvertible map.[15] Structures of the feasible set like those shown in figure 4(c) have important consequences for practical considerations. In such a situation, we have distinct sets of initial conditions for the fish stock and its predicted value which lead to a sustainable fish stock in the long run. They are embedded, however, in the set of combinations which lead to extinction of the fish population (in finite or infinite time). This contrasts sharply with the cases described by Mäler (2000), Sethi and Somanathan (1996), Deissenberg et al. (2003) or others, since in their models only one critical threshold value between the "good" and the "bad" outcome exists. If basins or feasible sets have a complicated structure like in figure 4(c), there are several such threshold values which relate to each other in a non-systematic fashion. To strengthen our argument, we show one final scenario in figure 5, where $a_1 = a_2 = 5.5$, $b_1 = b_2 = 1$, $\lambda = 0.3$,

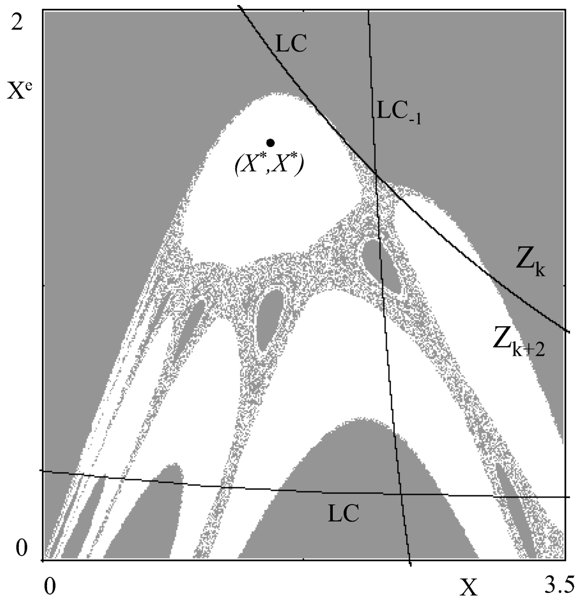

Figure 5. The non-cooperative case. The values of the parameters are $a_1 = a_2 = 5.5$, $b_1 = b_2 = 1$, $\lambda = 0.3$, $\gamma_1 = \gamma_2 = 2.5735$. The critical curves LC_{-1} and LC are shown.

$\gamma_1 = \gamma_2 = 2.5735$. In contrast to the cases discussed before, now even a very small difference in the value of either the fish stock or its prediction can have serious consequences with respect to the long run fate of the fish population. This kind of complexity of the feasible set is particularly relevant in fishery economics, because very small differences in initial conditions (or slight displacements due to noise) can lead to vastly different outcomes, namely conservation or extinction of the species. Authors have used the term *final-state sensitivity* to describe such situations (see Grebogi et al., 1983; Brock and Hommes, 1997).

5.2. *The full cooperation case*

For the sake of comparison, we briefly discuss the full cooperation case and compare the results with the observations of the previous subsection. We first consider the same parameter values as in figure 2(c) and examine the extent of the feasible set of the stable positive equilibrium in the full cooperation case (figure 6(a)). The feasible set is much larger than in the non-cooperative case. Moreover, $X_2^{*V} = 2.03$ and, since $H'(X_2^{*V}) = 0.175$ and $\alpha - 2\beta X_2^{*V} = -1.07$, Proposition 4.1 ensures that the positive equilibrium is stable for each $\lambda = [0, 1]$. Figure 6(b) is obtained with the same parameters as those used in figure 5. Also in this case, the equilibrium fish stock is larger, namely $X^{*V} = 1.71 > X^* = 1.51$. The most evident difference however, is that the feasible set is

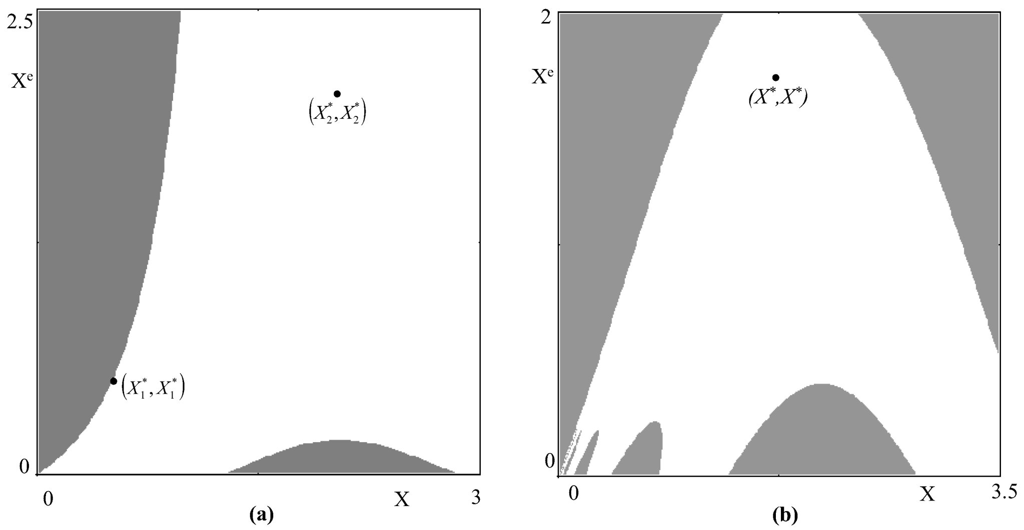

Figure 6. The full-cooperation case. The white region represents the feasible set of the equilibrium (X_2^*, X_2^*). The region in grey represents points leading to extinction of the fish population. (a) the same set of parameters as in figure 2(c). (b) the same parameters as in figure 5.

larger and any complexity of its structure is lost. Also in this case, since $H'(X^{*V}) = 0.77$ and $\alpha - 2\beta X_2^{*V} = -0.42$, Proposition 4.1 ensures that the positive equilibrium is stable for each $\lambda = [0, 1]$. These two examples illustrate that cooperation among players leads to a considerable enlargement of the feasible set. Given initial players' forecasts of the fish stock and actual fish stock levels, the likelihood of obtaining a feasible trajectory is higher if players behave cooperatively.

6. Conclusions and discussion

In this paper we have introduced a game-theoretic model, where a fish population is subject to the harvesting activities of n fishermen who sell their catch on m different markets. In contrast to the existing literature, we have assumed that fishermen do not know the biological growth function of the resource. Instead they use an adaptive forecasting scheme to update their estimates of the fish stock. We have considered the cases of full cooperation and non-cooperation and have provided existence and local stability conditions. Furthermore, we have studied the global dynamics of our model. For the non-cooperative case we have illustrated that feasible sets can be non-connected and multiply connected, and that this complexity vanishes if players cooperate. We have also observed that for the same set of parameters the feasible sets of the equilibrium is larger when players cooperate.

From this comparison, we can conclude that our results are in line with the existing literature on the role of cooperation in fishery economics; see Clarke (1990), Levhari and Mirman (1982), and Szidarovszky and Okuguchi (2000). There it is shown that cooperation among the players lead to a higher long run fish stock and, as far as the stability extent of the positive stable equilibrium is concerned, cooperation leads to more stability. In this paper we have shown that these results carry over to a two-dimensional framework. On the other hand, our simulations also show that the structure of the sets which lead to a sustainable fish population can be quite complex, a property which has not been observed before in fishery economics, but seems to be particularly relevant for it.

To conclude our paper, we would like to discuss the issue of measuring "stability" in higher-dimensional models, which is an important topic and not only relevant in the context of fishery economics. Put differently, what do we mean when we talk about the "extent of the feasible sets or the basins of attraction of a stable equilibrium"? Comparing the figures obtained for the non-cooperative and the full cooperation case (e.g. figures 2(c) and 6(a)), one can observe that the stable equilibrium in the former case is closer to the boundary of the feasible set than in the latter. Moreover, in figures 2(d), 3(a), 4(a)—although the feasible set is large and simply connected—the equilibrium is nevertheless quite close to its boundary. Such a situation might not be considered as being stable despite a large feasible set, since displacements of the trajectory—which might be due to small perturbations or small errors in the prediction of the fish stock—might result in extinction of the fish population. Small mistakes can cause large differences in outcomes. The situation is different for the same sets of parameters for the full cooperation case, where the stable equilibrium is further away from the boundary. Accordingly, in addition to having a larger set of initial conditions which lead to a sustainable long run steady state, we would consider such a situation as being more stable in the sense that it is more robust to exogenous influences or prediction errors (for a more rigorous treatment of these ideas, see McDonald et al., 1985). This certainly is a topic for future research.

Appendix 1: Positivity of profits and quantities

In this appendix we provide conditions such that the profits, the supplied and the harvested quantities of the players are positive.

The non-cooperative case

- *Player k's harvest*
  From (13) we get using (14)

$$h_k = \frac{A - H}{1 + 2B\gamma_k/X^e} = \frac{A}{(1 + 2B\gamma_k/X^e)(1 + f(X^e))}$$

which is always positive.

- *Total quantity supplied to the i-th market*
  By adding the individual quantities in (12) for all players k, we get

$$s_i = \frac{na_i}{b_i} - ns_i - \frac{2}{b_i X^e} \sum_{k=1}^{n} \gamma_k h_k$$

Therefore,

$$s_i = \frac{1}{(n+1)b_i} \left(na_i - \frac{2}{X^e} \sum_{k=1}^{n} \gamma_k h_k \right)$$

For $X^e \to \infty$, we have $f(X^e) \to n$ and $h_k \to A/(n+1)$. Hence, the expression in the parenthesis tends to na_i. Consequently, if X^e is sufficiently large, then $s_i > 0$. Furthermore, if $a_i \equiv a$ for all i, then

$$\begin{aligned} \frac{2}{X^e} \sum_{k=1}^{n} \gamma_k h_k &= \frac{2}{X^e} \sum_{k=1}^{n} \frac{\gamma_k A}{(1 + 2B\gamma_k / X^e)(1 + f(X^e))} \\ &< \frac{2}{X^e} \sum_{k=1}^{n} \frac{\gamma_k A}{2B\gamma_k / X^e} = \frac{nA}{B} = na \end{aligned}$$

since $A = aB$. This gives a sufficient condition for $s_i > 0$.

- *Individual quantities of players*
  In addition to $a_i \equiv a$ for all i, we now assume that players use the same fishing technology, so that $\gamma_k = \gamma$ for all k. Note that due to the latter assumption, the game becomes symmetric, since all n players face the same costs and supply the same m markets. Therefore, we have $x_{ki} = x_i$ (all the players offer the same quantity on market i) and $h_k = h = H/n$. Then from (12) we get

$$\begin{aligned} x_{ki} = x_i &= \frac{a}{b_i} - s_i - \frac{2}{b_i X^e} \gamma h = \\ &= \frac{a}{b_i} - \frac{1}{(n+1)b_i} \left(na - \frac{2}{X^e} n \gamma h \right) - \frac{2}{b_i X^e} \gamma h \\ &= \frac{1}{(n+1)b_i} \left(a - \frac{2}{X^e} \gamma h \right) \end{aligned}$$

where we used the expression for s_i from above. Since

$$\frac{2}{X^e} \gamma h = \frac{2\gamma A X^e}{X^e((n+1)X^e + 2B\gamma)} < \frac{A}{B} = a$$

it follows that $x_{ki} = x_i > 0$.

- *Player k's profit*
  Assume that $x_{ki} \geq 0$ for all k and i. Then for $c_k = 0$ we have

$$\begin{aligned}
\pi_k^e &= \sum_{i=1}^{m}(a_i - b_i s_i)x_{ki} - \gamma_k \frac{h_k^2}{X^e} \\
&= \sum_{i=1}^{m}\left[a_i - \frac{1}{(n+1)}\left(na_i - \frac{2}{X^e}\sum_{l=1}^{n}\gamma_l h_l\right)\right]x_{ki} - \gamma_k \frac{h_k^2}{X^e} \\
&= \sum_{i=1}^{m}\frac{1}{(n+1)}\left[a_i + \frac{2}{X^e}\sum_{l=1}^{n}\gamma_l h_l\right)\Bigg]x_{ki} - \gamma_k \frac{h_k^2}{X^e} \\
&> \frac{2h_k}{(n+1)X^e}\sum_{l=1}^{n}\gamma_l h_l - \gamma_k \frac{h_k^2}{X^e} \\
&= \frac{h_k}{(n+1)X^e}\left(2\sum_{l=1}^{n}\gamma_l h_l - (n+1)\gamma_k h_k\right)
\end{aligned}$$

  The last expression is clearly positive in the symmetric case, $\gamma_k = \gamma$ for all k. If the cost values γ_k of the players are very close to each other, then the player's harvests h_k are also close to each other. Hence, the expression in the last line above remains positive, since we have $2n$ positive terms and only $(n+1)$ negative terms.

The full cooperation case

- *Player k's harvest*
  From (18) and (19), we get

$$h_k = \frac{(A - 2H^V)X^e}{2B\gamma_k} = \frac{AX^e}{2\gamma_k(CX^e + B)}$$

  which is clearly positive.

- *Total quantity supplied to the i-th market*
  From (17), we obtain

$$s_i = \frac{1}{2b_i}\left(a_i - \frac{2\gamma_k h_k}{X^e}\right) = \frac{1}{2b_i}\left(a_i - \frac{A}{CX^e + B}\right)$$

  which is positive if and only if $a_i > \frac{A}{CX^e+B}$. If $a_i B \geq A$ or X^e is sufficiently large, then $s_i > 0$. If $a_i \equiv a$ for all i, then $s_i > 0$.

- *Individual quantities of players*
  Recall that

$$\sum_{k=1}^{n} x_{ki} = s_i \qquad i = 1, 2, \ldots, m$$
$$\sum_{i=1}^{m} x_{ki} = h_k \qquad k = 1, 2, \ldots, n$$

  and $\sum_{i=1}^{m} s_i = \sum_{k=1}^{n} h_k = H$; see (4). Therefore, from the theory of linear programming (the transportation problem) we know that there is always a set of positive solutions x_{ki} if $s_i > 0$ and $h_k > 0$.

- *Player k's profit*
  Assume that $x_{ki} \geq 0$ for all k and i. Then for $c_k = 0$ we obtain by using the expressions for h_k and s_i given above

$$\begin{aligned}\pi_k^e &= \sum_{i=1}^{m}(a_i - b_i s_i)x_{ki} - \gamma_k \frac{h_k^2}{X^e} \\ &= \sum_{i=1}^{m}\left(\frac{a_i}{2} + \frac{A}{2(CX^e + B)}\right)x_{ki} - \frac{A^2 X^e}{4\gamma_k (CX^e + B)^2} \\ &> \frac{A}{2(CX^e + B)}h_k - \frac{A^2 X^e}{4\gamma_k (CX^e + B)^2} = 0.\end{aligned}$$

  Therefore, the individual profits are always positive. As a consequence, the industry profit is also positive.

Appendix 2: Noninvertible maps and critical curves

In this appendix, we give some basic definitions and a minimal vocabulary concerning noninvertible maps of the plane and the method of critical curves.[16]

Let us consider a two-dimensional map $T : (x(t), y(t)) \rightarrow (x(t+1), y(t+1))$ written in the form

$$(x(t+1), y(t+1)) = T(x(t), y(t)) \tag{25}$$

where $(x, y) \in \mathbb{R}^2$ and the two components of T are assumed to be real valued continuous functions $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$. The point $(x(t+1), y(t+1)) \in \mathbb{R}^2$ is the rank-1 image of the point $(x(t), y(t))$ under T, and $(x(t), y(t))$ is called rank-1 preimage of the point $(x(t+1), y(t+1))$. The point $(x(t+s), y(t+s)) = T^s(x(t), y(t))$, $s \in \mathbb{N}$, is called image of rank-s of the point $(x(t), y(t))$, where T^0 is identified with the identity map and $T^t(\cdot) = T(T^{t-1}(\cdot))$. A point (x, y) such that $T^s(x, y) = (x_s, y_s)$ is called rank-s preimage of (x_s, y_s).

The map T is said to be noninvertible (or "many-to-one") if distinct points $(x_a, y_a) \neq (x_b, y_b)$ exist which have the same image, $T(x_a, y_a) = T(x_b, y_b) = (x, y)$. This can be equivalently stated by saying that points (x, y) exist which have several rank-1 preimages, i.e. the inverse relation $T^{-1}(x, y)$ is multi-valued.

As the point (x, y) varies in the plane, the number of its rank-1 preimages can change, and according to the number of distinct rank-1 preimages associated with each point of $\mathbb{R}^2$, the plane can be subdivided into regions, denoted by Z_k, whose points have k distinct preimages. Generally pairs of real preimages appear or disappear as the point (x, y) crosses the boundary separating regions characterized by a different number of rank-1 preimages. Accordingly, such boundaries are generally characterized by the presence of two coincident (merging) preimages. This leads us to the definition of *critical curves*, one of the distinguishing features of noninvertible maps. The critical curve of rank-1, denoted by LC (from the French "Ligne Critique") is defined as the locus of points having two, or more, coincident rank-1 preimages. These preimages are located in a set called critical curve of rank-0, denoted by LC_{-1}. The curve LC is the two-dimensional generalization of the notion of critical value (local minimum or maximum value) of a one-dimensional map, and LC_{-1} is the generalization of the notion of critical point (local extremum point). From the definition given above it is clear that the relation $LC = T(LC_{-1})$ holds, and the points of LC_{-1} in which the map is continuously differentiable are necessarily points where the Jacobian determinant vanishes:

$$LC_{-1} \subseteq \{(x, y) \in \mathbb{R}^2 | \det DT = 0\} \tag{26}$$

In fact, as LC_{-1} is defined as the locus of coincident rank-1 preimages of the points of LC, in any neighborhood of a point of LC_{-1} there are at least two distinct points mapped by T in the same point near LC. This means that the map T is not locally invertible in the points of LC_{-1} and, if the map T is continuously differentiable, it follows that $\det DT$ necessarily vanishes along LC_{-1}. Portions of LC separate regions Z_k of the phase space characterized by a different number of $rank - 1$ preimages, for example Z_k and Z_{k+2} (this is the standard occurrence). This property is at the basis of the contact bifurcations which give rise to complex topological structures of the basins, like those formed by non connected sets or multiply connected sets. In fact, if a parameter variation causes a crossing between a basin boundary and a critical set which separates different regions Z_k so that a portion of a basin enters a region where an higher number of inverses is defined, then new components of the basin may suddenly appear at the contact.

Geometrically, the action of a noninvertible map T can be expressed by saying that it "folds and pleats" the plane, so that two or more distinct points are mapped into the same point, or, equivalently, that several inverses are defined which "unfold" the plane. So, the backward iteration of a noninvertible map *repeatedly unfolds* the phase plane, and this implies that a basin may be non-connected, i.e. formed by several disjoint portions.

The map T defined in (16) is a noninvertible map. In fact, given a point $(X(t+1), X^e(t+1))$ several distinct points may exist which are mapped into it. Put differently, several preimages can be obtained by solving (16) with respect to $(X(t), X^e(t))$, so that the inverse relation $(X(t), X^e(t)) = T^{-1}(X(t+1), X^e(t+1))$ is a multi-valued function. In fact, from

$$\begin{cases} X(t+1) = X(t)(1+\alpha-\beta X(t)) - H(X^e(t)) \\ X^e(t+1) = \lambda X(t) + (1-\lambda)X^e(t) \end{cases}$$

we obtain $X(t) = \frac{1}{\lambda}(X^e(t+1) - (1-\lambda)X^e(t))$, where $X^e(t)$ can be obtained as a function of $X(t+1)$, $X^e(t+1)$ by solving the equation

$$\frac{\beta(1-\lambda)}{\lambda^2}(X^e)^2 + \frac{1-\lambda}{\lambda}\left(\frac{2\beta}{\lambda}X^e(t+1) - (1+\alpha)\right)X^e \\ - X(t+1) + \frac{1+\alpha}{\lambda}X^e(t+1) - \frac{\beta}{\lambda}X^{e^2}(t+1) = H(X^e).$$

We are interested in the real and positive solutions, that are located at the intersections (if any) of a parabola and the increasing and concave function H. Hence, we can have no solutions or at two real and positive solutions (indeed, there are several negative solutions, but we can neglect these preimages).

As the map T in (16) is continuously differentiable, it is easy to obtain the equation of LC_{-1}, since it is included in the set of points at which the determinant of the Jacobian vanishes, i.e. det $DT(X, X^e) = 0$. For $\lambda < 1$ the locus LC_{-1} of merging preimages has equation

$$X = \frac{1}{2\beta}\left[1+\alpha+\frac{\lambda}{1-\lambda}H'(X^e)\right]. \tag{27}$$

The graph of the curve LC_{-1} is characterized by a vertical asymptote at $X = (1+\alpha)/(2\beta)$ and crosses the X axis at $X = (1+\alpha)/(2\beta) + \lambda AC/(4(1-\lambda)B\beta)$. Applying the map T to the points of LC_{-1} one gets the critical curve $LC = T(LC_{-1})$, that can be used to identify regions of the (X, X^e)-plane whose points have different number of preimages, just as the critical points of a one-dimensional map can be used to locate regions with different preimages (see e.g. the quadratic map). These critical curves separate the phase plane into regions Z_k whose points have k preimages, or, equivalently, where k distinct inverses of T are defined (see e.g. Mira et al., 1996; or Agliari, Bischi, and Gardini, 2002).

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Notes

1. For criticism and a suggestion of a different modeling approach than in our paper, see e.g. Clark and Kirkwood (1986).

2. Multiple equilibria may be due convex-concave production functions, externalities, or expectational phenomena. For a recent survey and counter-intuitive results, see Deissenberg et al. (2003).
3. If each player represents a different country, and each country has its own market, then $m = n$, and each player is assumed to supply its domestic country as well as the $(n - 1)$ foreign countries. This is the point of view assumed in Szidarovszky and Okuguchi (1998, 2000) and in Bischi and Kopel (2002).
4. See Clark (1990) and Szidarovszky and Okuguchi (1998). This type of cost function is derived from a "production function" of the Cobb-Douglas-type with fishing effort and biomass as the two inputs.
5. The dynamical behavior of this equation has been studied extensively by May (1976, 1987) and May and Oster (1976).
6. The basin of attraction of the positive attractor is bounded by the unstable fixed point $X_0^* = 0$ and its rank-1 preimage $(X_0^*)_{-1} = (1 + \alpha)/\beta$.
7. In the economics literature, an adaptive scheme of the form used here is referred to as "adaptive expectations", and it is seen as a slightly more sophisticated form of learning or adjustment than "naive" expectations for obvious reasons.
8. However, the reader should be aware that the stability properties are different. For a comparison of the stability properties of the case of perfect stock information and the 2-D model we refer to Bischi and Kopel (2002).
9. For finding the equilibrium harvest we could also formulate an optimization problem for each player. The objective function is given by (11). The (linear) constraints may be given by the non-negativity conditions of quantities, the requirement that fish stock has to be positive, or that prices have to be positive. It can be shown that the set of these optimization problems is equivalent to a quadratic programming problem with linear constraints, which can be derived by using the Kuhn-Tucker conditions for the individual optimization problems of the n players.
10. More specific results can be proven for the two cases when all players have the same harvesting costs, $\gamma_k \equiv \gamma$ for all k. Under this additional assumption the games are symmetric. Because of space restrictions we do not present the results here, but they are available upon request.
11. For definitions of terms like "immediate basin", or "holes", see e.g. Abraham, Gardini, and Mira (1997), Mira et al. (1996), or Bischi and Kopel (2001).
12. This property also holds for the one-dimensional map which gives the dynamics under perfect stock information (7). It is easy to see that this map has an unimodal graph (see Bischi and Kopel, 2002). The map is a $Z_0 - Z_2$ noninvertible map, i.e. a point of its codomain may have two preimages or no preimage.
13. The values of the parameters have been chosen for illustrative purposes. For $\alpha = 3$, $\beta = 1$, the dynamics of the unharvested fish population which evolves according to (5) would exhibit chaotic oscillations around the carrying capacity $K = 3$ (see Conrad and Clark, 1987). We wanted to check how the two-dimensional system behaves for these parameter values. We are aware, however, that the value of the intrinsic growth rate is unusually high.
14. From the definition of a rank-1 preimage, we can alternatively say that the points of this hole are mapped into H_0 after one iteration.
15. For high values of the parameter λ any complexity in the topological structure of the basins is lost. This is due to the fact that as $\lambda \to 1$ (the naive expectation case) the Jacobian determinant never vanishes, so no critical curves exist; see Appendix 2, equation (27). It is also interesting to note that under the assumption of perfect stock information, with the same parameter values as in the present scenario, no such complexities arise.
16. For a deeper treatment see Mira et al. (1996), see also Puu (2000), or Agliari, Bischi, and Gardnini (2000), for several applications of the method of critical curves to noninvertible maps arising in dynamic economic modeling.

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