Piecewise linear

discontinuous 1D maps

(A) increasing -increasing case

(B) increasing -decreasing case

(B1) offsets of the same sign

(B2) offsets of opposite sign

$$x' = f(x) = \begin{cases} f_L(x) = a_L x + \mu_L \quad , & \text{if } \ x < 0 \\ f_R(x) = a_R x + \mu_R \quad , & \text{if } \ x > 0 \end{cases}$$

Theory: L. Gardini, F. Tramontana, V. Avrutin, M. Schanz "Border Collision Bifurcations in 1D PWL map and Leonov's approach." International Journal of Bifurcation & Chaos, 2010 (to appear).

L. Gardini, F. Tramontana "Border Collision Bifurcations in 1D PWL map with one discontinuity and negative jump. Use of the first return map" International Journal of Bifurcation & Chaos, 2010 (to appear).

L. Gardini, F. Tramontana, "Border Collision Bifurcation curves and their classification in a family of one-dimensional discontinuous maps", submitted.

Application to economics: G.I. Bischi, L. Gardini, F. Tramontana "Bifurcation curves in discontinuous maps", Discrete and Continuous Dynamical Systems B, 13(2) 2010, 249-267.

Fabio Tramontana, Laura Gardini, Piero Ferri "The dynamics of the NAIRU model with two switching regimes" Journal of Economic Dynamic and Control 34 (2010) 681–695.

Fabio Tramontana, Laura Gardini, Anna Agliari "Endogenous cycles in discontinuous growth models", Mathematics and Computers in Simulation (to appear).

Application to social sciences:

G.I. Bischi, L. Gardini, U. Merlone "Impulsivity in binary choices and the emergence of periodicity", Discrete Dynamics in Nature and Society, vol. 2009, Article ID 407913, 22 pages, 2009. doi:10.1155/2009/407913, http://www.hindawi.com/journals/ddns/2009/407913.html

G.I. Bischi, L. Gardini, U. Merlone "Periodic cycles and bifurcation curves for one-dimensional maps with two discontinuities", Journal of Dynamical Systems and Geometric Theory 7(2), 2009, 101-123.

Application to financial markets:

F.Tramontana, F.Westerhoff, L.Gardini "On the complicated price dynamics of a simple one-dimensional discontinuous financial market model with heterogeneous interacting traders", Journal of Economic Behavior and Organization, 74 (2010) 187–205.

F.Tramontana, L.Gardini and F.Westerhoff "Intricate asset price dynamics and one-dimensional discontinuous maps" in "Advances in nonlinear economic dynamics", T. Puu and A. Panchuck (Editors), Nova Science Publishers 2010.

F.Tramontana, L.Gardini, F.Westerhoff "Heterogeneous speculators and asset price dynamics: further results from a one-dimensional discontinuous piecewise-linear" in "Advances in nonlinear economic dynamics" JEE (submitted).

Generalizations:

L. Gardini, F. Tramontana, I. Sushko "Border Collision Bifurcations in one-dimensional linear-hyperbolic maps" (to appear).

F. Tramontana, L. Gardini "Border Collision Bifurcations in discontinuous one-dimensional linear-hyperbolic maps", Communications in Nonlinear Science and Numerical Simulation (to appear).

(A) increasing -increasing case

Improvement of the Leonov's approach for analytical BCB curves.

The object of this lesson is to give a new interpretation and improvements of some results, associated with the dynamics of the piecewise linear map

$$x' = f(x) = \begin{cases} f_L(x) = a_L x + \mu_L & , \quad \text{if} \quad x < 0 \\ f_R(x) = a_R x + \mu_R & , \quad \text{if} \quad x > 0 \end{cases}$$

The first works associated with the piecewise linear discontinuous map $f(x)$ are due to Leonov :

N.N. Leonov, Map of the line onto itself, Radiofisica 3 (3) (1959) 942-956.

N.N. Leonov, Discontinuous map of the straight line, Dohk. Ahad. Nauk. SSSR. 143 (5) (1962) 1038-1041.

and were known to a few researchers (and their collaborators):

C. Mira, Sur les structure des bifurcations des diffeomorphisme du cercle, C.R.Acad. Sc. Paris 287 Series A (1978) 883-886.

C. Mira, Chaotic dynamics, World Scientific, Singapore, 1987.

Y.L. Maistrenko, V.L. Maistrenko, L.O. Chua, Cycles of chaotic intervals in a time-delayed Chua's circuit, Internat. J. Bifur. Chaos 3 (6) (1993) 1557-1572.

Y.L. Maistrenko, V.L. Maistrenko, S.I. Vikul, L.O. Chua, Bifurcations of attracting cycles from time-delayed Chua's circuit, Internat. J. Bifur. Chaos 5 (3) 653-671.

Y.L. Maistrenko, V.L. Maistrenko, S.I. Vikul, On period-adding sequences of attracting cycles in piecewise linear maps, Chaos Solitons Fractals 9 (1) (1998) 67-75.

however, his results were unknown to a wide public, and when this research subject became of interest in several applied sciences (engineering, physics, and social sciences), an increasing number of papers have been written on this topic, and also other old results by Feigen (obtained in 1975) were rediscovered by di Bernardo and J. Hogan, but the pioneering works by Leonov are still mainly unknown.

M. Di Bernardo, M.I. Feigen, S.J. Hogan, M.E. Homer, Local analysis of C-bifurcations in n-dimensional piecewise smooth dynamical systems, Chaos Solitons Fractals 10 (11) (1999) 1881-1908.

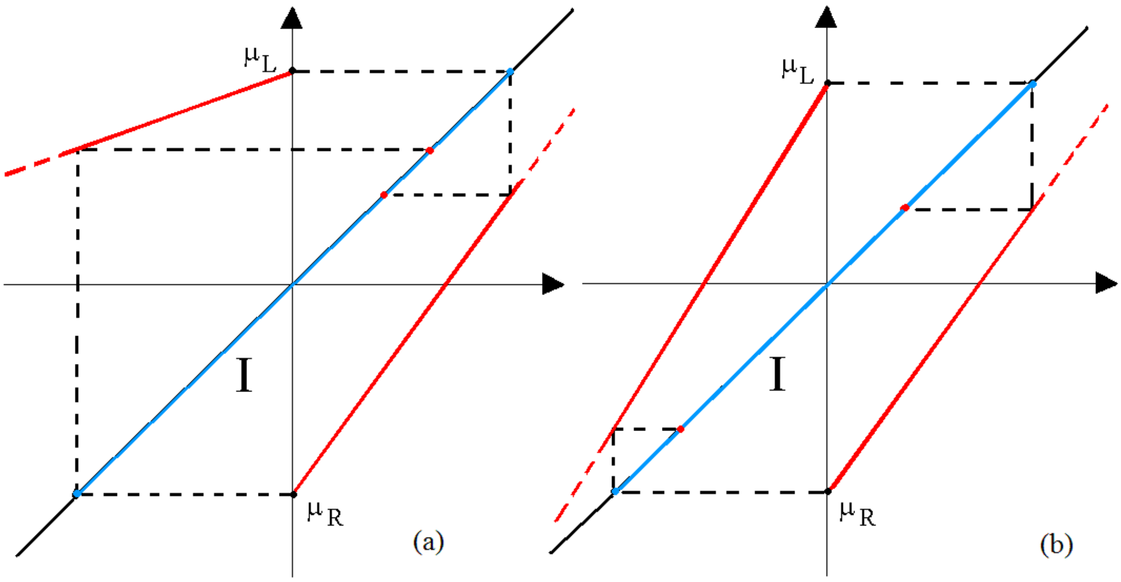
The good results already written in 1959 are really remarkable: a map in the coefficients with which it is possible to get the analytical BCB curves of several complexity levels, when in the more recent literature the analytical results stop at the BCB of the first level only.

Let us first consider the simple goal to recall Leonov's work. However, in doing this, it will be clear that it is possible to go bejond his results, obtaining an improvement in the application of his technique.

Leonov already noticed that the 4 parameters can be reduced to 3 by rescaling x and using the parameter $\mu = \mu_L/\mu_R$. However, as we shall, see the 4-parameter notation is helpful.

Let us start with the following ranges of the parameters, for which the map has no fixed points

$$a_L > 0 \quad , \quad a_R > 0 \quad , \quad \mu_R < 0 < \mu_L \tag{1}$$

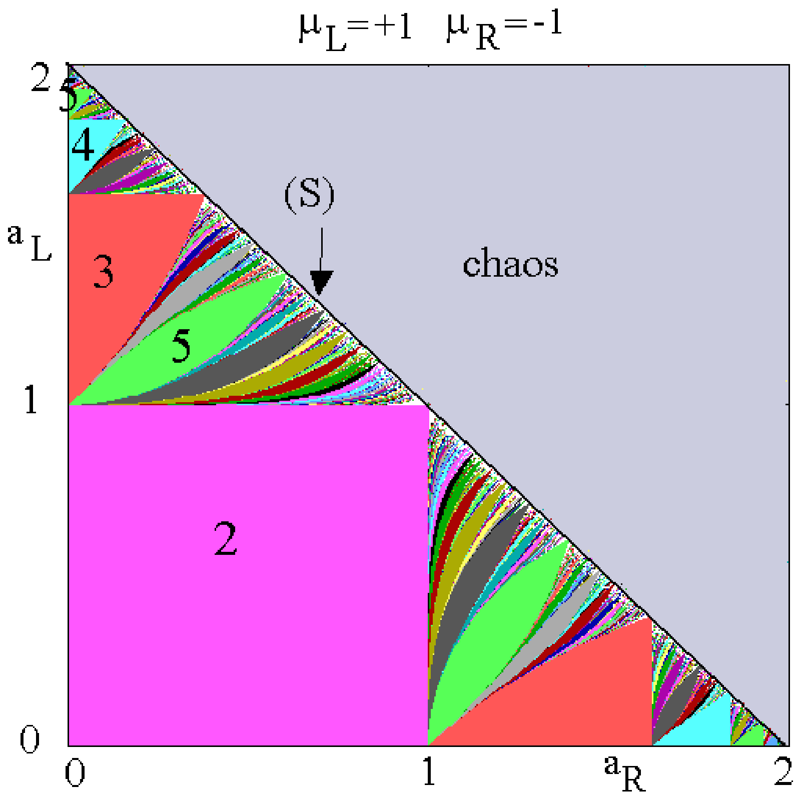


it is clear that as long as the slopes are less than 1 only stable cycles can exist. However the stability range is much wider, as already proved by Keener (in 1980):

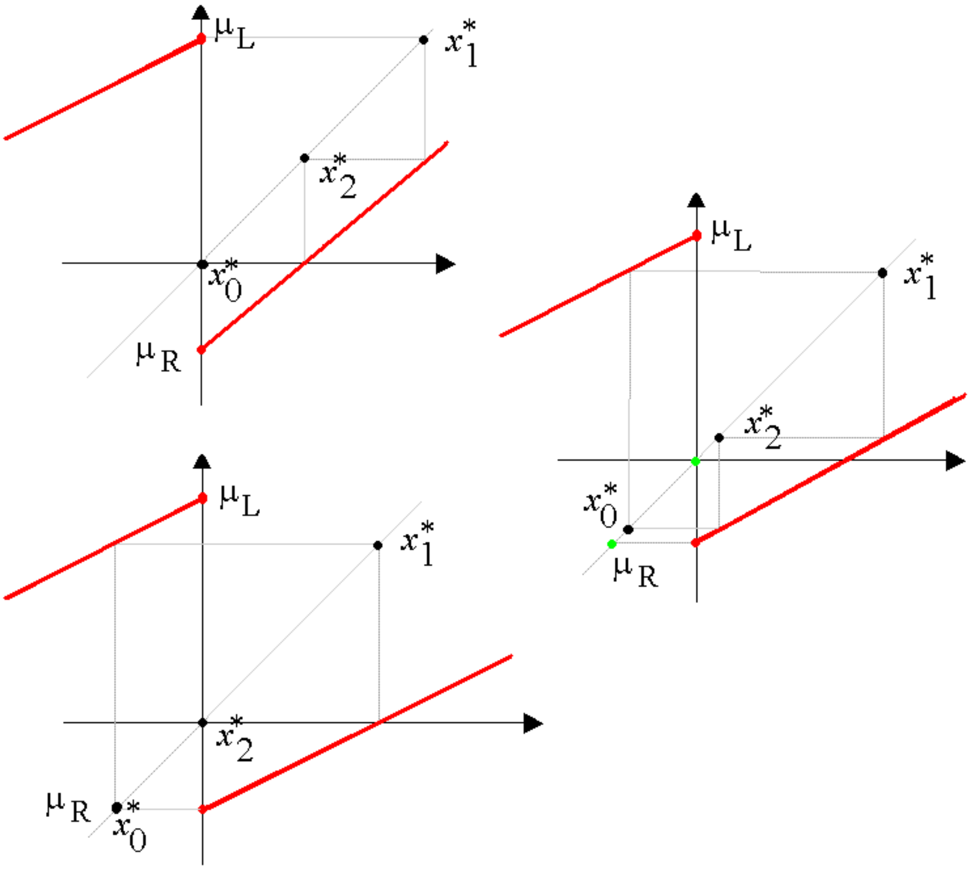
J.P. Keener, Chaotic behavior in piecewise continuous difference equations, Trans. Amer. Math. Soc. 261(2) (1980) 589-604.

the main point is the invertibility or non invertibility of the map in its absorbing invariant interval $I = [f_R(0), f_L(0)] = [\mu_R, \mu_L]$. The bifurcation condition is given by:

$$(S): \ \mu_L(1 - a_R) - \mu_R(1 - a_L) = 0 \qquad (2)$$



The simplest cycles to analyze are those called by Leonov of *first level of complexity* (also called *maximal* or *principal cycles*) of the so-called symbol sequence LR^{n_1}, for $n_1 \geq 1$.



For such a cycle of period $(n_1 + 1)$ we can order the periodic points. Let us define $x_0^* < 0$ and $x_1^* > ... > x_{n_1}^* > 0$. Then the x_i^* are $n_1 + 1$ fixed points of the map $f^{n_1+1}(x)$ and only one point is in the negative side, x_0^*, which is a fixed point of the linear function $f_R^{n_1} \circ f_L(x)$, and its range of existence as a fixed point of f^{n_1+1} in I is $f_R(0) = \mu_R \leq x_0^* \leq 0$, the related equalities denote the BCB leading to its appearance/disappearance. So from the function:

$$f_R^{n_1} \circ f_L(x) = (a_R^{n_1} a_L)x + (\mu_L a_R^{n_1} + \mu_R \frac{1 - a_R^{n_1}}{1 - a_R}) \quad (3)$$

we obtain its fixed point, which is the periodic point x_0^* for f, which exists for:

$$f_R(0) = \mu_R \leq x_0^* = \frac{1}{1 - a_R^{n_1} a_L} [\mu_L a_R^{n_1} + \mu_R \frac{1 - a_R^{n_1}}{1 - a_R}] \leq 0.$$

The two BCB curves bounding the region of existence of the cycle are denoted by $\xi_{LR^{n_1}}^l$ (resp. $\xi_{LR^{n_1}}^r$):

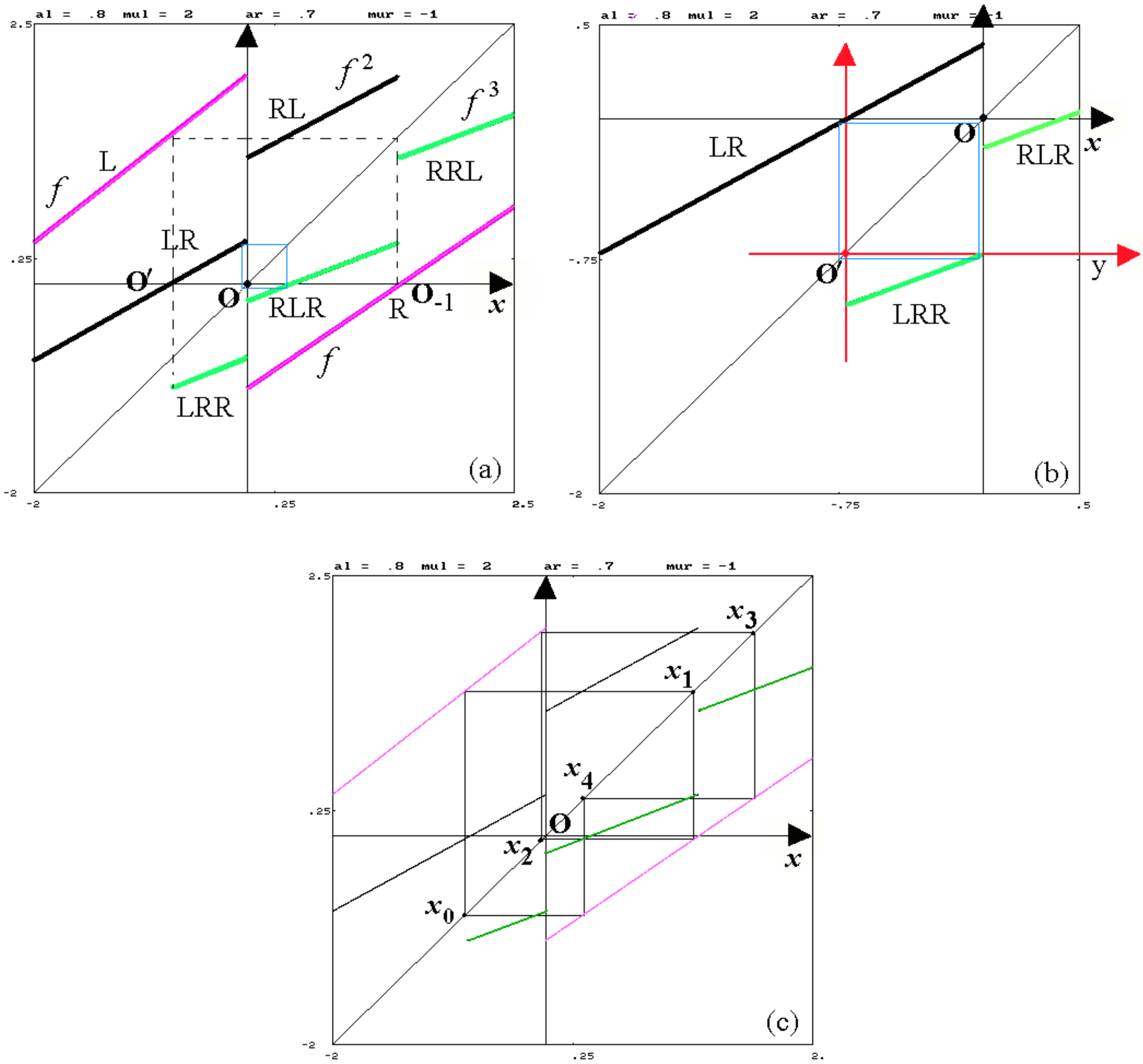
$$\begin{aligned} \xi_{LR^{n_1}}^l &: \mu_L \leq -\mu_R \varphi_{n_1}^R \quad , \quad \varphi_{n_1}^R = \frac{1 - a_R^{n_1}}{(1 - a_R) a_R^{n_1}} \\ \xi_{LR^{n_1}}^r &: \mu_L \geq -\mu_R (a_L + \varphi_{n_1 - 1}^R). \end{aligned}$$

We can reason in some sense "symmetrically" (with respect to the L and R sides) looking for cycles having the symbol sequence RL^{n_1}. It is clear, due to the structure of the initial map, that it is enough the exchange the symbols L and R in the above calculations, and reverse the related inequalities. Notice in fact that changing sign in the previous sequence we get the new one: $x_0^* > 0$, $x_1^* < ... < x_{n_1}^* < 0$. Changing the letters and the inequalities in the previous constraint $\mu_R = f_R(0) \leq x_0^* \leq 0$ we get the new one: $\mu_L = f_L(0) \geq x_0^* \geq 0$. Thus we have:

$$\begin{aligned}
\mu_L &\geq x_0^* = \frac{1}{1-a_L^{n_1}a_R}[\mu_R a_L^{n_1} + \mu_L \frac{1-a_L^{n_1}}{1-a_L}] \geq 0 \\
\xi^r_{RL^{n_1}} &: \mu_R \geq -\mu_L \varphi^L_{n_1} \quad , \quad \varphi^L_{n_1} = \frac{1-a_L^{n_1}}{(1-a_L)a_L^{n_1}} \\
\xi^l_{RL^{n_1}} &: \mu_R \leq -\mu_L (a_R + \varphi^L_{n_1-1})
\end{aligned}$$

There is now a suitable discussion *to show that the periodicity regions are all disjoint* (for which we refer to the paper).

Then we can find the BCB associated with all the other complexity level, by using an iterative process in the map coefficients. The simple idea comes from the fact that some powers of the map are in the same situation as the original branches



so considering any two consecutive cycles with symbol sequence LR^{n_1} and LR^{n_1+1}, or equivalently with symbol sequence LR^{n_1} and RLR^{n_1}, in order to avoid the change of coordinate, we consider the function $T_L(x) = f_R^{n_1} \circ f_L(x)$ on the L side of $x = 0$ and

$T_R(x) = f_R^{n_1} \circ f_L \circ f_R(x)$ on the R side of $x = 0$, and we have the needed operator:

$$x' = T(x) = \begin{cases} T_L(x) = A_L x + M_L & , \quad \text{if} \quad x < 0 \\ T_R(x) = A_R x + M_R & , \quad \text{if} \quad x > 0 \end{cases} \tag{4}$$

where:

$$\begin{aligned} A_L &= a_L a_R^{n_1} \quad , \quad M_L = \mu_L a_R^{n_1} + \mu_R \frac{1 - a_R^{n_1}}{1 - a_R} \\ A_R &= a_L a_R^{n_1+1} \quad , \quad M_R = A_L \mu_R + M_L \end{aligned} \tag{5}$$

Now it is only a matter of applications of the previous results. We can immediately have the periodic points and the BCB curves of the cycles obtained as fixed points for the functions in the form $T_R^{n_2} \circ T_L(x)$ and $T_L^{n_2} \circ T_R(x)$, and this first level of T corresponds to the (old level $+1$) for the map f. So we get everything for the two families of cycles having the symbol sequence

$$T_L T_R^{n_2} \text{ and } T_R T_L^{n_2}$$

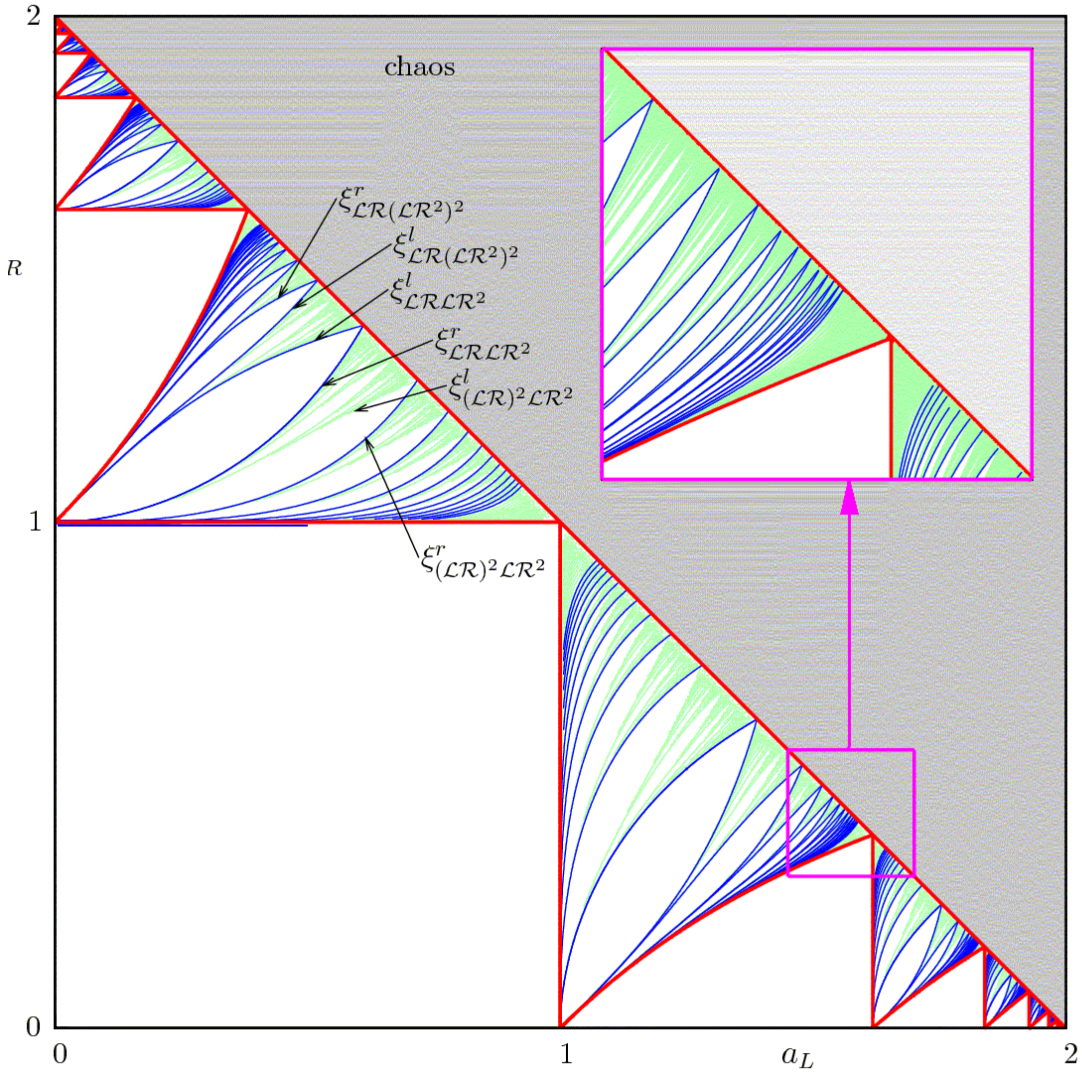
$$LR^{n_1}(LR^{n_1+1})^{n_2} \text{ and } LR^{n_1+1}(LR^{n_1})^{n_2}$$

Similarly we can reason considering any two consecutive cycles with symbol sequence LR^{n_1} and LR^{n_1+1}, or equivalently with symbol sequence LR^{n_1} and RLR^{n_1}, considering the function $T_L(x) = f_R^{n_1} \circ f_L(x)$ on the L side of $x = 0$ and $T_R(x) = f_R^{n_1} \circ f_L \circ f_R(x)$ on the R side of $x = 0$, we have the needed operator. However we can also use the previous computations, just exchanging L and R and reversing the inequalities, obtaining the BCB curves of two families of cycles having the symbol sequence

$$T_R T_L^{n_2} \text{ and } T_L T_R^{n_2}$$

$$RL^{n_1}(RL^{n_1+1})^{n_2} \text{ and } RL^{n_1+1}(RL^{n_1})^{n_2}$$

This process is thus generalized to any level of complexity, to get the analytic expression of the BCB curves



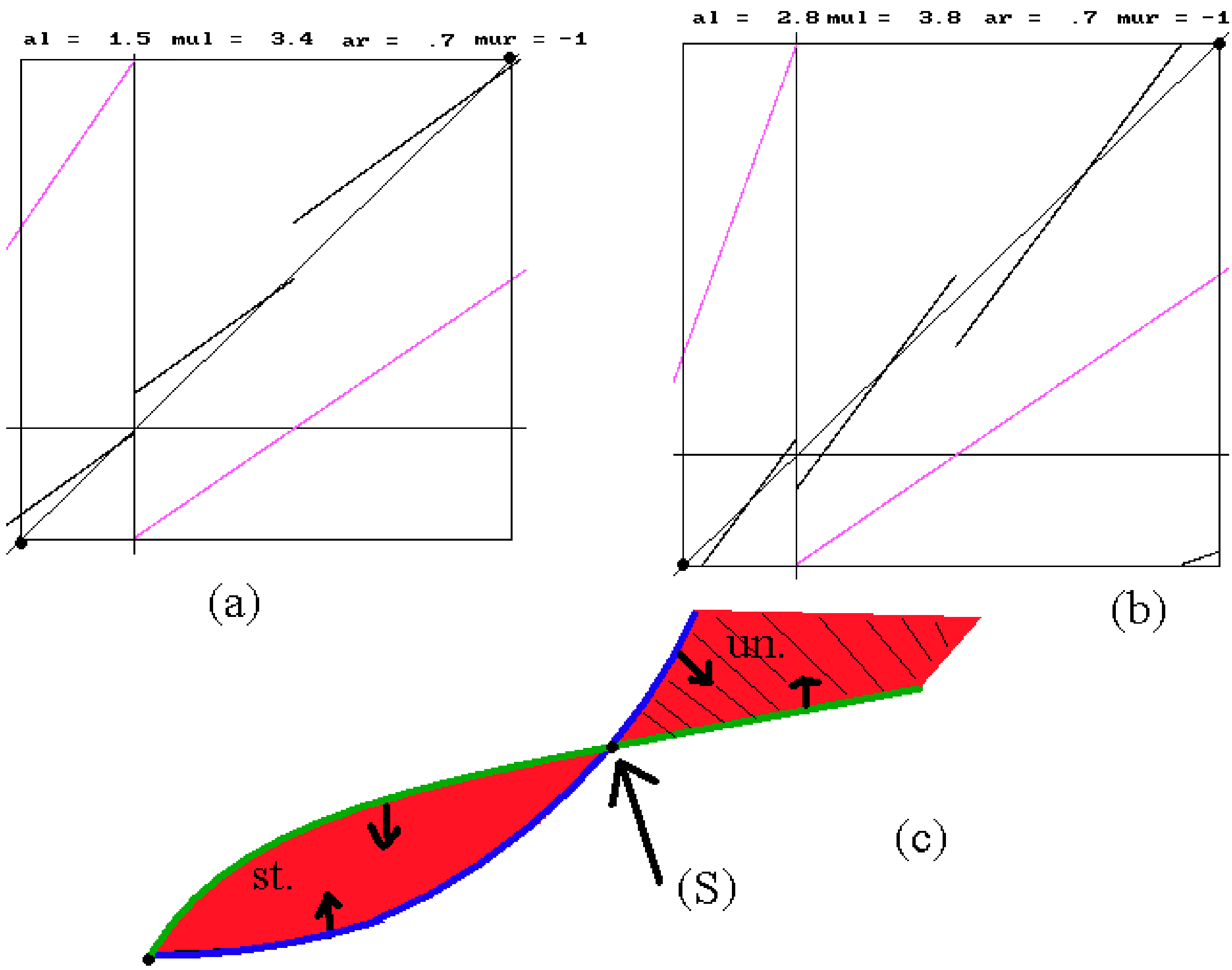
Change of stability on (S): The following

Proposition. *When the parameters satisfy*

$$\mu_L > a_L\left(\frac{-\mu_R}{1-a_R}\right) + \frac{\mu_R}{1-a_R} \quad \text{(resp. } < \text{)}$$

all the cycles are stable (resp. unstable).

is proved by using of the rotation numbers of the cycles

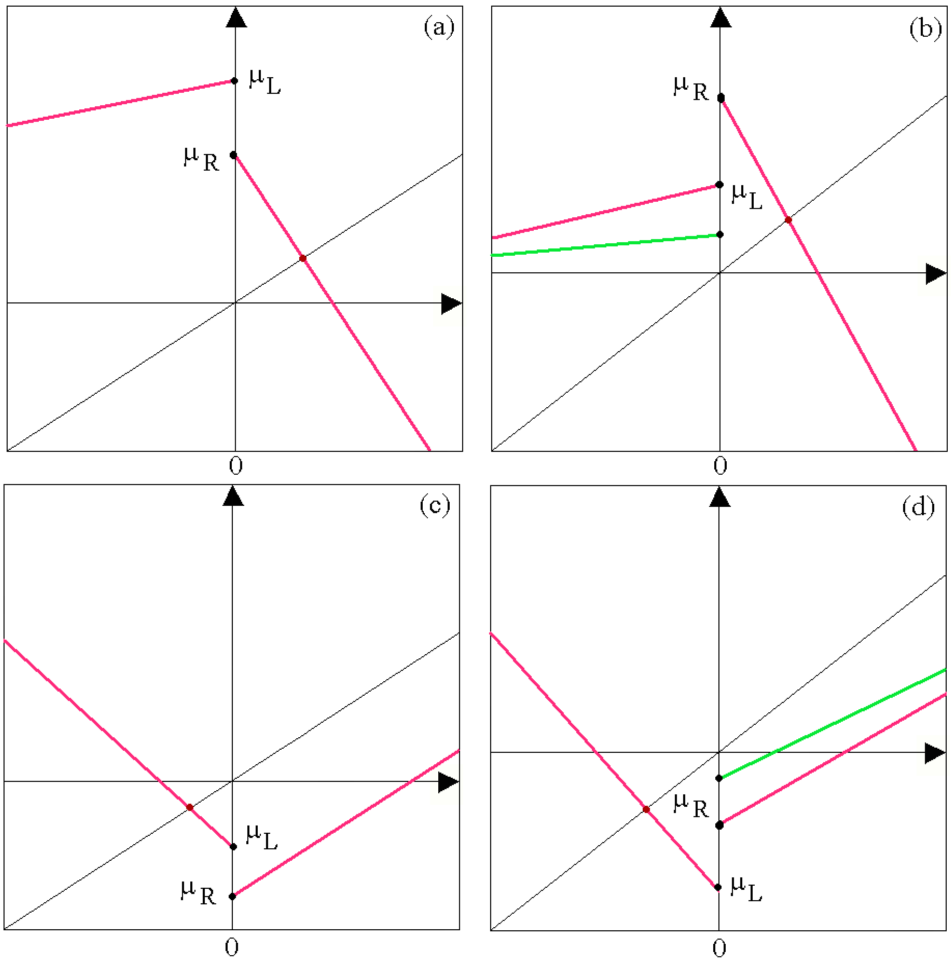


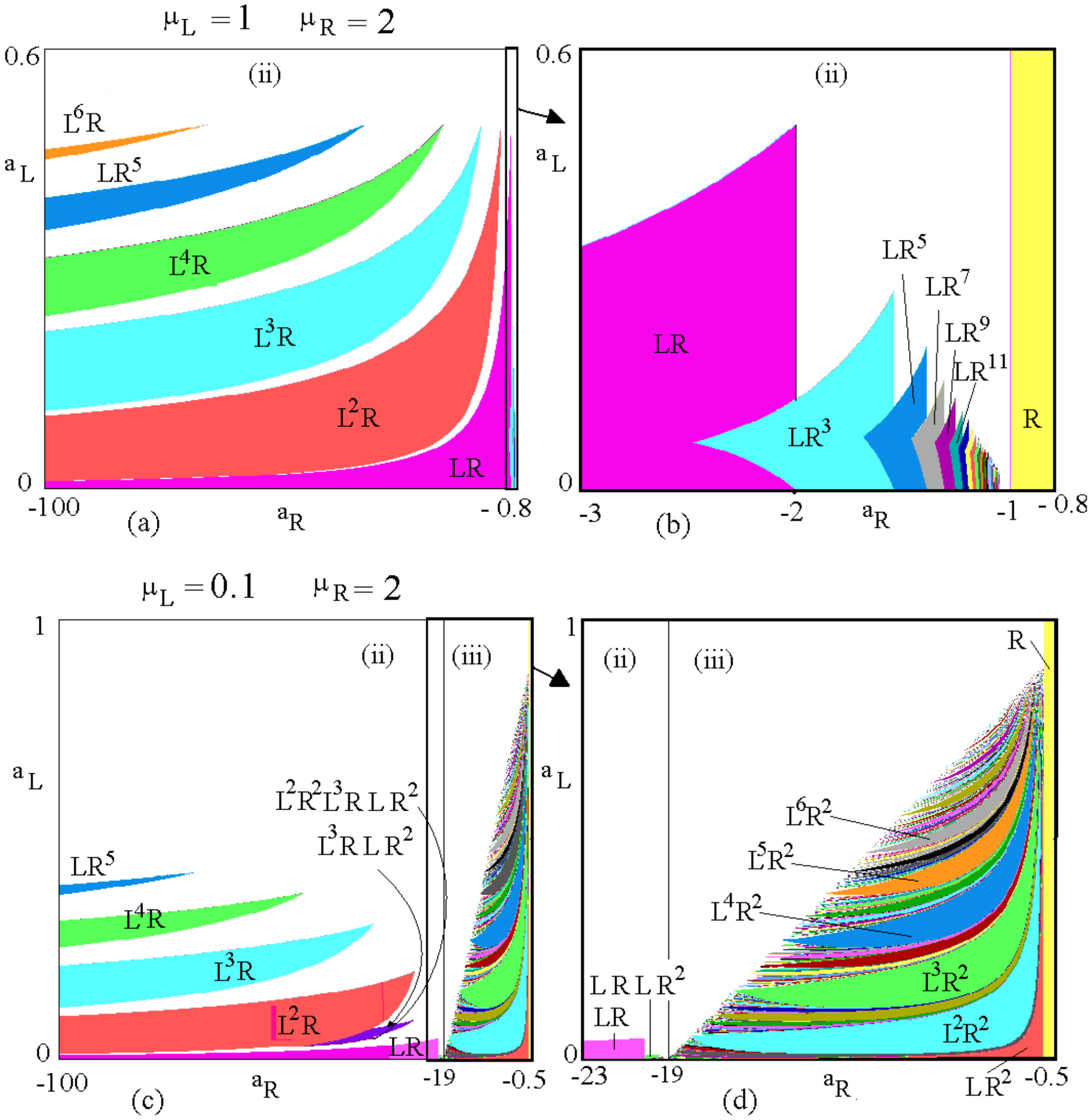
It is important to notice that the application of the Leonov approach (the map in the coefficients) is not limited to the range considered above: it can be used whenever the adding scheme exists.

Let us consider a different set of the parameters so that the map has a jump of different kind:

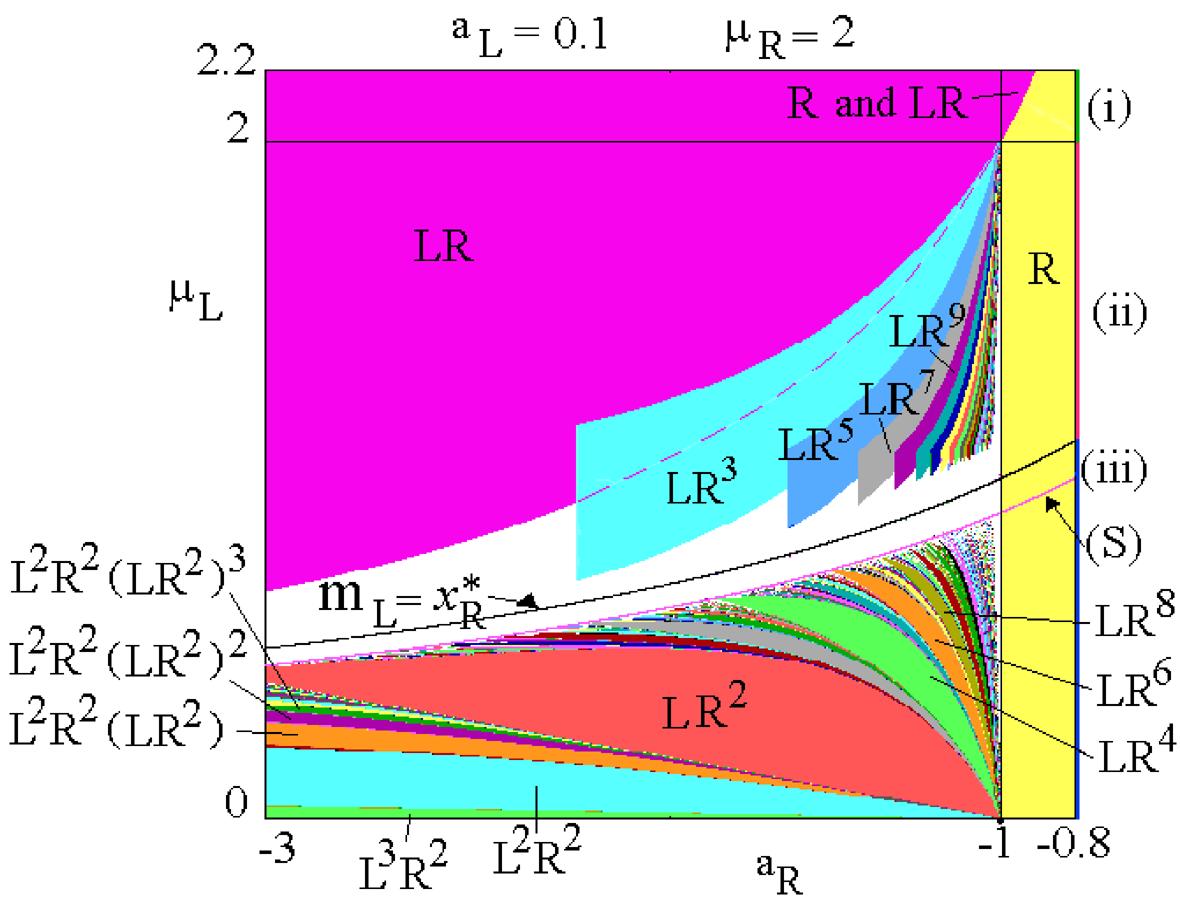
(B) increasing -decreasing case

(offsets of the same sign)

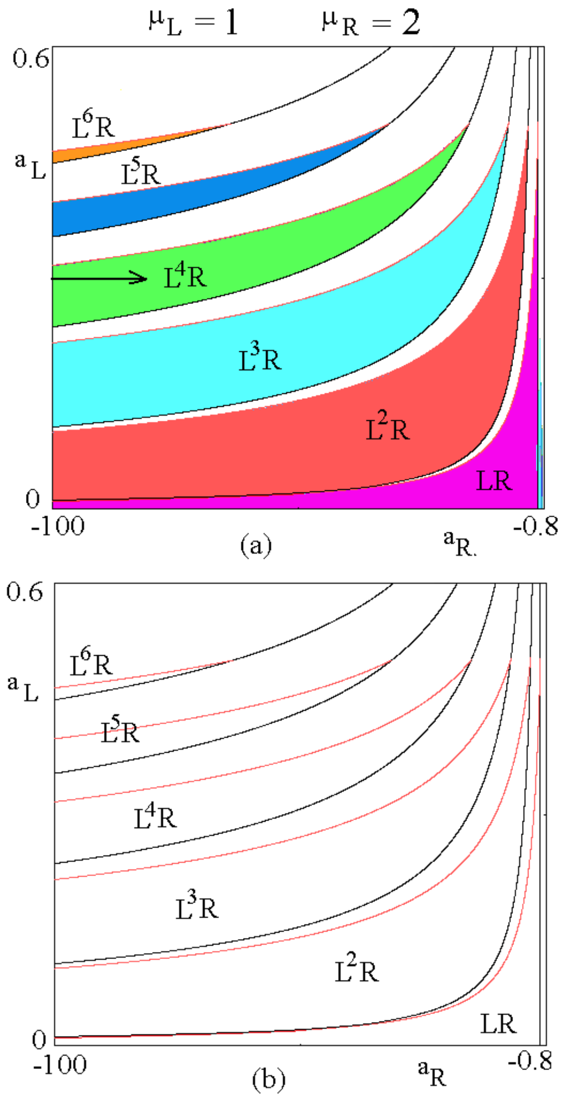




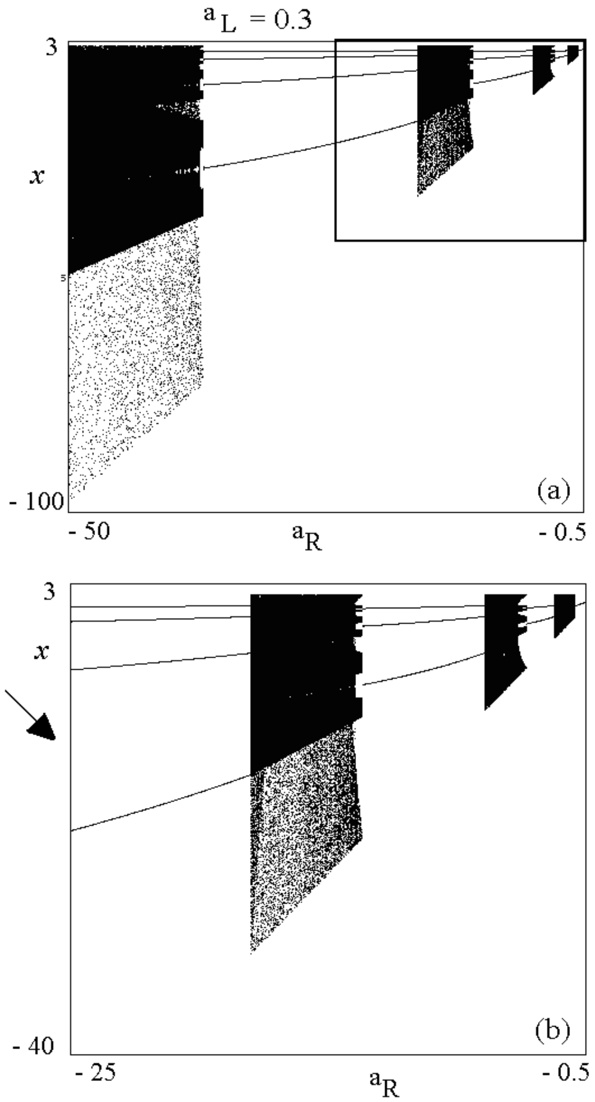
$\mu_L = 1 \quad \mu_R = 2$
$\mu_L = 0.1 \quad \mu_R = 2$
(a)
(b)
(c)
(d)
(ii)
(iii)
a_L
a_R
0.6
0
1
-100
- 0.8
-3
-2
-1
-19
-0.5
-23
L^6R
LR^5
L^4R
L^3R
L^2R
LR
LR^3
LR^7
LR^9
LR^{11}
R
$L^2R^2L^3RLR^2$
L^3RLR^2
L^6R^2
L^5R^2
L^4R^2
L^3R^2
L^2R^2
LR^2
$LRLR^2$



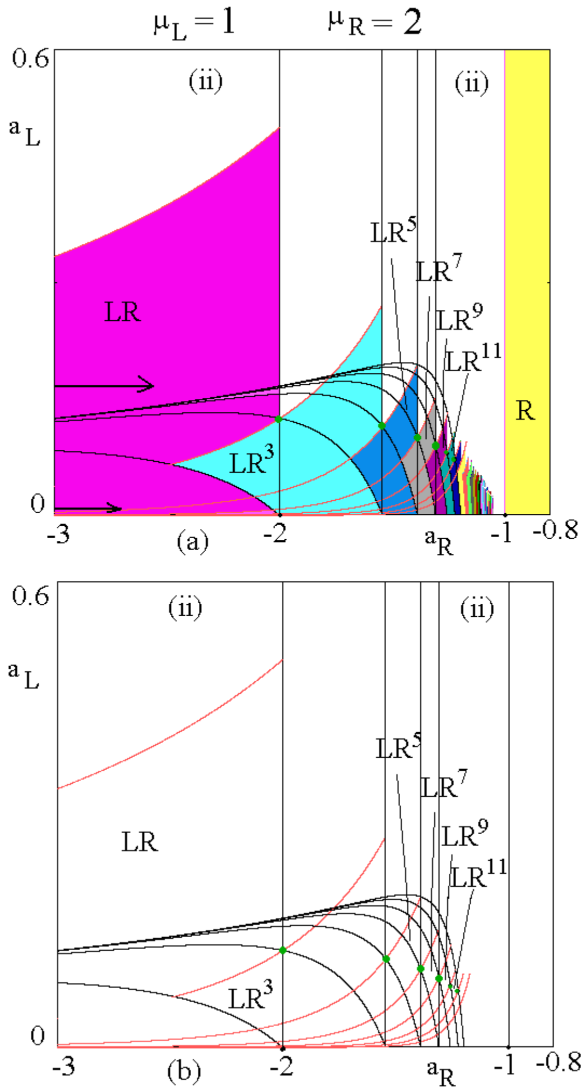
$a_L = 0.1$
$\mu_R = 2$
2.2
2
0
μ_L
R and LR
(i)
LR
R
(ii)
LR^9
LR^7
LR^5
LR^3
(iii)
(S)
$m_L = x_R^*$
$L^2R^2(LR^2)^3$
$L^2R^2(LR^2)^2$
$L^2R^2(LR^2)$
LR^8
LR^6
LR^4
LR^2
-3
L^3R^2
L^2R^2
a_R
-1
-0.8

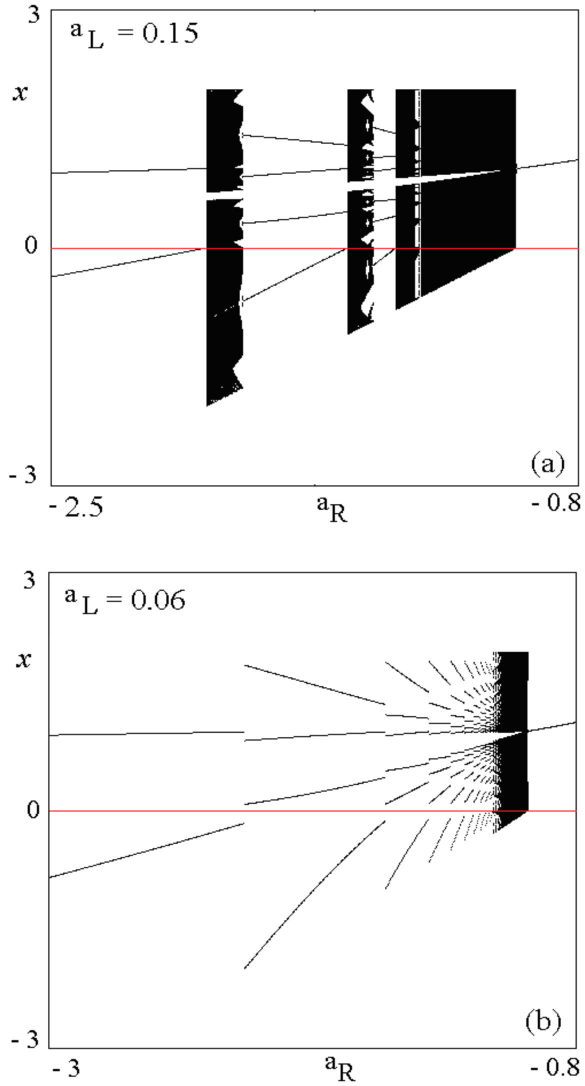


$\mu_L = 1$ $\mu_R = 2$
0.6
a_L
0
-100
a_R
-0.8
L^6R
L^5R
L^4R
L^3R
L^2R
LR
(a)
0.6
a_L
0
-100
a_R
-0.8
L^6R
L^5R
L^4R
L^3R
L^2R
LR
(b)

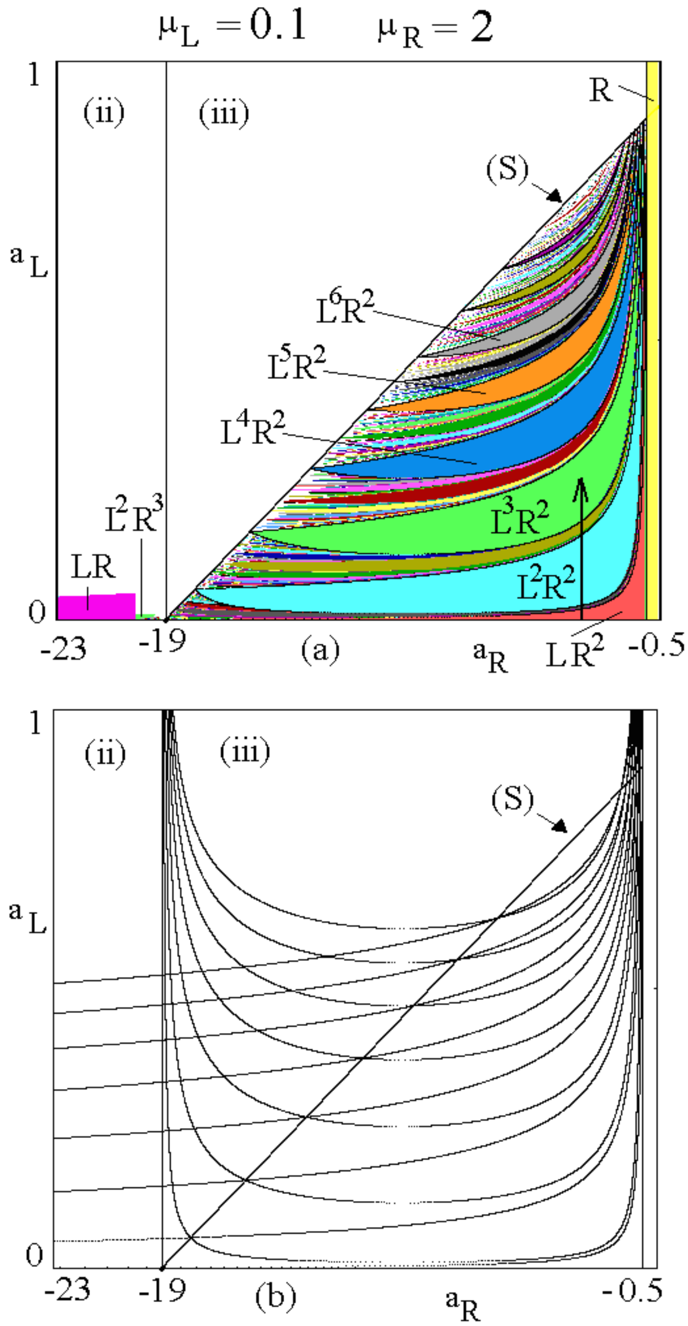


a L = 0.3
3
x
- 100
(a)
- 50
a R
- 0.5
3
x
- 40
(b)
- 25
a R
- 0.5

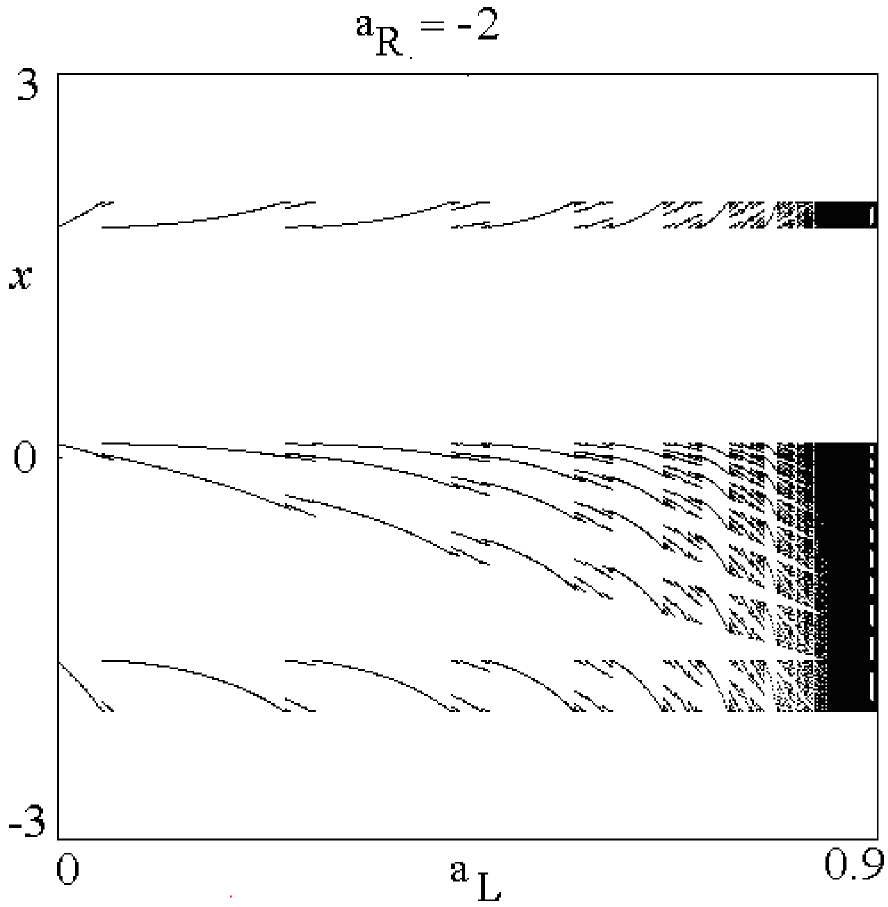

$\mu_L = 1$
$\mu_R = 2$
0.6
a_L
(ii)
(ii)
LR
LR^3
LR^5
LR^7
LR^9
LR^{11}
R
0
-3
(a)
-2
a_R
-1
-0.8
0.6
a_L
(ii)
(ii)
LR
LR^3
LR^5
LR^7
LR^9
LR^{11}
0
-3
(b)
-2
a_R
-1
-0.8

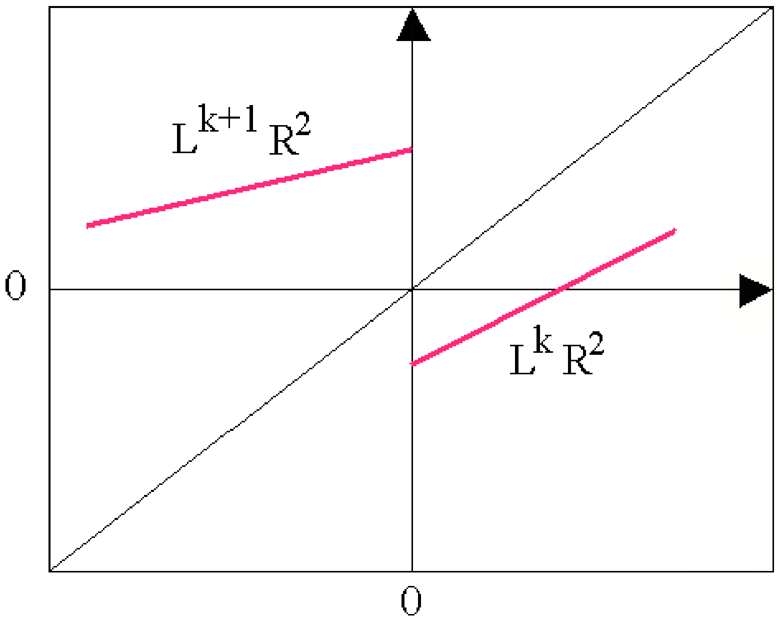


3
$a_L = 0.15$
x
0
- 3
(a)
- 2.5
a_R
- 0.8
3
$a_L = 0.06$
x
0
- 3
(b)
- 3
a_R
- 0.8

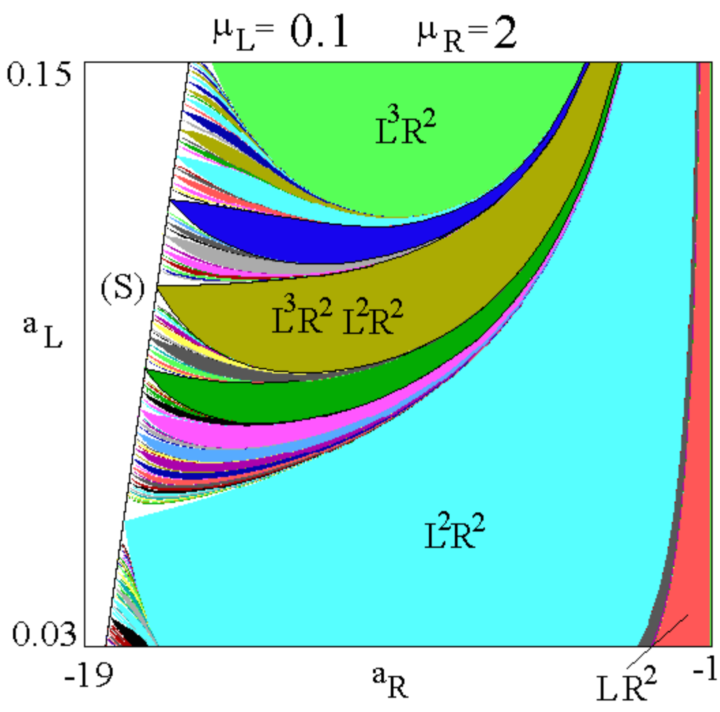


$\mu_L = 0.1$ $\mu_R = 2$
1
(ii)
(iii)
R
(S)
a_L
L^6R^2
L^5R^2
L^4R^2
L^2R^3
L^3R^2
LR
L^2R^2
0
-23
-19
(a)
a_R
LR^2
-0.5
1
(ii)
(iii)
(S)
a_L
0
-23
-19
(b)
a_R
-0.5

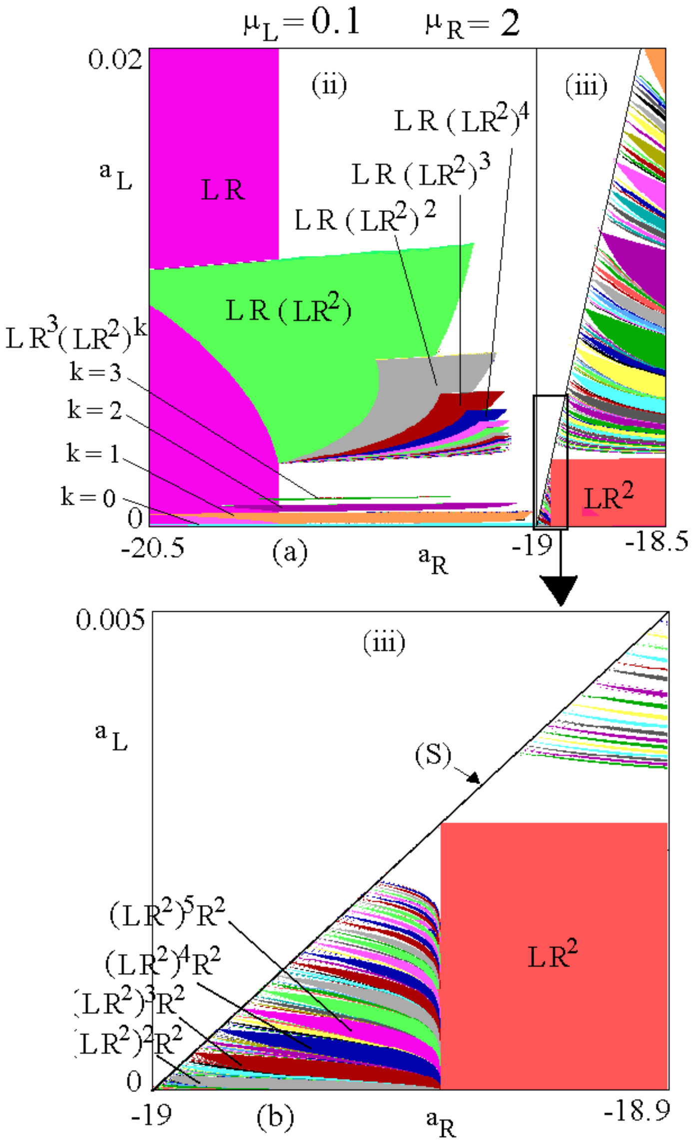

$a_R = -2$
3
x
0
-3
0
a_L
0.9



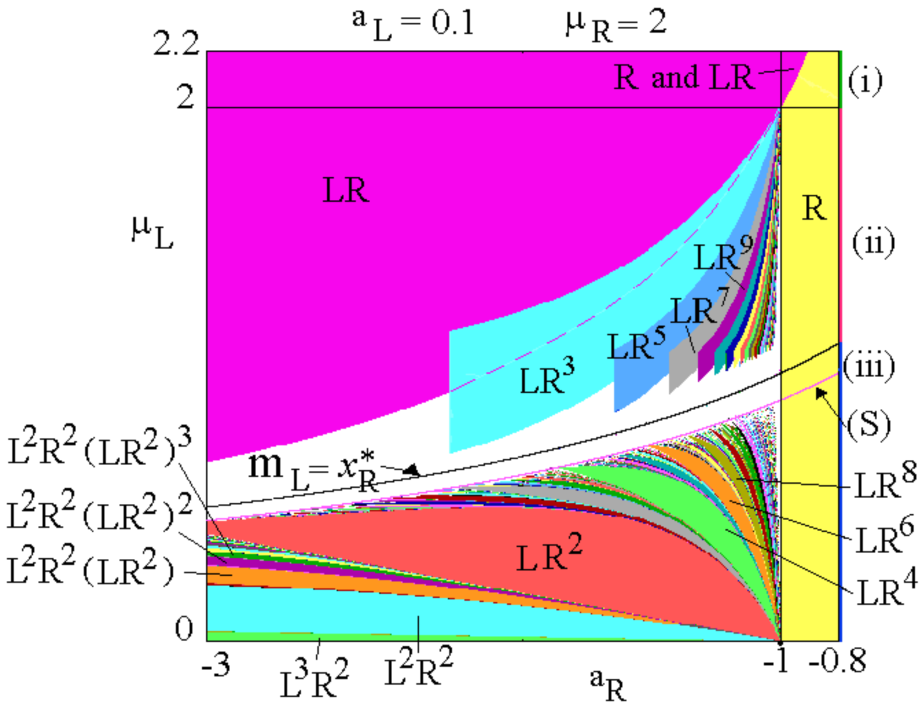
$L^{k+1}R^2$
L^kR^2
0
0



$\mu_L = 0.1$ $\mu_R = 2$
0.15
0.03
a_L
a_R
-19
-1
(S)
L^3R^2
$L^3R^2L^2R^2$
L^2R^2
LR^2



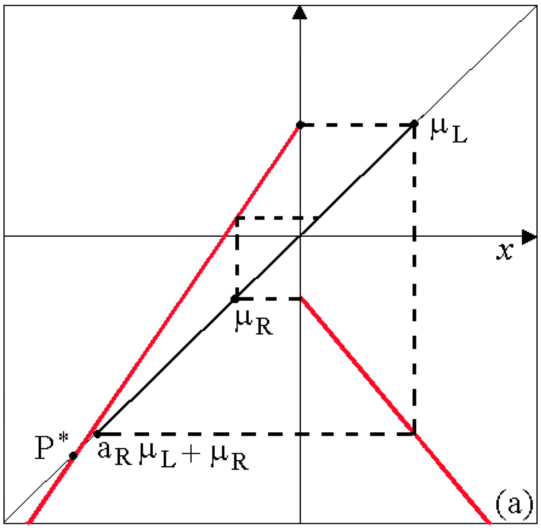
$\mu_L = 0.1$ $\mu_R = 2$
0.02
a_L
(ii)
(iii)
LR
$LR(LR^2)^4$
$LR(LR^2)^3$
$LR(LR^2)^2$
$LR(LR^2)$
$LR^3(LR^2)^k$
k=3
k=2
k=1
k=0
0
LR^2
-20.5
(a)
a_R
-19
-18.5
0.005
(iii)
a_L
(S)
$(LR^2)^5R^2$
$(LR^2)^4R^2$
$(LR^2)^3R^2$
$(LR^2)^2R^2$
LR^2
0
-19
(b)
a_R
-18.9

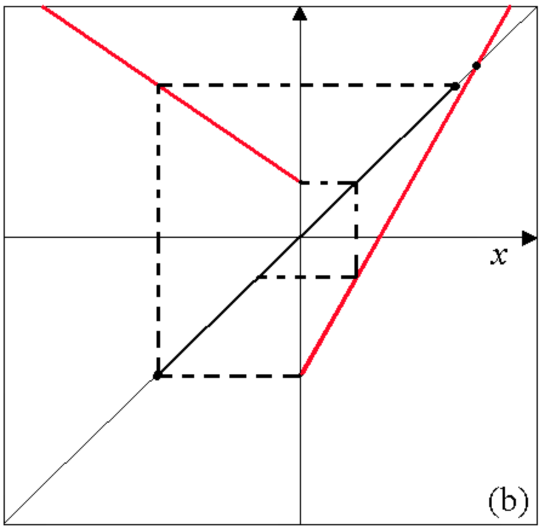


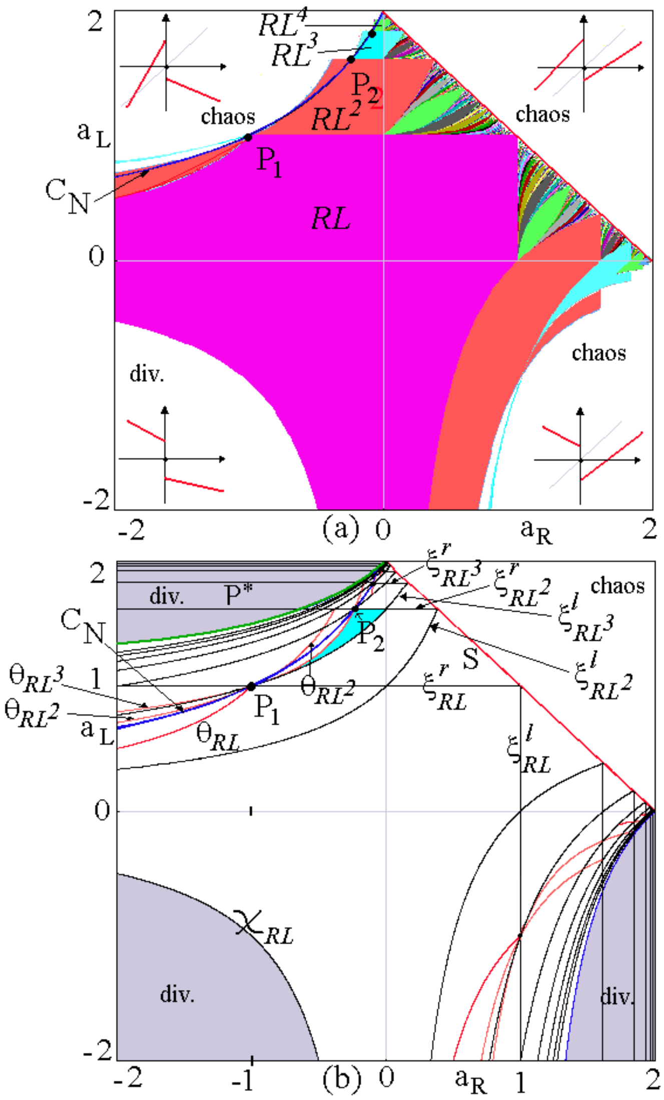
$a_L = 0.1$
$\mu_R = 2$
2.2
2
0
-3
-1
-0.8
μ_L
a_R
R and LR
(i)
LR
R
(ii)
LR^9
LR^7
LR^5
LR^3
(iii)
(S)
$m_L = x_R^*$
LR^8
LR^6
LR^4
LR^2
$L^2R^2(LR^2)^3$
$L^2R^2(LR^2)^2$
$L^2R^2(LR^2)$
L^3R^2
L^2R^2

(B) increasing -decreasing case

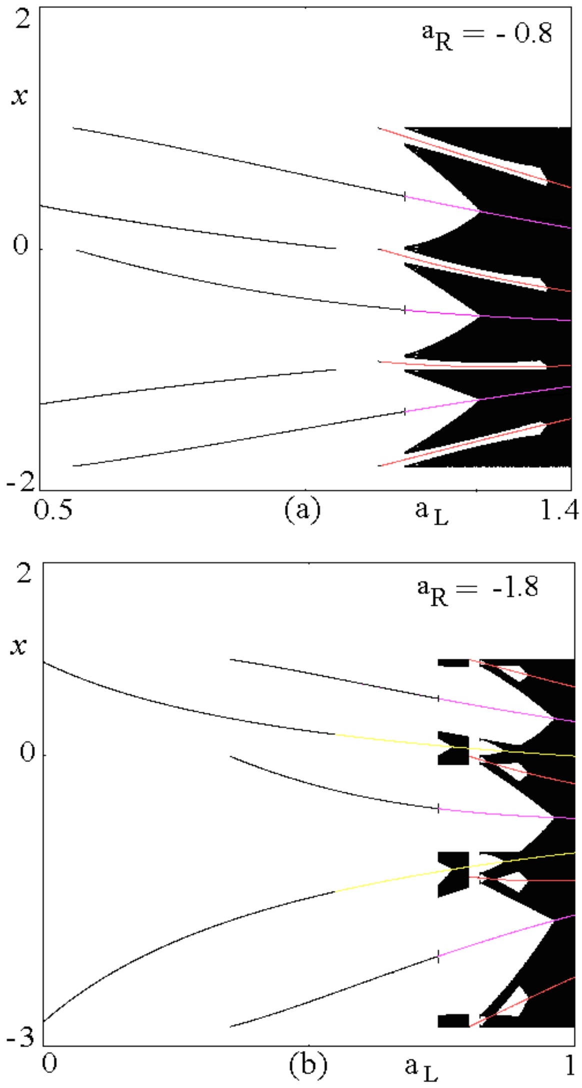
(offsets of opposite sign)

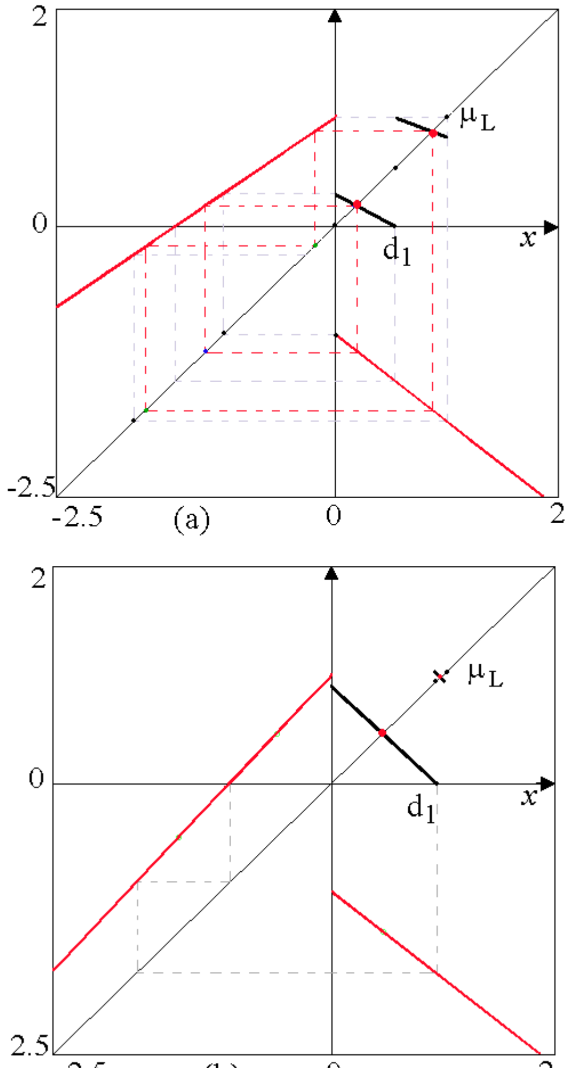






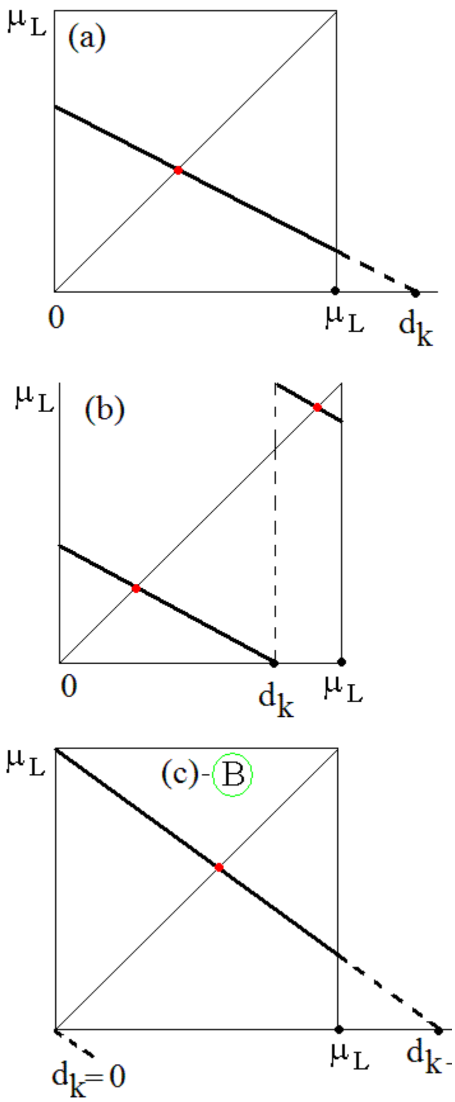
(a)
RL^4
RL^3
P_2
RL^2
chaos
chaos
a_L
P_1
C_N
RL
chaos
div.
a_R
(b)
div.
P^*
C_N
$\xi^r_{RL^3}$
$\xi^r_{RL^2}$
$\xi^l_{RL^3}$
$\xi^l_{RL^2}$
chaos
S
θ_{RL^3}
θ_{RL^2}
θ_{RL}
P_1
P_2
ξ^r_{RL}
ξ^l_{RL}
χ_{RL}
div.
div.
a_L
a_R


2
a_R = - 0.8
x
0
-2
0.5
(a)
a_L
1.4
2
a_R = -1.8
x
0
-3
0
(b)
a_L
1

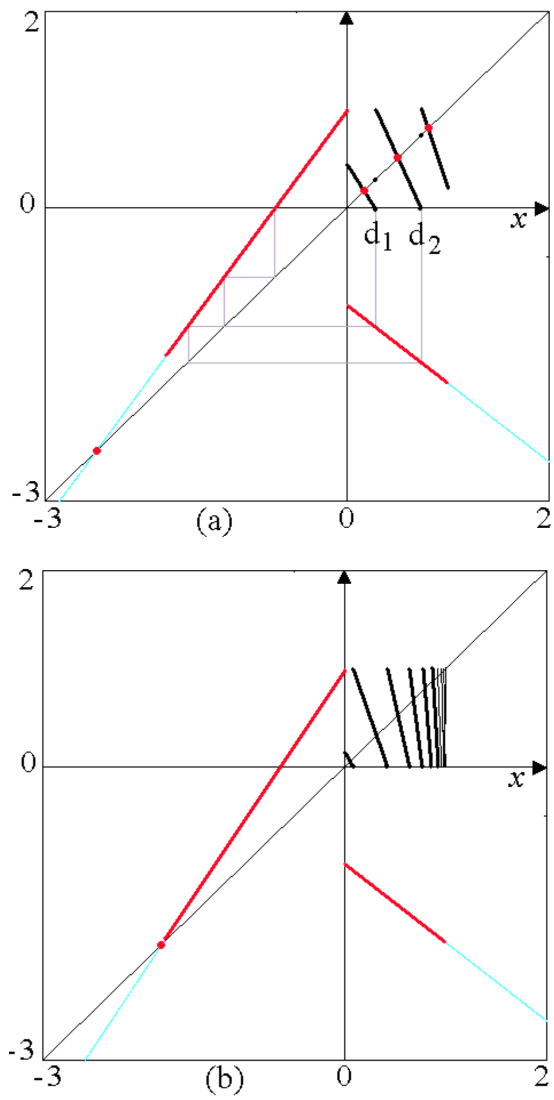


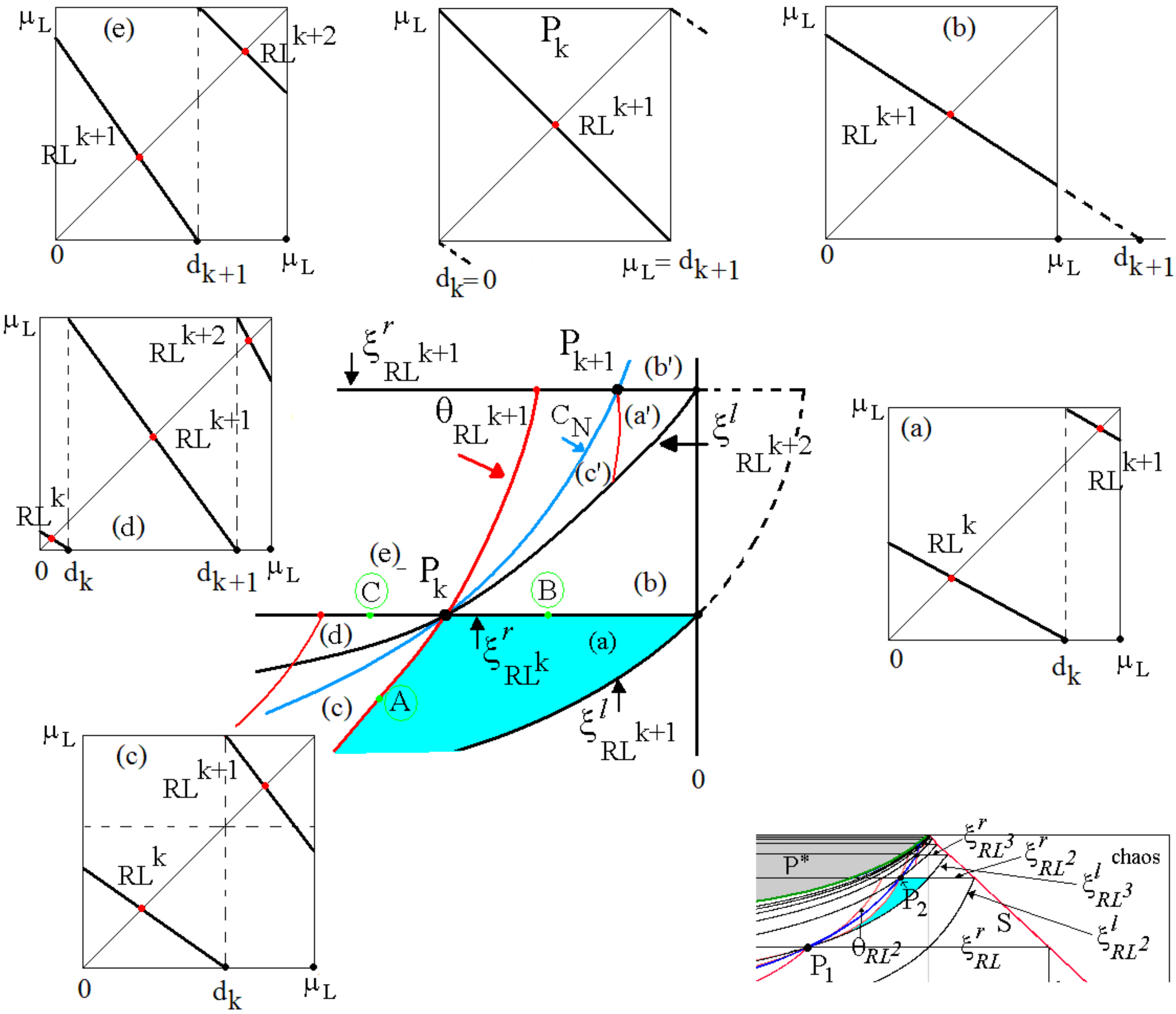
(a)

(b)



μ L
(a)
0
μ L
d k
μ L
(b)
0
d k
μ L
μ L
(c)-B
d k= 0
μ L
d k-


2
0
-3
x
d_1
d_2
-3
(a)
0
2
2
0
-3
x
-3
(b)
0
2


(e)
RL^{k+2}
RL^{k+1}
d_{k+1}
μ_L
P_k
$d_k = 0$
$\mu_L = d_{k+1}$
(b)
d_{k+1}
(d)
RL^k
d_k
$\xi^r_{RL^{k+1}}$
P_{k+1}
(b')
(a')
$\theta_{RL^{k+1}}$
C_N
(c')
$\xi^l_{RL^{k+2}}$
C
B
A
(a)
(c)
$\xi^r_{RL^k}$
$\xi^l_{RL^{k+1}}$
0
P*
P_1
P_2
S
chaos
θ_{RL^2}
$\xi^r_{RL^3}$
$\xi^r_{RL^2}$
$\xi^l_{RL^3}$
$\xi^l_{RL^2}$
ξ^r_{RL}