MDEF2014

8th Workshop MDEF
Modelli Dinamici in Economia e Finanza
Dynamic Models in Economics and Finance

An Evolutionary Approach to Nonlinear Heterogeneous Oligopolies

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Heterogeneous oligopolies

Research aim

GOAL

Investigate the evolutionary behavior of cournotian heterogeneous oligopolies of generic size N, under the effect of evolutionary pressure in a **play the field** context

Each firm select a behavioral rule from a set of *two* different rules. We focus on the evolutionary fractions framework, namely

- the firms can choose between different behavioral rules, adapting their adjustment mechanism endogenously. (**Evolutionary fractions**)

The rules we focus on are based on best response mechanisms and differentiate because of the rationality degree of agents.

Questions

Oligopoly size: N

Oligopoly composition : ω_t is the fraction of oligopolists that adopted the first mechanism at time t

Evolutionary pressure: β, is the intensity of choice, the propension to for a firm to change its behavioral rule

Informational costs : C, are the cost associated to the most rational firm, to take into account its informational effort

Does increasing N always lead to instability?

How local stability is affected by rationality (in terms of evolutionary pressure β, C?

Behavioral rules

Best Response mechanisms

We consider **best response** mechanisms with **different rationality degrees**

Rational (R) player

- full informational and computational capabilities
- complete knowledge of economic setting (demand and cost functions)
- endowed with *perfect foresight*, also of oligopoly composition
- able to optimally respond to the other players strategies

Best Response (BR) player

- complete knowledge of economic setting (demand and cost functions)
- NOT endowed with perfect foresight, *static expectation*
- able to optimally respond to the other players (expected) strategies

Local Monopolistic Approximation (LMA) player

- incomplete knowledge of economic setting (market price p_t, the produced quantity Q_t, local knowledge of the demand function in (p_t, Q_t))
- conjecture a demand function (local linear approximation), solve optimization

Homogeneous oligopolies

All firms adopt the *same decisional rule*.
Several works focus on stability thresholds with respect to oligopoly size

- Linear demand function: Palander (1939), Theocharis (1959), Canovas et al (2008).
- Isoelastic demand function: Puu (1991), Lampart (2012).
- LMA adjustment: Bischi et al.(2007) and Naimzada and Tramontana (2009).

Common behavior: **increasing** oligopoly **size** leads to **instability**. LMA is "more stable" than Best Response.

Heterogeneous oligopolies

Several couplings of different adjustment mechanisms for **duopolistic** markets: Agiza and Elsadany (2003,2004), Angelini et. al (2009), Tramontana(2010), C. and Naimzada (2014).
Droste et al. (2002) (linear demand function, no oligopoly size, only evolutionary fractions), Hommes et al. (2011) (linear demand function), Bischi et al. (2014)

Economic setting

Isoelastic (inverse) demand function (Cobb-Douglas preferences)

$$p(Q) = \frac{1}{Q}$$

Constant marginal costs c_i:

$$C(q_i) = c_i q_i$$

Identical marginal costs for firms adopting the **same rule**

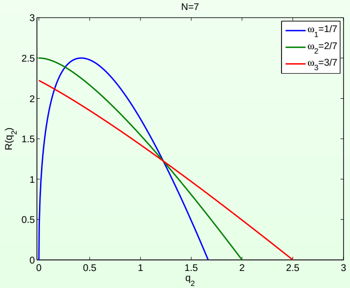
Behavioral rules

Generic R player

- → Compute the best response to the (correctly foreseen) strategies and oligopoly composition at time $t+1$ of remaining R players and F_2 players
- → The strategies of R players are identical: compute a (pseudo) best response to the (correctly foreseen) strategies at time $t+1$ of F_2 players

$$q_1^t = R(q_2^t, \omega_t) = \max\left\{ \frac{(\omega_t N - 1) - 2c_1\omega_t(1-\omega_t)N^2 q_2^t \sqrt{\Delta(q_2^t, \omega_t)}}{2c_1\omega_t^2 N^2}, 0 \right\}$$

where $\Delta(q_2^t, \omega_t) = (\omega_t N - 1)^2 + 4c_1\omega_t(1-\omega_t)N^2 q_2^t$.



Behavioral rules

Generic LMA player:

Approximated best response depends on own LMA player strategy q_t^t and on aggegated strategy Q^t (which depends on the oligopoly composition ω_t at time t)

$$q_2^{t+1} = L(q_2^t, Q^t(\omega_t)) = \max\left\{\frac{1}{2}q_2^t + \frac{1}{2}\left(1 - c_2 Q^t\right)Q^t, 0\right\}.$$

Generic BR player

Classical best response to the others' past time aggregated strategy Q_{-i}^t (static expectations) $i = 1, 2$ (which depends on the oligopoly composition ω_t at time t)

$$q_i^{t+1} = B(Q_{-i}^t(\omega_t)) = \max\left\{\sqrt{\frac{Q_{-i}^t}{c_i}} - Q_{-i}^t 0\right\},$$

Switching mechanism

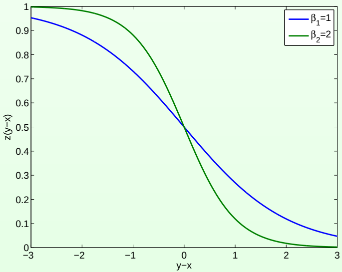
How do the firms select the rule to adopt? We assume that

- the decisional rules past performance and costs are commonly known
- the firms choose their decisional mechanism on the base of the previous period performance
- according to this choice, they decide their strategy

We investigate the dynamics given by the logit choice rule

$$z(x,y) = \frac{\exp(\beta x)}{\exp(\beta x) + \exp(\beta y)} = \frac{1}{1 + \exp(\beta \Delta xy)},$$

where $\Delta xy = y - x$.



Performance evaluation

How do the firms select decide which is the most performing mechanism?
They compare past period net profits $\pi_i - C_i$, in which C are **informational** supplementary costs associated to the most rational firms (without loss of generality, we assume that informational costs of the least rational firm are null)
Profit

$$\pi_i = \frac{q_i}{Q} - c_i q_i$$

Switching mechanism

$$\omega_{t+1} = \frac{1}{1 + \exp(\beta \Delta \pi^t)}.$$

where $\Delta\pi^t = \pi_2^t - \pi_1^t + C_1$.

- $\beta = 0 \;\rightarrow\; \omega_{t+1} = 1/2$, for any initial partitioning of the decisional mechanism, the players do not consider and compare the performance but they (immediately) uniformly distribute between the two decisional mechanisms.
- $\beta \rightarrow +\infty$, the logit function approaches a step function, so the firms are inclined to rapidly switch to the most profitable mechanism and ω_{t+1} approaches 0 or 1.

Profits positivity

Due to the supplementary informational costs, profits of the most rational firms can become negative.
This is acceptable only for short periods.
To check this, we introduce **cumulative** profits, namely the sum of the net profits

$$\Pi_i^{t+1} = \Pi_i^t + \pi_i^{t+1} - C_i,$$

We couple numerical investigations with cumulative profits diagrams, to prove that the results are economically meaningful.

First evolutionary fraction model: Rational vs. LMA

Two dimensional model

$$q_2^{t+1} = \max\left\{\frac{1}{2}q_2^t + \frac{1}{2}\left(1 - c_2 Q^t\right)Q^t, 0\right\},$$

$$\omega_{t+1} = \frac{1}{1 + \exp(\beta(\pi_2^{LMA,t}(q_1^t, q_2^t, \omega_t) - \pi_2^{R,t}(q_1^t, q_2^t, \omega_t) + C))}$$

where $Q^t = \omega_t N R(q_2^t, \omega_t) + (1 - \omega_t) N q_2^t$ and

$$\pi_1^{R,t}(q_1^t, q_2^t, \omega_t) = \frac{R(q_2^t, \omega_t)}{Q^t} - c_1 R(q_2^t, \omega_t),$$

$$\pi_2^{LMA,t}(q_1^t, q_2^t, \omega_t) = \frac{q_2^t}{Q^t} - c_2 q_2^t$$

We focus on identical marginal costs $c = c_1 = c_2$

Proposition

The Nash equilibrium $q_i^\star = (N-1)/(N^2 c)$ is the only positive steady state, to which corresponds the equilibrium fraction $\omega^\star = 1/(1 + \exp\beta C)$

Evolutionary fractions Rational vs. LMA

Local Stability

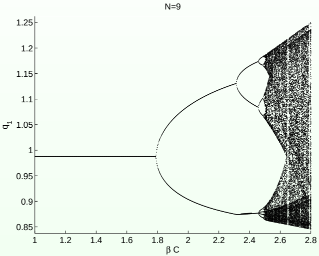
Proposition

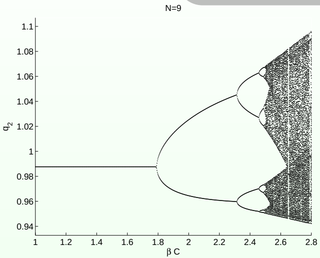
If $N \leq 5$ the Nash equilibrium is always stable. When $N > 5$, the Nash equilibrium is stable provided that

$$\beta C < \log\left(\frac{3N-3}{N-5}\right).$$

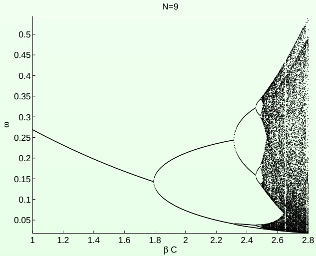
- Oligopolies with $N \leq 5$ have stable equilibrium independently of the equilibrium fraction ω^*, regulated by βC.
- For $N > 5$, we always have suitable C and β that make stable the equilibrium.
- As C and β sufficiently increase, the equilibrium become unstable.
- However, if $\beta C \leq \log(3)$, the equilibrium is stable for all the oligopoly sizes N.
- When stability condition is violated, equilibrium loses stability through a flip bifurcation.

Evolutionary fractions Rational vs. LMA



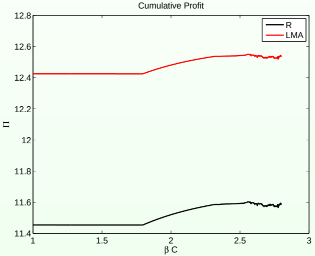


Bifurcation diagrams of strategies q_i for $c = 0.1$

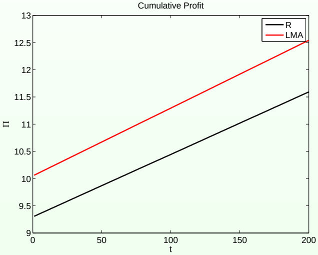


Bifurcation diagram of oligopoly composition ω for $c = 0.1$

Evolutionary fractions Rational vs. LMA



Cumulative profits of R and LMA players. For each value of βC, we plot the value of Π after $T = 1000$.



Time serie of cumulative profits of R and LMA players for $\beta C = 2.8$ (most unstable situation). After a transient of $T = 800$ iterations, the last 200 values are reported.

Second model: Evolutionary fractions BR vs. LMA

Three dimensional system with inertial mechanims (inertia α_i)

$$\begin{cases} q_1^{t+1} = q_1^t + \alpha_{BR}\left(\sqrt{\dfrac{Q_{-1}^t}{c_1}} - q_1^t\right) - Q_{-1}^t, \\ q_2^{t+1} = q_2^t + \alpha_{LMA}\left(\dfrac{1}{2}q_2^t + \dfrac{1}{2}\left(1 - c_2 Q^t\right)Q^t - q_2^t\right) \\ \omega_{t+1} = 1/(1 + \exp(\beta(\pi_2^{LMA,t}(q_1^t, q_2^t, \omega_t) - \pi_2^{R,t}(q_1^t, q_2^t, \omega_t) + C))) \end{cases}$$

where $Q_{-1}^t = (\omega_t N - 1)q_1^t + (1 - \omega_t)Nq_2^t$, $Q^t = \omega_t N q_1^t + (1 - \omega_t)Nq_2^t$ and

$$\pi_1^{BR,t}(q_1^t, q_2^t, \omega_t) = q_1^t/Q^t - c_1 q_1^t, \quad \pi_2^{LMA,t}(q_1^t, q_2^t, \omega_t) = q_2^t/Q^t - c_2 q_2^t$$

We focus on identical marginal costs $c = c_1 = c_2$.
Inertia has to be considered, otherwise only for small oligopolies ($N < 5$) equilibrium can be stable.

Proposition

The Nash equilibrium $q_i^\star = (N-1)/(N^2 c)$ is the only positive steady state, to which corresponds the equilibrium fraction $\omega^\star = 1/(1 + \exp \beta C)$

Evolutionary fractions Best Response vs. LMA

Proposition

For $N > 2$, let us define

$$\tilde{\gamma} = \frac{(N\alpha_1 - 4)(N-1)(4-\alpha_2)}{(4N - N\alpha_1 - 4)(\alpha_2 - N\alpha_2 + 4)}, \qquad \gamma(\alpha_2) = \frac{2(N-1)(8 - N\alpha_2)}{N(N(2-\alpha_2)+\alpha_2)}.$$

Then, setting $\hat{\alpha}_{BR} = 4/N$ and $\hat{\alpha}_{LMA} = 4/(N-1)$, we have

- *E^* is stable $\forall \beta C > 0 \Leftrightarrow$*

$$\begin{cases} N < 5, \\ \alpha_i \in (0,1], \end{cases} \qquad \text{or} \qquad \begin{cases} N \geq 5 \\ \alpha_{BR} \in (0, \gamma(\alpha_{LMA})], \alpha_{LMA} \in (0, \hat{\alpha}_{LMA}] \\ (\alpha_{BR}, \alpha_{LMA}) \neq (\gamma(\alpha_2), \hat{\alpha}_{LMA}). \end{cases}$$

- *E^* is unstable $\forall \beta C > 0 \Leftrightarrow N \geq 5$ and $\alpha_{BR} \in [\gamma \alpha_{LMA}, 1], \alpha_{LMA} \in [\hat{\alpha}_{LMA}, 1]$.*
- *E^* is conditionally stable on ω for*

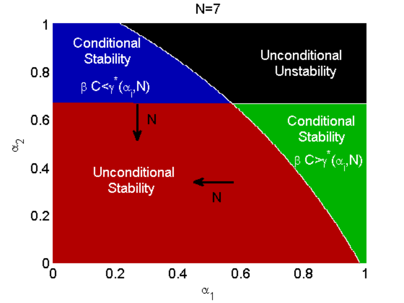
$$\beta C > \tilde{\gamma} \Leftrightarrow \begin{cases} N \geq 5, \\ \alpha_{BR} \in (\gamma(\alpha)_{LMA}, 1], \\ \alpha_{LMA} \in \left(\dfrac{-2N^2 + 16N - 16}{N^2 - N}, \hat{\alpha}_{LMA} \right), \end{cases} \qquad \beta C < \log(\tilde{\gamma}) \Leftrightarrow \begin{cases} N \geq 5, \\ \alpha_{BR} \in (0, \gamma(\alpha)_{LMA}), \\ \alpha_{LMA} \in (\hat{\alpha}_{LMA}, 8/N]. \end{cases}$$

Evolutionary fractions BR vs. LMA

Stability

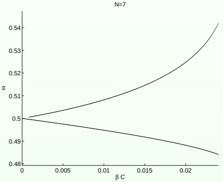
Four different situations are possible:

- Equilibrium is unconditionally stable , independently of evolutionary pressure β and informational costs C;
- Equilibrium is unconditionally unstable , independently of evolutionary pressure β and informational costs;
- Evolutionary pressure β and informational costs C are destabilizing
- Evolutionary pressure β and informational costs C are stabilizing

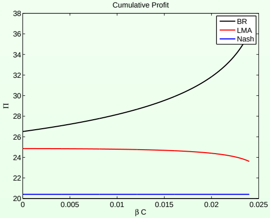


Evolutionary fractions BR vs. LMA

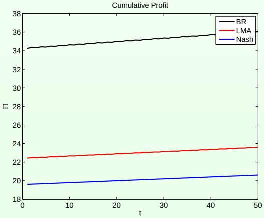
Neutrally unstable scenario ($c = 0.2, \alpha_{BR} = 0.8, \alpha_{LMA} = 0.5$)



Bifurcation diagrams of oligoply composition ω



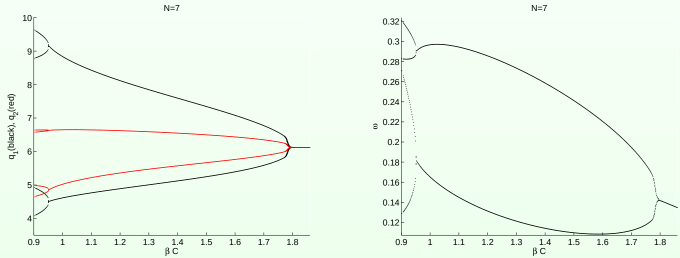
Cumulative profits of BR and LMA players. For each value of βC, we plot the value of Π after $T = 1000$.



Time serie of cumulative profits of BR and LMA players for $\beta C = 0.024$ (most unstable situation). After a transient of $T = 950$ iterations, the last 50 values are reported.

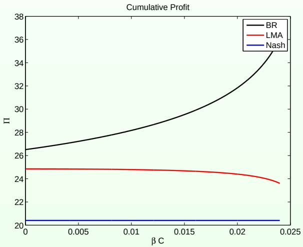
Evolutionary fractions BR vs. LMA

Stabilizing scenario ($c = 0.2, \alpha_{BR} = 1, \alpha_{LMA} = 0.5667$)

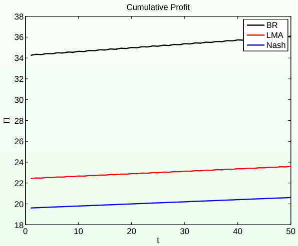


Left: bifurcation diagrams of strategies q_1 (black), q_2 (red). Right bifurcation diagram of oligopoly composition ω.

Stabilizing scenario ($c = 0.2, \alpha_{BR} = 1, \alpha_{LMA} = 0.5667$)



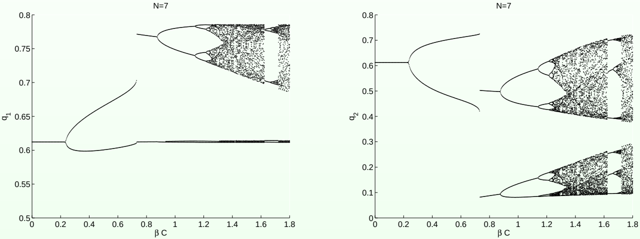
Cumulative profits of BR and LMA players. For each value of βC, we plot the value of Π after $T = 1000$.



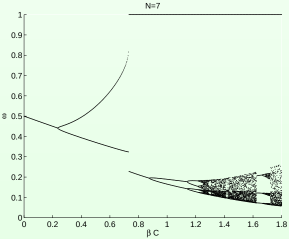
Time serie of cumulative profits of BR and LMA players for $\beta C = 0.9$ (most unstable situation). After a transient of $T = 950$ iterations, the last 50 values are reported.

Evolutionary fractions BR vs. LMA

Destabilizing scenario ($c = 0.2, \alpha_{BR} = 0.27, \alpha_{LMA} = 0.9$)



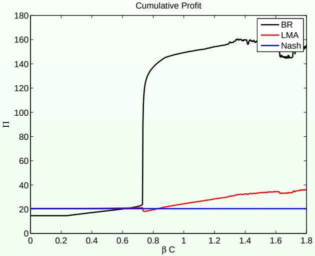
Bifurcation diagrams of strategies q_i.



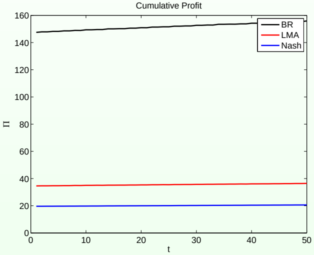
Bifurcation diagram of oligopoly composition ω

Evolutionary fractions BR vs. LMA

Simulations



Cumulative profits of BR and LMA players. For each value of βC, we plot the value of Π after $T = 1000$.



Time serie of cumulative profits of BR and LMA players for $\beta C = 1.8$ (most unstable situation). After a transient of $T = 950$ iterations, the last 50 values are reported.

Third model: Evolutionary fractions BR vs. BR

Two dimensional model

$$q_2^{t+1} = \max\left\{\sqrt{\frac{Q_{-2}^t}{c_2}} - Q_{-2}^t, 0\right\},$$

$$\omega_{t+1} = \frac{1}{1 + \exp(\beta(\pi_2^{LMA,t}(q_1^t, q_2^t, \omega_t) - \pi_2^{R,t}(q_1^t, q_2^t, \omega_t) + C))}$$

where $Q^t = \omega_t NR(q_2^t, \omega_t) + (1 - \omega_t)(N - 1)q_2^t$ and

$$\pi_1^{R,t}(q_1^t, q_2^t, \omega_t) = \frac{R(q_2^t, \omega_t)}{Q^t} - c_1 R(q_2^t, \omega_t),\ \pi_2^{BR,t}(q_1^t, q_2^t, \omega_t) = \frac{q_2^t}{Q^t} - c_2 q_2^t$$

We consider **different marginal costs**, we focus on $c_1 \geq c_2$

Proposition

The Nash Equilibirum is the only positive steady state. The explicit expression for equilibrium fraction is not available.

Evolutionary fractions Rational vs. BR

$c_1 = c_2$

Analytical results for **identical** marginal costs
Equilibrium fraction

$$\omega^\star = \frac{1}{1 + \exp \beta C},$$

Proposition

Equilibrium is stable provided that

$$\beta C < \log\left(\frac{3N - 4}{N - 4}\right)$$

- Oligopolies with $N \leq 4$ have stable equilibrium independently of the equilibrium fraction, regulated by βC.
- For $N > 4$, we always have suitable C and β that make stable the equilibrium
- As C and β sufficiently increase, the equilibrium become unstable.
- However, if $\beta C \leq log(3)$, the equilibrium is stable for all the oligopoly sizes N.
- When stability condition is violated, equilibrium loses stability through a flip bifurcation.

FIXED FRACTIONS Rational vs. Best Response

We recall stability result for fixed fraction situation

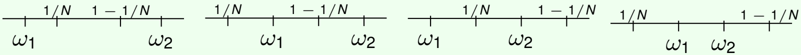
Proposition

Let

$$\omega_{1,2} = \frac{c_2\left(3c_1N - 2c_1 - c_2N - 2c_2 \pm \sqrt{2\widetilde{\Delta}}\right)}{2c_1c_2N + c_1^2N - 3c_2^2N},$$

where $\widetilde{\Delta} = c_2^2N^2 + 2c_1^2N^2 + c_1^2 + c_2^2 - 2c_1^2N - 2c_1c_2N^2 + 2c_1c_2 - 2c_2^2N$. *Then equilibrium is stable provided that* $\omega \in (\omega_1, \omega_2)$*.*

With respect to the R player fraction, four possible scenarios arise



Neutrally stable | Stabilizing | Destabilizing | Mixed

Question:
Do we have, also in the evolutionary fractions framework with different marginal costs, TWO stability thresholds, with βC acting as ω?

Evolutionary fractions Rational vs. BR revisited

Also in the evolutionary fractions R vs. BR model, we need to introduce a function to limit output levels variations to preserve strategies positivity

Improved model

Two dimensional model

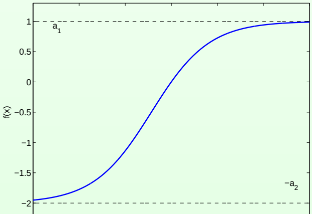
$$q_2^{t+1} = q_2^t + f(\gamma(BR(Q_{-2}^t) - q_2^t)),$$

$$\omega_{t+1} = \frac{1}{1 + \exp(\beta(\pi_2^{LMA,t}(q_1^t, q_2^t, \omega_t) - \pi_2^{R,t}(q_1^t, q_2^t, \omega_t) + C))}$$

where f is an increasing, sign preserving, bounded function and γ is the reaction speed of the BR agents.

Example: sigmoid function

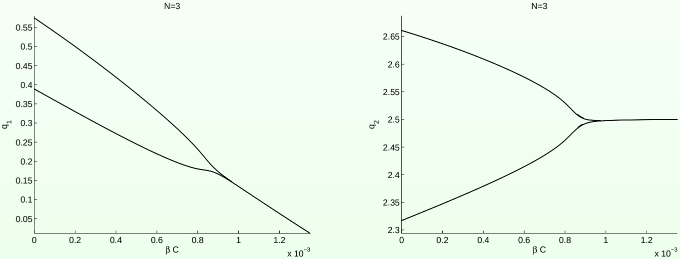
$$f(x) = a_2 \left(\frac{a_1 + a_2}{a_2 + a_1 \exp(-x)} - 1 \right)$$



Evolutionary fractions BR vs. LMA

Simulations

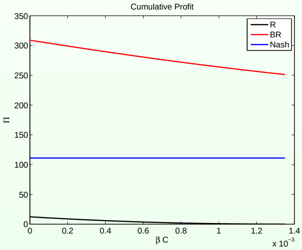
Stabilizing scenario ($a_1 = 3, a_2 = 1, \gamma = 2.65, c_1 = 0.2, c_2 = 0.1$)



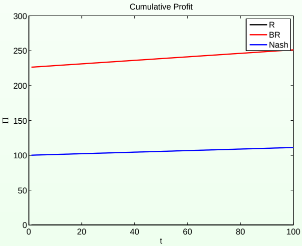
Bifurcation diagrams of strategies q_i

Evolutionary fractions R vs. BR

Stabilizing scenario ($a_1 = 3, a_2 = 1, \gamma = 2.65, c_1 = 0.2, c_2 = 0.1$)



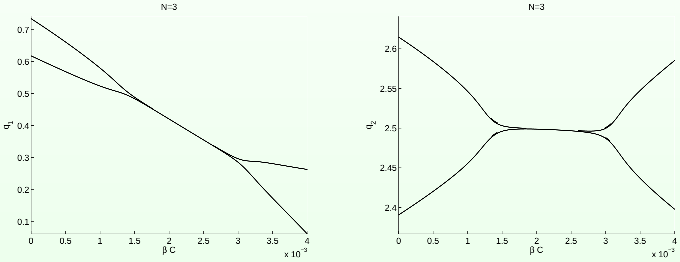
Cumulative profits of R and BR players. For each value of βC, we plot the value of Π after $T = 1000$



Time serie of cumulative profits of R and BR players for $\beta C = 0.01$ (most unstable situation). After a transient of $T = 900$ iterations, the last 100 values are reported.

Evolutionary fractions R vs. BR

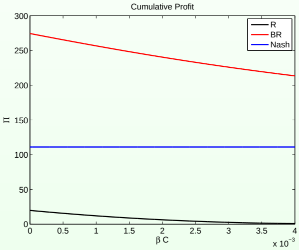
Mixed scenario ($a_1 = 3, a_2 = 1, \gamma = 2.665, c_1 = 0.18, c_2 = 0.1$)



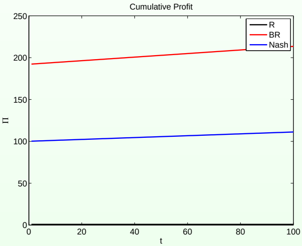
Bifurcation diagrams of strategies q_i

Evolutionary fractions R vs. BR

Mixed scenario ($a_1 = 3, a_2 = 1, \gamma = 2.665, c_1 = 0.18, c_2 = 0.1$)



Cumulative profits of R and BR players. For each value of βC, we plot the value of Π after $T = 1000$

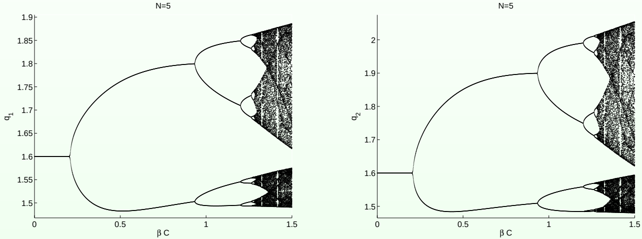


Time serie of cumulative profits of R and BR players for $\beta C = 0.01$ (most unstable situation). After a transient of $T = 900$ iterations, the last 100 values are reported.

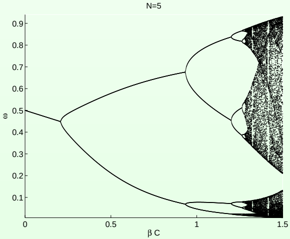
Evolutionary fractions R vs. BR

Simulations

Destabilizing scenario ($a_1 = 3, a_2 = 1, \gamma = 2.5, c_1 = 0.1, c_2 = 0.1$)

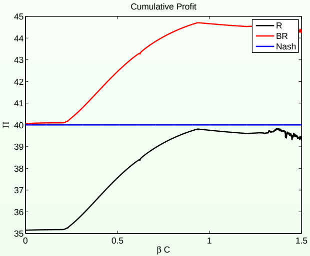


Bifurcation diagrams of strategies q_i.

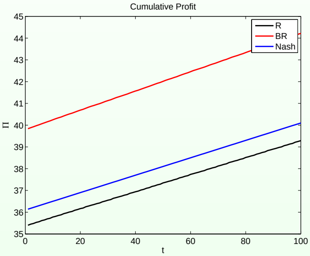


Bifurcation diagram of oligopoly composition ω

Evolutionary fractions R vs. BR



Cumulative profits of R and BR players. For each value of βC, we plot the value of Π after $T = 1000$.



Time serie of cumulative profits of R and BR players for $\beta C = 1.8$ (most unstable situation). After a transient of $T = 950$ iterations, the last 50 values are reported.