

Discrete Dynamical Systems with Memory in Social Science



Gian Italo Bischi and Francesca Grassetti

Abstract This paper explores the role of memory in discrete dynamical systems applied to the social sciences. The authors extend classical evolutionary models by incorporating both finite and infinite forms of memory, representing agents' dependence on past experiences. These memory mechanisms influence the stability and complexity of social dynamics without altering the equilibrium points of the original system. Analytical results and numerical simulations reveal that memory can produce non-monotonic effects on stability, generating bifurcations and coexistence of multiple attractors. The study highlights that the inclusion of memory introduces richer and more realistic dynamics in models of human decision-making, emphasizing the need for both local and global analyses to fully understand the long-term behavior of social and economic systems.

Keywords Discrete dynamical systems · Social dynamics · Memory effects · Stability · Bifurcations · Replicator dynamics

1 Introduction

Discrete dynamic models are usually represented by iterated functions mapping the state of a system at time t into its state at time $t + 1$, e.g. from today's state into tomorrow's one. However, often the determination of the future state involves knowledge of several past states, in other words some memory of past history of the system. This may occur in the dynamic modeling of systems in physics, engineering, natural sciences (see e.g. [18, 33, 38]). However, this may be particularly important in modelling the time evolution of social and economic systems, involving human agents that learn from their past experiences (see e.g. [3, 6, 12, 29, 35]). In social sciences,

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and economics in particular, the inclusion of memory in modeling human decisions may sometimes be considered as a method to represent learning processes (see [3, 12]) or discounted averaging of the past experiences (see e.g. [7, 22, 30]). Related strands of research have also emphasized mechanisms such as sticky expectations, bounded rationality, or experience-based learning (see, for instance [13, 24]), which share with our framework the idea that past information or limited cognitive processing may shape current decisions. Beyond economics, research in the social sciences emphasizes how memory and past experiences shape collective behavior [20]. Classic studies on technology adoption and diffusion highlight the role of prior beliefs and social influence [4, 32]. In political science, the bounded rationality tradition and the concept of path dependence stress how historical experiences condition present decisions [19, 31]. Similarly, in finance, experience-based learning has been shown to affect expectations and preferences [34]. In parallel, a systematic line of research by Alonso-Sanz has investigated the role of memory in discrete dynamical systems more broadly. In his monograph [1], memory is introduced in classical nonlinear maps, highlighting how even simple deterministic dynamics are profoundly altered once past states are incorporated into the updating rule. Specific contributions include the analysis of the logistic map with memory [2], which illustrates how memory changes the bifurcation structure and the onset of chaos, and applications to Cournot games with heterogeneous players [26], where memory affects strategic interaction and market outcomes.

The inclusion of past history in the modeling of the time evolution of a real system adds nontrivial complexities, thus introducing a trade-off between the advantage of dealing with more realistic models and the drawbacks of dynamic models which are more difficult to be studied. However, it is interesting to discuss the influence of memory effects on stability and complexity both in general and in particular dynamic models.

In this paper we start from a general one-dimensional discrete dynamic model, whose time evolution is driven by human decisions which depend on some index of performance or a payoff or a fitness measure or an increase of utility function, and we introduce two different methods to include memory effects in a model without memory. The first one is based on a moving weighted average involving a finite number M of states of the past, thus giving rise to an $M + 1$ dimensional dynamical system, see e.g. [5, 10]. The other one, following the path indicated by [22], see also [9], generalizes the evolutionary model without memory by the inclusion of a weighted sum of all previously observed payoffs, with exponentially (or geometrically) distributed weights that discount past outcomes. This is obtained by a recursive method which increases the dimension of the model by only one first order difference equation with an added dynamic variable, which represents a sort of accumulated payoff over time. Both these generalizations of the evolutionary model without memory are performed in a way that does not modify the equilibrium points of the original model, and only affects their stability properties.

After the introduction of these forms of memory in the computation of the index of performance, we give some general results about the effects of memory on the existence and stability of equilibrium points. To illustrate these general results we

apply them to a particular model of binary choice based on a replicator dynamics driven by comparisons of a given payoff as index of performance. In this particular model a population of human agents can choose between two different behavioral strategies, with the possibility of switching their choices over time following an “imitate the better performing strategy”, according to a replicator dynamics in discrete time. The payoffs obtained by each agent as a consequence of the chosen option are affected by the number of agents currently making the same choice, thus giving rise to nonlinearities and feedback phenomena. Such models are particularly meaningful in a dynamic setting, where an evolutive adaptive process is introduced to mimic how agents choose between alternative strategies (see e.g. [5, 10, 27, 36, 37]). By using this simple social model we confirm the stability results proved and we stress some drawbacks and possible oversimplified conclusions based on the analytical results obtained, due to some complexities arising as a result of nonlinearities which are typical of the dynamic models used in social sciences. We also show that the problem of the effects of memory on the stability properties of equilibrium points is a quite intriguing topic. Indeed, the effect of memory reveals to be not monotonic, thus explaining why several ambiguous conclusions can be found in the literature (a comparison of the titles of Refs. [15, 17] is quite emblematic). Although our focus in this paper is on replicator dynamics, it is important to stress that the introduction of memory has also been explored in other areas of economic dynamics. Diks et al. [21], for instance, incorporate memory into an evolutionary learning framework, showing that even simple adaptive mechanisms can be substantially modified once past performance is taken into account. Cavalli and Naimzada [16] analyze a tâtonnement process with fading memory, where the auctioneer adjusts prices not only on the basis of current excess demand but also considering past imbalances. These studies confirm that memory can be embedded in very different modeling traditions, influencing long-run outcomes.

The structure of the paper is the following. In Sect. 2 we introduce a general framework for modeling the time evolution of a one-dimensional dynamical system involving human decisions. In Sect. 3 a finite memory is introduced involving M states of the past in order to decide next period strategy, and we provide some general results on the effects of memory intensity for the case $M = 1$, giving rise to an equivalent two-dimensional dynamical system. In Sect. 4 a form of infinite memory is proposed through a recursive scheme that allows, again, to reduce the model with memory to a tractable two-dimensional dynamical system. Section 5 concludes.

2 A General Discrete-Time Model for Social Systems Without Memory

Let us consider a dynamic social system whose state is represented by a single real variable $x(t)$ which is a function of time. In the dynamic modelling of a social system, time is often considered as discrete, i.e. its values are expressed by multiples of a

finite time interval Δt , because changes in the state of the system are caused by human decisions which cannot occur at any time (event-driven time). If Δt is taken as the time unit, i.e. $\Delta t = 1$, the law of motion of such a system can be represented by a first-order difference equation in the following standard form

$$x(t + 1) = f(x(t), g(x(t))) \quad (1)$$

where $x \in \mathbb{R}$ is the dynamic variable depending on agents' decisions, $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ represents the law of motion, $g : \mathbb{R} \rightarrow \mathbb{R}$ is an index of performance (utility gain, fitness, profit...) that determines (or, at least, influences) the decisions leading the state variable x from its value at time period t into the next value, at time period $t + 1$.

As an example, let us consider a social system where a population of N agents faces a binary choice between strategies R or L , and let us denote by $x(t) \in [0, 1]$ the fraction of the population choosing R at time period t , the complementary fraction $(1 - x(t))$ being the portion of population making the opposite choice L (this kind of binary games is intensively studied in the literature, see e.g. [5, 10, 27, 36, 37]). If we denote by $R(t) = R(x(t))$ the payoff obtained by an agent adopting the strategy R at time t , and by $L(t) = L(x(t))$ the payoff of an agent choosing L at the same time, the dynamics of $x(t)$ can be represented by the well known Replicator Dynamics in discrete time (see e.g. [14, 28])

$$x(t + 1) = \frac{x(t) \exp(\alpha R(x(t)))}{x(t) \exp(\alpha R(x(t))) + (1 - x(t)) \exp(\alpha L(x(t)))}$$

where $\alpha > 0$ represents a speed of reaction. This difference equation can be rewritten as

$$x(t + 1) = f(x(t), g(x(t))) = \frac{x(t)}{x(t) + (1 - x(t)) \exp(-\alpha g(x(t)))} \quad (2)$$

where $g(x) : [0, 1] \rightarrow \mathbb{R} = R(x) - L(x)$ represents the “gain” of an agent choosing R when population state is at x .

It can be noticed that $x = 0$ and $x = 1$ are always fixed points of the recurrence (2), denoted as boundary fixed points, representing limiting cases in which all agents choose the same strategy (case of a pure strategy in the whole population), whereas an interior (or mixed strategy) fixed point is given by $x^* \in (0, 1)$ such that the two payoffs are identical, $L(x^*) = R(x^*)$, i.e. $g(x^*) = 0$, as in Fig. 1, where the payoff functions $R(x)$ and $L(x)$ represent a typical minority game, i.e. the strategy R is preferred when a small fraction of the population (a minority) chooses it.

In the following, we generalize the model (1) by considering two methods to include the effects of past payoffs on the current choices, both based on a weighted average of the past outcomes. In other words, some memory is introduced. Both these generalizations are performed in a way that does not modify the equilibrium points of the model without memory, and only affects their stability properties.

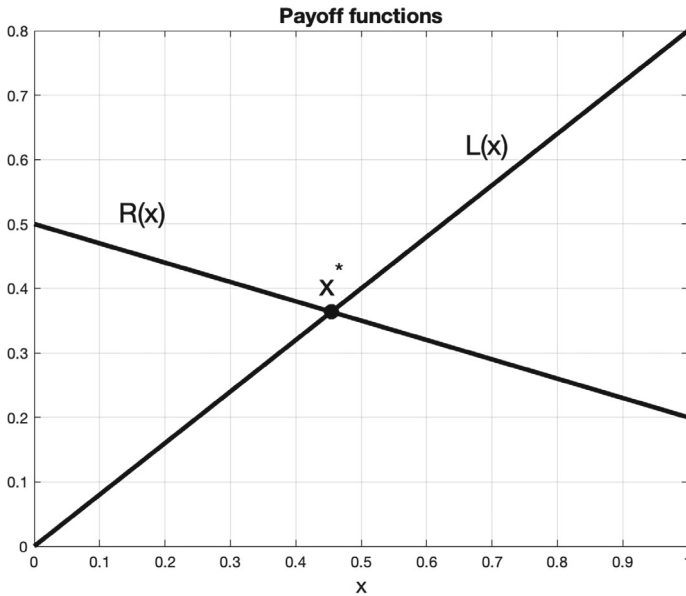


Fig. 1 Graphical identification of the interior fixed point $x^* \in (0, 1)$, where the two payoff functions $L(x)$ and $R(x)$ intersect, i.e. $g(x^*) = L(x^*) - R(x^*) = 0$

3 The Model with Finite Memory

In the model (1) the players' decisions about the next period strategy choice are only based on the knowledge of current index of performance. A generalization of this assumption consists in adding to the current index of performance a weighted average of some of the performances observed in the past (a form of memory):

$$x(t+1) = f(x(t), U(t)) \quad (3)$$

with

$$U(t) = \sum_{k=0}^M \omega_k g(x(t-k)) \quad (4)$$

where M is referred to as the *length of memory*, and the coefficients ω_k represent normalized weights, satisfying the condition $\sum_{k=0}^M \omega_k = 1$. It can be seen as an *accumulated gain*, or a moving average of the gains obtained in the current and M past periods, with $U(0) = g(x(0))$ if one assumes $g(x(t)) = 0$ for $t < 0$, according to the convention that the system dynamics start at $t = 0$. We shall denote this recursive scheme as a *finite memory of length M* . For $M = 0$ the case with no memory (1)

is obtained, whereas for $M > 0$ the weights modulate the “shape” of memory. The model (3) with (4) becomes a difference equation of order $M + 1$:

$$x(t + 1) = f(x(t), \sum_{k=0}^M \omega_k g(x(t - k))) \quad (5)$$

equivalent to a $M + 1$ dimensional discrete dynamical system.

Some properties concerning existence and stability of equilibrium points can be stated and compared with the corresponding ones in the model with no memory (1). Indeed, if x^* is a fixed point for (1) then it is also a fixed point for the model with finite memory, according to the following proposition:

Proposition 1 *Let $x^* = f(x^*, g(x^*))$ be an equilibrium point of (1), then $(x^*, \dots, x^*) \in \mathbb{R}^{M+1}$ is a fixed point of the $M + 1$ order difference equation (5) and vice-versa.*

Proof If $x(0) = x(1) = \dots = x(M) = x^*$, then $U(t) = \sum_{k=0}^M \omega_k g(x^*) = g(x^*) \sum_{k=0}^M \omega_k = g(x^*)$ for each $t \geq M$ and (3) reduces to (1) with $x(t) = x^*$ so that both are satisfied for each $t \geq M$. $\square$

A question that naturally arises is the following: if x^* is stable for the model without memory, i.e. $\left| \frac{df}{dx}(x^*) \right| < 1$, where $\frac{df}{dx}(x^*) = \frac{\partial f}{\partial x}(x^*) + \frac{\partial f}{\partial g}(x^*)g'(x^*)$, what can be said about its stability for the model (5)?

This question is not trivial in general, but can be easily addressed in the case $M = 1$, i.e. with $U(t) = (1 - \omega)g(x(t)) + \omega g(x(t - 1))$, so that the dynamical system (5) becomes two-dimensional

$$\begin{cases} x(t + 1) = f(x(t), (1 - \omega)g(x(t)) + \omega g(y(t))) \\ y(t + 1) = x(t) \end{cases} \quad (6)$$

and the memory parameter ω is such that for $\omega = 0$ the case without memory is obtained, for $\omega = \frac{1}{2}$ we have a uniform average, whereas in the other limiting case $\omega = 1$ we have $x(t + 1) = f(x(t), g(x(t - 1)))$, i.e. the computation of $x(t + 1)$ is based on the gain obtained in the previous period (ignoring the current one). The recurrence (6) has been considered in particular dynamic economic models, see e.g. [5, 10, 11]. Here we give a general statement about the local stability of a fixed point of (6) according to the following proposition:

Proposition 2 *If x^* is a fixed point of (1) then it is a local asymptotically stable fixed point for (6) provided that:*

- (i) $\frac{df}{dx}(x^*) < 1$
- (ii) $\frac{df}{dx}(x^*) > -1 + 2\omega \frac{\partial f}{\partial g}(x^*)g'(x^*)$
- (iii) $\omega \frac{\partial f}{\partial g}(x^*)g'(x^*) > -1$

Proof The Jacobian matrix of the two-dimensional discrete dynamical system (6) can be written as

$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} + (1 - \omega) \frac{\partial f}{\partial U} g'(x) & \omega \frac{\partial f}{\partial U}(x) g'(y) \\ 1 & 0 \end{bmatrix}$$

where $\frac{\partial f}{\partial U}$ indicates the partial derivative of the function f with respect to its second argument, i.e. it corresponds to $\frac{\partial f}{\partial g}$ if we refer to (1).

The Trace and Determinant of the Jacobian of (6) at the fixed point (x^*, x^*) are:

$$Tr = \frac{\partial f}{\partial x}(x^*) + (1 - \omega) \frac{\partial f}{\partial g} g'(x^*);$$

$$Det = -\omega \frac{\partial f}{\partial g}(x^*) g'(x^*).$$

So, the standard sufficient conditions for local asymptotic stability of the fixed point (see e.g. [23, 25]) become:

$$1 - Tr + Det > 0 \quad \text{iff} \quad \frac{\partial f}{\partial x}(x^*) + \frac{\partial f}{\partial g} g'(x^*) = \frac{df}{dx}(x^*) < 1$$

$$1 + Tr + Det > 0 \quad \text{iff} \quad \frac{\partial f}{\partial x}(x^*) + \frac{\partial f}{\partial g} g'(x^*) - 2\omega \frac{\partial f}{\partial g} g'(x^*) = \frac{df}{dx}(x^*) - 2\omega \frac{\partial f}{\partial g} g'(x^*) > -1$$

$$Det < 1 \quad \text{iff} \quad \omega \frac{\partial f}{\partial g}(x^*) g'(x^*) > -1. \quad \square$$

Notice that when $\omega \rightarrow 0$ the stability conditions (i), (ii), reduce to the same stability conditions of the model without memory, i.e. $-1 < \frac{df}{dx}(x^*) < 1$. Condition (iii), instead, becomes $0 > -1$, which is always satisfied and therefore redundant.

Instead, when the memory parameter $\omega > 0$, its influence on the stability properties of a fixed point of (6) can be studied by the following two corollaries of Proposition 2.

Corollary 1 Let x^* be a stable fixed point of (1), i.e. $\left| \frac{df}{dx}(x^*) \right| < 1$, and $\frac{\partial f}{\partial g}(x^*) g'(x^*) > 0$. Then the fixed point (x^*, x^*) of (refMSP1) may lose stability for increasing values of ω through a flip bifurcation.

Proof Under the given assumptions, the stability conditions (i) and (iii) of Proposition 2 are always satisfied, whereas (ii) is violated if $2\omega \frac{\partial f}{\partial g}(x^*) g'(x^*) > \frac{df}{dx}(x^*) + 1$, i.e.

$$\omega > \omega_f = \frac{\frac{df}{dx}(x^*) + 1}{2 \frac{\partial f}{\partial g}(x^*) g'(x^*)} \quad (7)$$

□

Corollary 2 Let x^* be a stable fixed point of (1), i.e. $\left| \frac{df}{dx}(x^*) \right| < 1$, and $\frac{\partial f}{\partial g}(x^*) g'(x^*) < 0$. Then the fixed point (x^*, x^*) of (6) may lose stability for increasing values of ω through a Neimark-Sacker bifurcation. Moreover, if $\frac{\partial f}{\partial g}(x^*) g'(x^*) < 0$ and x^* is an unstable fixed point of (1) with $\frac{df}{dx}(x^*) < -1$, then the fixed point (x^*, x^*) of (6) may become stable for increasing values of ω through a backward flip bifurcation.

Proof If $\left| \frac{df}{dx}(x^*) \right| < 1$, and $\frac{\partial f}{\partial g}(x^*)g'(x^*) < 0$ then the stability conditions (i) and (ii) of Proposition 2 are always satisfied, whereas (iii) is violated if

$$\omega > \omega_h = -\frac{1}{\frac{\partial f}{\partial g}(x^*)g'(x^*)} \quad (8)$$

provided that $\omega_h < 1$. Instead, if $\frac{df}{dx}(x^*) < -1$ from (ii) is not satisfied for $\omega = 0$, but it becomes true for $\omega > \omega_f > 0$, provided that $\omega_f < 1$. $\square$

Remark If $\frac{\partial f}{\partial g}(x^*)g'(x^*) < 0$ and $\frac{df}{dx}(x^*) < -1$ and $0 < \omega_f < \omega_h < 1$, the fixed point (x^*, x^*) of (6) is unstable for $\omega = 0$ (i.e. the model without memory), it gains stability for increasing values of the memory parameter ω through a backward flip bifurcation, and if ω is further increased it loses stability again through a Neimark-Sacker bifurcation. This dynamic scenario shows a non monotonic effect of memory.

For example, in the case of the model (2), we have

$$f(x, g(x)) = \frac{x}{x + (1-x)\exp(-\alpha g(x))},$$

hence $\frac{\partial f}{\partial x} = \frac{\exp(-\alpha g(x))}{(x+(1-x)\exp(-\alpha g(x)))^2}$ and $\frac{\partial f}{\partial g} = \frac{x(1-x)\alpha \exp(-\alpha g(x))}{(x+(1-x)\exp(-\alpha g(x)))^2}$. So, in an interior fixed point x^* , characterized by $g(x^*) = 0$, we have $\frac{\partial f}{\partial x}(x^*) = 1$ and $\frac{\partial f}{\partial g}(x^*) = \alpha x^*(1-x^*)$. For an interior fixed point (x^*, x^*) of the corresponding model with finite memory of order $M = 1$, equivalent to the 2-dimensional dynamical system

$$\begin{cases} x(t+1) = \frac{x(t)}{x(t)+(1-x(t))e^{-\alpha((1-\omega)g(x(t))+\omega g(y(t)))}} \\ y(t+1) = x(t) \end{cases} \quad (9)$$

the conditions for local asymptotic stability given in Proposition 2 become

- (i) $\alpha x^*(1-x^*)g'(x^*) < 0$
- (ii) $(1-2\omega)\alpha x^*(1-x^*)g'(x^*) > -2$
- (iii) $\alpha\omega x^*(1-x^*)g'(x^*) > -1$

If $g'(x^*) > 0$ then (x^*, x^*) is unstable, whereas if $g'(x^*) < 0$ then (x^*, x^*) is asymptotically stable if $\omega_f < \omega < \omega_h$, with

$$\omega_f = \frac{1}{2} + \frac{1}{\alpha x^*(1-x^*)g'(x^*)}; \quad \omega_h = -\frac{1}{\alpha x^*(1-x^*)g'(x^*)}$$

and if ω exits the stability interval $[\omega_f, \omega_h]$ decreasing through ω_f then P loses stability through a flip bifurcation, if increasing through ω_h then it loses stability through a Neimark-Sacker bifurcation.

For example, consider the case where $R(x) = -0.9x + 0.9$ and $L(x) = 0.7x + 0.2$, so that $g(x) = R(x) - L(x) = -1.6x + 0.7$, and $\alpha = 8$. Then, for $\omega = 0$, i.e.,

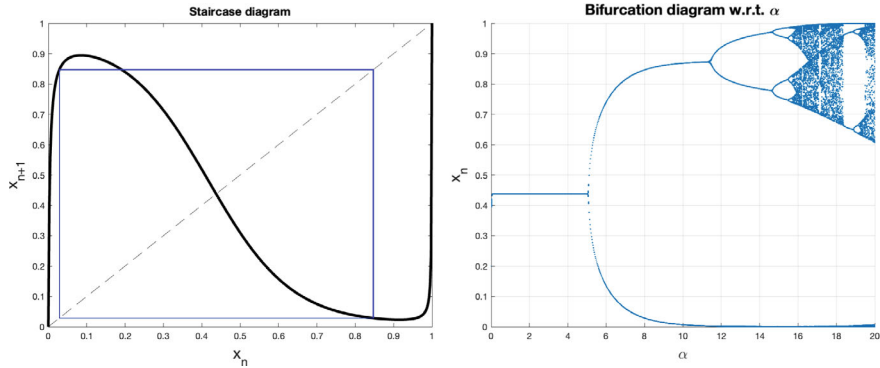


Fig. 2 Left: Graph of the iterated function $f(x, g(x))$ with a stable period-2 cycle for the memoryless case $\omega = 0$ with parameters $a = -0.9$, $b = 0.9$, $c = 0.7$, $d = 0.2$, and $\alpha = 8$. Right: Bifurcation diagram with respect to α for the same parameter set, revealing successive period-doubling bifurcations and the presence of chaotic windows

the standard replicator model without memory, the dynamics exhibit a stable period-2 cycle, as shown in the left panel of Fig. 2. This already departs from convergence to a fixed point, indicating the onset of nontrivial oscillatory behavior. In the right panel of the same figure, we present the bifurcation diagram of the system with respect to the parameter α . The diagram reveals a sequence of period-doubling (flip) bifurcations as α increases, leading to the emergence of chaotic windows and complex dynamical regimes. These transitions exemplify the rich structure of the system even in the absence of memory effects and underscore the sensitivity of its long-term behavior to the strength of selection.

For the same parameter set of previous figure, we show some numerical explorations in the presence of memory, i.e. for $\omega > 0$. Figure 3 confirms the flip bifurcation leading to stability of the fixed point for increasing values of the memory parameter ω , and the Neimark-Sacker bifurcation causing a stability loss again for larger values of ω , as stated in the Corollaries given above.

However, local analysis is not enough: the global dynamics of the system reveal the coexistence of multiple attractors (as shown in Fig. 4) and intricate basin structures (see Fig. 5), which cannot be captured by linearization around fixed points. Numerical simulations show that depending on the value of the memory parameter ω , the system exhibits a rich variety of dynamical behaviors. In the left panel of Fig. 5, for $\omega = 0.28$ (and the same parameter value as in Fig. 4) we observe the coexistence of a fixed point with a 3-period cycle. In the right panel, for $\omega = 0.32$ (and baseline parameter values), a stable period-3 cycle coexists with a quasi-periodic attractor, located around the unstable fixed point, generated by the Neimark-Sacker bifurcation. The corresponding basins of attraction are quite intertwined and cause strong path dependence. In order to further explore the complexity arising from the interplay between memory and nonlinear dynamics, we focus on a narrower interval of the memory parameter, namely $\omega \in [0.25, 0.35]$. Figure 4 shows two bifurcation

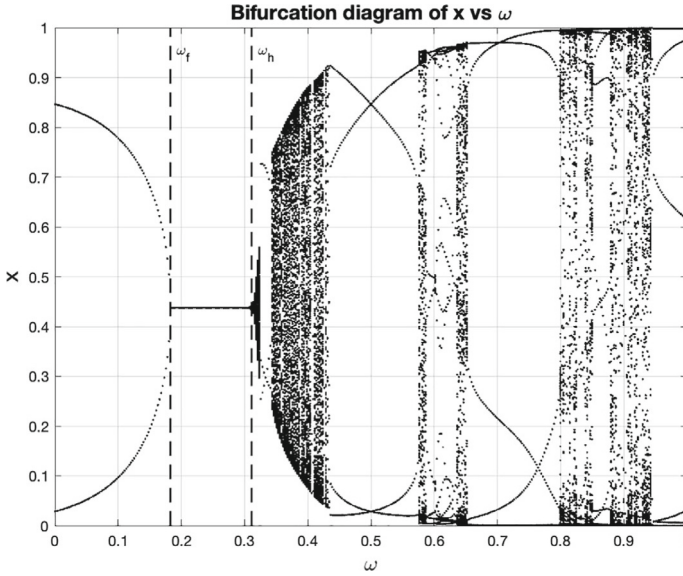


Fig. 3 Bifurcation diagram of x with respect to the memory weight parameter ω in the model (9), obtained by iterating the system and plotting the last 200 values of $x(t)$ after discarding the transient. The vertical dashed lines indicate the theoretical thresholds ω_f and ω_h , where flip and Neimark-Sacker bifurcations occur, respectively. Between the two vertical dashed lines the equilibrium is locally asymptotically stable

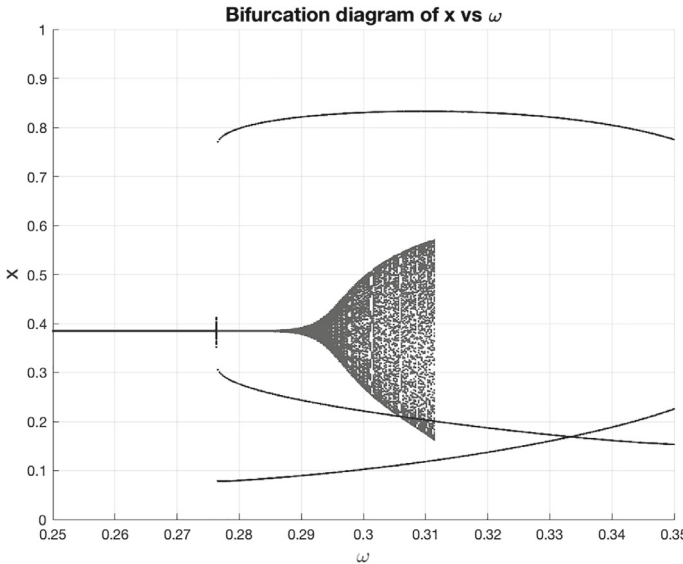


Fig. 4 Bifurcation diagrams for x versus ω for the replicator model with finite memory in the interval $\omega \in [0.25, 0.35]$, for different initial conditions. The system parameters are $a = -0.5$, $b = 0.5$, $c = 0.8$, $d = 0.0$, and $\alpha = 11$

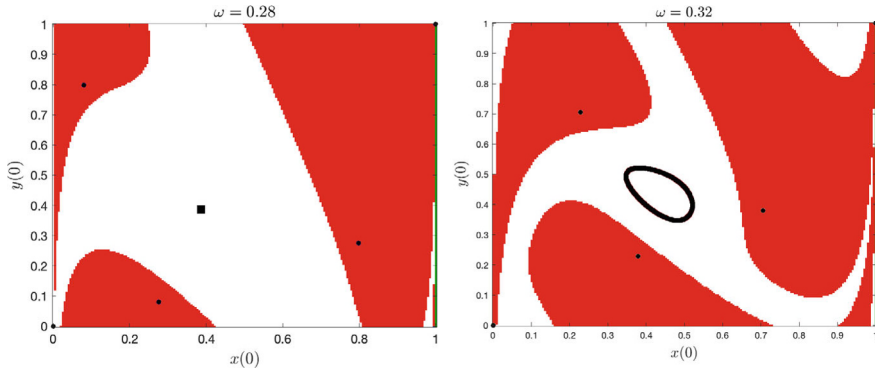


Fig. 5 Basins of attraction for different values of the memory parameter ω . Left: for $\omega = 0.28$, a period-3 cycle coexists with a fixed point, each with its own basins of attraction. Right: for $\omega = 0.32$, the period-3 cycle (black points in the red region) coexists with an attractive limit cycle, each with its own basin of attraction (red and white respectively)

diagrams obtained by initializing the system from two different initial conditions. The diagrams reveal a particularly subtle dynamical scenario, in which different attractors coexist. In the lower part of the bifurcation window, a stable period-3 cycle coexists with a stable fixed point. This case is particularly deceptive from a dynamical perspective: the apparent convergence to a fixed point might mask the presence of a different coexisting attractor (periodic or chaotic), depending on the initial state. As ω increases, the fixed point loses stability, and the system displays the coexistence of the period-3 cycle with a higher-period or possibly chaotic attractor. These results confirm that local stability analysis alone is insufficient to fully characterize the model's global behavior. Coexisting attractors and intricate basins of attraction imply strong path dependence and the presence of regions in parameter space where very different dynamical outcomes are possible.

4 The Model with Infinite Fading Memory

Following [22], see also [5, 8, 10], we introduce a way to model players' memory by considering a measure of performance that represents a discounted sum of the performances observed along the whole story of the system. This is obtained by considering again (3) with an index of performance $U(t)$ obtained recursively at each time step as a convex combination:

$$U(t) = (1 - \omega) g(x(t)) + \omega U(t - 1) \quad (10)$$

where $\omega \in [0, 1]$ and $U(0) = g(x(0))$. Again, the parameter $\omega \in [0, 1]$ gives a measure of the memory, as $U(t) = g(x(t))$ for $\omega = 0$ (no-memory case), and, by backward induction reasoning, we get

$$U(t) = (1 - \omega) \sum_{k=0}^{t-1} \omega^k g(x(t-k)) + \omega^t U(0) \quad (11)$$

i.e. $U(t)$ is a discounted average (with exponentially fading weights) of all the past performances along the whole story of the dynamical system as, at any time step the sum of the weights is normalized to 1:

$$(1 - \omega) \sum_{k=0}^{t-1} \omega^k + \omega^t = (1 - \omega) \frac{1 - \omega^t}{1 - \omega} + \omega^t = 1$$

If the recursive scheme (10) is plugged into the dynamic model (3) then a two-dimensional dynamical is obtained

$$\begin{cases} x(t+1) = f(x(t), U(t)) \\ U(t+1) = (1 - \omega) g(x(t+1)) + \omega U(t) \end{cases} \quad (12)$$

where $g(x(t+1)) = g(f(x(t), U(t)))$. The equilibrium points of the model with infinite memory (12) are the same as for the model with no memory (1), according to the following proposition:

Proposition 3 *If $x^* = f(x^*, g(x^*))$ is a fixed point of the model without memory (1), then $(x^*, g(x^*))$ is a fixed point of the map (12) and vice-versa.*

Proof Replacing x with x^* and U with $g(x^*)$ in (12) an identity is obtained. $\square$

Hence the questions of a comparison of the stability conditions for the model without memory with the ones of the same equilibrium under (12) naturally arises, as stated in the following proposition:

Proposition 4 *Let x^* be a fixed point of (1). Then $(x^*, g(x^*))$ is a locally asymptotically stable fixed point of (12) provided that the following conditions hold:*

- (i) $(1 - \omega) \frac{df}{dx}(x^*) + \omega < 1$
- (ii) $\frac{df}{dx}(x^*) + \omega \left(1 + \frac{\partial f}{\partial x}(x^*) - \frac{\partial f}{\partial g}(x^*) g'(x^*) \right) > -1$
- (iii) $\omega \frac{\partial f}{\partial x}(x^*) < 1$

where $\frac{df}{dx}(x^*) = \frac{\partial f}{\partial x}(x^*) + \frac{\partial f}{\partial g}(x^*) g'(x^*)$.

Proof The Jacobian matrix of the two-dimensional discrete dynamical system (12) can be written as

$$J(x, U) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial U} \\ (1 - \omega)g'(x)\frac{\partial f}{\partial x} & (1 - \omega)g'(x)\frac{\partial f}{\partial U} + \omega \end{bmatrix}$$

where $\frac{\partial f}{\partial U}$ indicates the partial derivative of the function f with respect to its second argument, i.e. it corresponds to $\frac{\partial f}{\partial g}$ if we refer to (1).

The Trace and Determinant of the Jacobian of (6) at the fixed point (x^*, x^*) are:

$$Tr = \frac{\partial f}{\partial x}(x^*) + (1 - \omega)\frac{\partial f}{\partial g}g'(x^*) + \omega;$$

$$Det = \omega\frac{\partial f}{\partial x}(x^*).$$

The standard sufficient conditions for local asymptotic stability of the fixed point (see e.g. [23, 25]) become:

$$1 - Tr + Det > 0 \text{ iff } (1 - \omega)\frac{df}{dx}(x^*) + \omega < 1$$

$$1 + Tr + Det > 0 \text{ iff } (1 - \omega)\frac{\partial f}{\partial g}g'(x^*) + (1 + \omega)\frac{\partial f}{\partial x}(x^*) + \omega > -1$$

$$Det < 1 \quad \text{iff } \omega\frac{\partial f}{\partial x}(x^*) < 1. \quad \square$$

Notice that when $\omega \rightarrow 0$ the stability conditions (i), (ii), (iii) reduce to the same stability conditions of the model without memory, i.e. $-1 < \frac{df}{dx}(x^*) < 1$.

Instead, when the memory parameter $\omega > 0$, its influence on the stability properties of a fixed point can be studied by the following corollary of Proposition 4.

Corollary 3 *The stability condition (i) is always satisfied when $\frac{df}{dx}(x^*) \geq 0$, whereas for $\frac{df}{dx}(x^*) < 0$ it is satisfied for $\omega < \omega_s$, where $\omega_s = \frac{\frac{df}{dx}(x^*)}{\frac{df}{dx}(x^*) - 1} \in (0, 1)$. The stability condition (iii) is always satisfied for $\frac{\partial f}{\partial x}(x^*) \leq 1$ whereas when $\frac{\partial f}{\partial x}(x^*) > 1$ the fixed point is stable for $\omega < \omega_n = \frac{1}{\frac{\partial f}{\partial x}(x^*)} \in (0, 1)$. Instead the role of the memory parameter ω in the condition (ii) is more difficult to be analyzed in general. However, the following considerations can be made:*

- *Condition (ii) is always satisfied if any of the following holds:*
 $\frac{\partial f}{\partial x}(x^*) > 0$ and $\frac{\partial f}{\partial g}(x^*)g'(x^*) > 0$,
 or $\frac{\partial f}{\partial x}(x^*) > -1 - \frac{\partial f}{\partial g}(x^*)g'(x^*) > 0$ and $\frac{\partial f}{\partial g}(x^*)g'(x^*) < 0$,
 or $\frac{\partial f}{\partial g}(x^*)g'(x^*) - 1 < \frac{\partial f}{\partial x}(x^*) < 0$ and $\frac{\partial f}{\partial g}(x^*)g'(x^*) > 0$,
 or $-1 - \frac{\partial f}{\partial g}(x^*)g'(x^*) < \frac{\partial f}{\partial x}(x^*) < 0$ and $\frac{\partial f}{\partial g}(x^*)g'(x^*) < 0$.
- *Condition (ii) is never satisfied if any of the following holds:*
 $\frac{\partial f}{\partial x}(x^*) < -\frac{\partial f}{\partial g}(x^*)g'(x^*) - 1 < 0$ and $\frac{\partial f}{\partial g}(x^*)g'(x^*) > 0$,
 or $\frac{\partial f}{\partial x}(x^*) < \frac{\partial f}{\partial g}(x^*)g'(x^*) - 1 < 0$ and $\frac{\partial f}{\partial g}(x^*)g'(x^*) < 0$.
- *In all other cases, the validity of condition (ii) depends on the value of ω . In particular, condition (ii) holds if one of the following is true:*
 $0 < \frac{\partial f}{\partial x}(x^*) < -1 - \frac{\partial f}{\partial g}(x^*)g'(x^*)$, $\frac{\partial f}{\partial g}(x^*)g'(x^*) < 0$ and $\omega > \omega_b$
 or $-(\frac{\partial f}{\partial g}(x^*)g'(x^*) + 1) < \frac{\partial f}{\partial x}(x^*) < \frac{\partial f}{\partial g}(x^*)g'(x^*) - 1 < 0$, $\frac{\partial f}{\partial g}(x^*)g'(x^*) > 0$
 and $\omega < \omega_b$

$$\text{or } \frac{\partial f}{\partial g}(x^*)g'(x^*) - 1 < \frac{\partial f}{\partial x}(x^*) < -(1 + \frac{\partial f}{\partial g}(x^*)g'(x^*)) < 0, \quad \frac{\partial f}{\partial g}(x^*)g'(x^*) < 0$$

and $\omega > \omega_b$

$$\text{where } \omega_b = -\frac{1 + \frac{\partial f}{\partial x}(x^*) + \frac{\partial f}{\partial g}(x^*)g'(x^*)}{1 + \frac{\partial f}{\partial x}(x^*) - \frac{\partial f}{\partial g}(x^*)g'(x^*)}.$$

For the evolutionary model (2) with infinite memory, equivalent to the two-dimensional dynamical system

$$\begin{cases} x(t+1) = \frac{x(t)}{x(t) + (1-x(t)) \exp(-\alpha U(t))} \\ U(t+1) = (1-\omega)g(x(t+1)) + \omega U(t) \end{cases} \quad (13)$$

the conditions for the local asymptotic stability of an interior fixed point become

- (i) $(1-\omega)\alpha x^*(1-x^*)g'(x^*) < 0$
- (ii) $(1-\omega)(1+\alpha x^*(1-x^*)g'(x^*)) > -1$
- (iii) $\omega < 1$

from which it is evident that (iii) is always true, hence Neimark-Sacker bifurcation cannot occur, whereas (i) is always satisfied when $g'(x^*) < 0$ and it is never satisfied for $g'(x^*) > 0$. So, in the case $g'(x^*) < 0$ the stability of the fixed point depends on (ii), that can be rewritten as

$$\frac{\omega - 2}{1 - \omega} < \alpha x^*(1 - x^*)g'(x^*)$$

As $\omega \in [0, 1]$, this stability condition is always satisfied for $\alpha < -\frac{2}{x^*(1-x^*)g'(x^*)}$ whereas for $\alpha > -\frac{2}{x^*(1-x^*)g'(x^*)}$ (i.e. a sufficiently fast replicator dynamics) the fixed point is stable provided that $\omega > \omega_b$, with $\omega_b = \frac{\alpha x^*(1-x^*)g'(x^*) + 2}{\alpha x^*(1-x^*)g'(x^*) - 2}$, i.e. for a sufficiently high level of memory. Thus, in this case memory has a stabilizing effect. Notice also that ω_b is an increasing function of α with $\lim_{\alpha \rightarrow \infty} \omega_b = 1$.

For the set of parameters $a = -0.9$, $b = 0.9$, $c = 0.7$, $d = 0.2$, we show some numerical explorations of system (13). Since $g'(x^*) < 0$, condition (i) for local asymptotic stability is satisfied, implying that the fixed point might be locally attracting.

Figure 6 confirms the presence of a flip bifurcation, which leads to the stabilization of the fixed point as the memory parameter ω increases, in agreement with the analytical threshold ω_f derived above. In the left panel, the bifurcation diagram with respect to ω displays a stable period-2 orbit for small memory values ($\omega < \omega_f$) that becomes a stable fixed point once $\omega > \omega_f$. In the right panel, we fix $\alpha = 20$ and explore the asymptotic dynamics as a function of ω . The diagram reveals a much richer behavior: successive period-doubling bifurcations and a transition to chaos, suggesting the existence of a classical *route to chaos* for decreasing values of the memory parameter ω . These dynamics underline the complex interplay between adaptation speed and memory in the system.

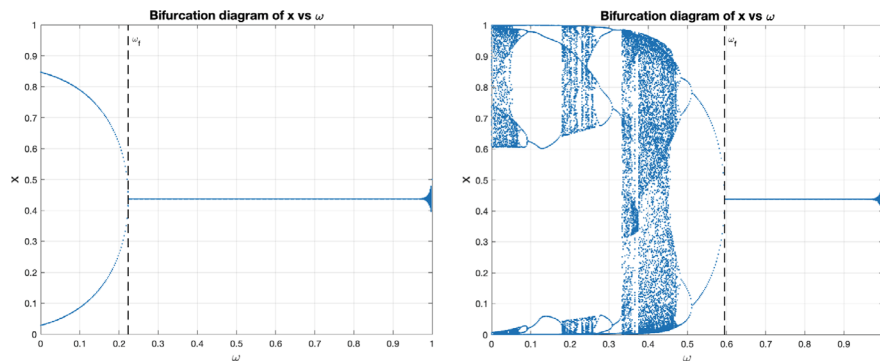


Fig. 6 Bifurcation diagrams for system (13) with $a = -0.9$, $b = 0.9$, $c = 0.7$ and $d = 0.2$. Left: For $\alpha = 8$ a stable period-2 orbit for low memory transits to a fixed point beyond the flip bifurcation threshold ω_b . Right: For $\alpha = 20$, period-doubling bifurcations and chaos appear as ω decreases

However, local analysis is not sufficient to capture the full complexity of the system's global dynamics. The coexistence of multiple attractors and the intricate geometry of their corresponding basins of attraction can lead to highly nontrivial outcomes, as illustrated in Fig. 7: we observe the coexistence of two periodic attractors, a period-3 cycle and a period-8 cycle. Although both attractors cover similar

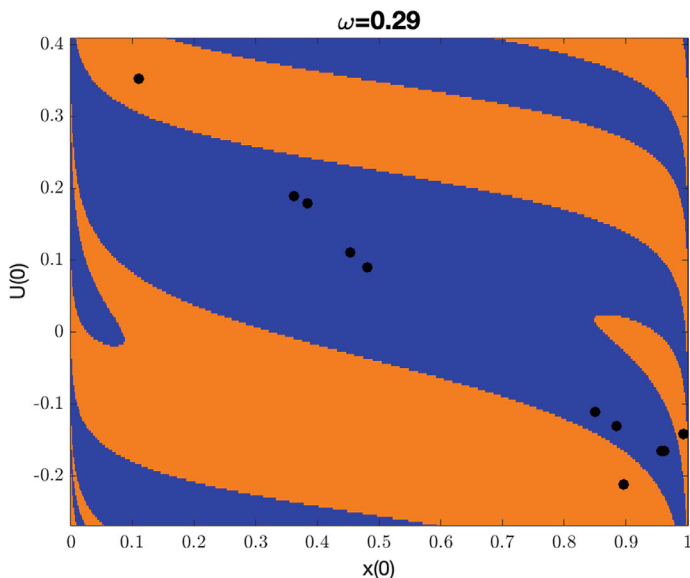


Fig. 7 Basins of attraction for the infinite-memory replicator dynamics for $\omega = 0.29$: two periodic attractors coexist—a period-8 cycle and a period-3 cycle—with interwoven basins. Other parameter values: $a = -0.35$, $b = 0.9$, $c = 0.7$, $d = 0.2$, $\alpha = 20$

regions of the phase plane and their basins are strongly interwoven, they remain distinct in terms of their dynamics. This fragmentation of the basin implies that slight variations in initial conditions can drastically alter the system's long-term behavior, emphasizing the path dependence of the dynamics.

5 Conclusions

In this paper we have considered a class of unidimensional discrete-time dynamic models, often used to describe social and economic systems characterized by event-driven time, as agents decide their strategies at given time periods by using past information. If the information used to reach a decision involves several states of the past, then we say that the dynamics involves memory.

The effects of memory on stability remain a rather controversial research topic (see e.g. [9, 15, 17, 30]). By using two tractable low-dimensional discrete dynamical system to represent models with memory effects, named finite and infinite respectively, we have given some general analytical results concerning the local asymptotic stability of the equilibrium points of the models with memory and we have compared them with the analogous results of the model without memory. The effects of the memory parameter on the long run outcomes of these two ways of considering the influence of the past states suggest a stabilization of the equilibrium when a uniform memory is considered. Moreover, non monotonic effects of memory have been stressed, as sometimes increasing memory leads to instability and in other cases memory reinforces stability. This result has consequences which go beyond the specific model we were considering. In fact, it shows once more that when considering nonlinear systems both local and global analysis are necessary to have better understanding of the model. Indeed, by some numerical simulations we have shown that a local analysis is not sufficient to get a meaningful comparison of the stability properties of the one-dimensional model with no memory with those of the two-dimensional models, with finite memory of length 1 and the one with infinite memory respectively. Indeed, such a comparison should be based on several criteria, such as the region of local stability in the space of parameters, the speed of convergence and the extension and shape of the basins. So, studies limited to local stability are not sufficient to give correct information, and sometimes it may even be misleading. Our numerical simulations provide further insight into nonlinear phenomena and the related effects of the presence of memory.

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