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A Hicksian multiplier-accelerator model with floor determined by capital stock

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Abstract

This article reconsiders the Hicksian multiplier-accelerator model with the "floor" related to the depreciation on actual capital stock. Through the introduction of the capital variable, a growth trend is created endogenously by the model itself, along with growth rate oscillations around it. The "ceiling" can be dispensed with altogether. As everything is growing in such a model, a variable transformation is introduced to focus relative dynamics of the income growth rate and the actual capital output ratio.

Keywords: Hicksian multiplier-accelerator model; Floor; Capital stock

1. Introduction

1.1. Business cycle theory

Business (or trade) cycle theory has been a most vital research area in economics for more than a Century, and the explanations offered have been very diverse. In various periods, general surveys have been produced, such as the excellent ones by von Haberler (1937)and

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several following editions. A recent general review is Glasner (1997). The more sophisticated business cycle theories use different mechanisms to explain upswing and downturn, involving both supply and demand side, the monetary issues, and business confidence and expectations.

The present paper has no intention to address such enormous diversity, but keeps to the Samuelson–Hicks model that grew out of Keynesian macroeconomics. This origin shows up in the fact that the cycles are generated solely through the demand side. The model distinguishes itself by letting one single mechanism be responsible for both upswing and downturn. This may be seen as a bit unsophisticated and mechanistic, but lends the model a certain elegance and unity of structure. It is also quite rich in potential results and has called forth a considerable literature since its evolution in the period 1939–1950. It is still interesting to see what it can do.

The present paper has a double scope. *First*, the Hicksian "floor" is related through a depreciation factor to the actual capital stock that results from the successive investments generated by the model. This seems to be a most reasonable assumption, as the "floor" is defined to be maximum disinvestment when there is no reinvestment at all. *Second*, as this, unlike the traditionally assumed fixed, "floor" can result in unbounded deviation of all variables, a new method of relative dynamics is suggested. The strategy is based on relative growth rates instead of the variables themselves. Using this we find that the model proposed can produce growth rate cycles of any periodicity. These growth rate cycles are always bounded (even without any "ceiling"), whereas the income variable itself shows exponential growth (with cyclic deviations). The method therefore also provides a synthesis between growth theory and business cycle theory.

1.2. The Samuelson–Hicks model

The objective of the present article thus is to reconsider the Hicksian multiplier-accelerator model of business cycles. This model, introduced by Samuelson (1939), was based on two interacting principles: consumers spending a fraction c of past income, $C_t = cY_{t-1}$, and investors aiming at maintaining a stock of capital K_t in given proportion a to the income Y_t to be produced. With an additional time lag for the construction period for capital equipment, net investments, by definition the change in capital stock, $I_t = K_t - K_{t-1}$, become $I_t = a(Y_{t-1} - Y_{t-2})$. As income is generated by consumption and investments, $Y_t = C_t + I_t$, a simple feed back mechanism $Y_t = (c + a)Y_{t-1} - aY_{t-2}$ was derived. It could generate growth or oscillations in income.

The consumption component was referred to in terms of the "multiplier" and the investment component in terms of the "principle of acceleration". To be quite true to history, two remarks should be added: first, Samuelson applied the accelerator to consumption expenditures only; the above described application to all expenditures is due to Hicks (1950), discussed in more detail below. The difference in terms of substance is marginal. Second, all these models contain an additional term called "autonomous expenditures" (i.e., government expenditures, any consumption independent of income, and investments not dependent on the business cycles generated by the model). The multiplier, $1/(1 - c)$ (i.e., the multiplicative factor applied to such autonomous expenditures), resulted in an equilibrium income, or a particular solution to the difference equation. Income could then be redefined as a deviation from this equilibrium income, and the original equation regained, though now in income deviations from the equilibrium. This makes sense of negative values of income that inevitably result from the above difference equation.

However, there was more to the Hicksian reformulation. He realized that there must be limits to the accelerator-based investment function. In a depression phase $I_t = a(Y_{t-1} - Y_{t-2}) < 0$, and it can even happen that income (=production) decreases at a pace so fast that more capital can be dispensed with than disappears through natural wear. As nobody actively destroys capital to such an end, there is a lower limit to disinvestment, the so called "floor", fixed at the (negative) net investment that occurs when no worn out capital is replaced at all.

At the same time Hicks suggested that there be a "ceiling" at full employment, when income could not be expanded any more. Hicks never assembled the pieces to a complete formal model. It is clear that the floor constraint is applied to the investment function, so it becomes something like $I_t = \max\{a(Y_{t-1} - Y_{t-2}), -I^{\mathrm{f}}\}$, where I^{f} is the absolute value of the floor disinvestment. On the other hand it is not quite clear what the ceiling is applied to. Most likely Hicks thinks of it as applied to income, so that the income formation equation is changed to $Y_t = \min\{cY_{t-1} + I_t, Y^{\mathrm{c}}\}$, where Y^{c} is the full employment capacity income. Gandolfo (1985) interpreted the model this way, and Hommes (1991) gave a more or less full analysis of this model. The above symbols conform to Hommes's notation. It seems that the first formalization in this format is due to Rau (1974).

It is not obvious in such a formulation which agents cut their expenditures when the ceiling is reached. As an alternative one might incorporate it in the investment function, along with the floor, thereby implying that it is the investors who abstain from further investment when they realize that full employment is reached. This was the choice of Goodwin (1951) and many other students of the Hicksian business cycle machine, including the present authors (see Puu, 1989; Sushko et al., 2003).

To the complete model also belong the autonomous expenditures which were already mentioned. These can be constant, or growing. In his verbal description Hicks seems to have been in favour of exponentially growing autonomous expenditures, as he obviously wanted to model both growth and cycles around a growing trend. Growth, however, was not created endogenously by the model, as the cycles were, but introduced ad hoc. To make this type of model suitable for analysis, the floor and ceiling must be assumed to be growing too, at the same rate as the autonomous expenditures, and this seems to have been Hicks's own tacit assumption. The assumption of equal growth rates is fairly arbitrary. Gandolfo modelled it this way, though Hommes preferred to analyze the stationary case where autonomous expenditures, floor, and ceiling were all constant. A recent mathematical analysis of Gandolfo's case may be found in Gallegati et al. (2003).

1.3. *A suggested reformulation*

The assumed growth of the floor along with the autonomous expenditures is particularly problematic. A growing capital stock, as a result of growing autonomous expenditures, should *increase* the absolute value of maximum disinvestment, so it is not only arbitrary to assume the floor to grow at the same rate as the autonomous expenditures, but the change even *goes in the wrong direction*. The floor would rather be *decreasing* with capital accumulation. For this reason it seems to be important to make capital an explicit variable in the model, and to relate the floor directly to capital stock; put $I_t^{\mathrm{f}} = rK_t$, where K_t is capital stock and r is the rate of depreciation. Just to avoid misunderstanding, it should be understood that, although the income variable, as we have seen, can be negative in the sense of a negative *deviation* from equilibrium, capital by necessity always is nonnegative.

Making this change to the model, we get the extra benefit that the growing trend need not be introduced exogenously. It would result *within* the model through capital accumulation. The model hence *explains* both the growth trend and the business cycles that take place around it.

As for the ceiling, at least for a start, we dispense with it. It was noted by Duesenberry (1950) in his review of Hicks's book 1950, that both floor and ceiling were not always needed for bounded motion and that in particular the ceiling could be dispensed with. Allen (1957, p. 220) gives a very clear account of the argument: "*On pursuing this point, as Duesenberry does, it is seen that the explosive nature of the oscillations is largely irrelevant, and no ceiling is needed. A first intrinsic oscillation occurs, the accelerator goes out in the downswing, and a second oscillation starts up when the accelerator comes back with new initial conditions. The explosive element never has time to be effective—and the oscillations do not necessarily hit a ceiling*".

Of course, with accumulating capital, the floor is no longer fixed, and the growth of capital stock allows increasing amplitude swings around the growth trend for which it is also responsible. Growth is something that economists regard as a good feature for a model, but it is no good for the use of mathematical methods which favour the study of fixed points and their destabilization, stationary cycles, quasiperiodicity, and chaos. To make the model suitable for standard analysis, we focus on relative dynamics, the rate of growth of income, and of the actual capital/output ratio.

Before this reduction to relative dynamics, we, however, have to state the complete model with the explicit inclusion of the stock of capital.

2. The model

2.1. The absolute growth dynamics

Let us first just restate the consumption function

$$C_t = cY_{t-1}, \tag{1}$$

and the investment function

$$I_t = \max\{a(Y_{t-1} - Y_{t-2}), -rK_{t-1}\}, \tag{2}$$

where c, a and r are parameters such that $0 < c < 1$, $a > 0$, $0 < r < 1$.

We now also need a relation for capital stock updating

$$K_t = K_{t-1} + I_t, \tag{3}$$

which just says that capital stock changes with net investments according to (2), accelerator generated as $K_t = K_{t-1} + a(Y_{t-1} - Y_{t-2})$, or, in the case when the floor is activated, just decays, like a radioactive substance as $K_t = (1-r)K_{t-1}$. As there is no ceiling, the income generation equation reads

$$Y_t = C_t + I_t. \tag{4}$$

Eqs. (1)–(4) now define the complete system. It is easy to see through numerical studies that the model can create a process of accumulating capital, along with a growth trend in income, and this without any growing autonomous expenditures at all. Further, the model creates growth cycles around these secular trends.

2.2. *Fixed points and their stability*

Let us first note that if

$$a(Y_{t-1} - Y_{t-2}) + rK_{t-1} \geq 0, \tag{5}$$

then the first alternative in (2) applies. Let us call the part of phase space where the inequality (5) is satisfied **Region I**. Then, eliminating the consumption and investment variables through substitution from (1) into (3) and (4), we see that in **Region I** the system is defined by

$$K_t = K_{t-1} + a(Y_{t-1} - Y_{t-2}), \tag{6}$$

$$Y_t = cY_{t-1} + a(Y_{t-1} - Y_{t-2}). \tag{7}$$

Let us now look at the other alternative, where the second branch of (2) is activated (i.e., $I_t = -rK_{t-1}$). This occurs when

$$a(Y_{t-1} - Y_{t-2}) + rK_{t-1} < 0. \tag{8}$$

Let us call this **Region II**. From (1) to (4) we get

$$K_t = (1-r)K_{t-1}, \tag{9}$$

$$Y_t = cY_{t-1} - rK_{t-1}. \tag{10}$$

It is easy to find the fixed points for (6)–(7). From (7), there is just one fixed point for income, $Y_t = Y_{t-1} = Y_{t-2} = 0$. Next, putting $Y_{t-1} = Y_{t-2}$ in (6), we conclude that $K_t = K_{t-1}$, that is *any* (positive) capital stock may be an equilibrium stock. Simulation experiments indicate that, depending on the dynamical process (i.e., on the initial conditions), the stock of capital may end up at different equilibrium values. As for stability, this also implies that, if there is some perturbation of the capital stock, then the process will again end up at a new equilibrium stock. However, income always goes to the single zero equilibrium. This, of course, is true only if the equilibrium is *stable*.

It is also obvious that the system (9)–(10) has only one fixed point located at the origin of phase space.

Let us first investigate the stability of the system (6)–(7). As (7) is independent of the capital stock, we can study this single equation alone. However, we have to observe that (7) is a linear *second order* difference equation. Writing down the Jacobian matrix of (7) and the corresponding characteristic equation, one can easily get its roots, or eigenvalues:

$$\lambda_{1,2} = \tfrac{1}{2}(a+c) \pm \tfrac{1}{2}\sqrt{(a+c)^2 - 4a}. \tag{11}$$

From (11) we can immediately see that the zero fixed point is a node if $(a+c)^2 - 4a > 0$ and a focus if $(a+c)^2 - 4a < 0$. It is stable iff $|\lambda_{1,2}| < 1$. The latter condition holds for the parameter ranges: $c < 1$, $a < 1$, $a > -(1+c)/2$. Taking into account the feasible parameter range, we conclude that the *stability region* for the fixed point of (7) is

$$0 < c < 1, \qquad 0 < a < 1.$$

Because of the linearity, the system (6)–(7) is a contraction for the above parameter range not only at the fixed point but in the entire **Region I**.

The system (6)–(10) becomes an expansion if $|\lambda_{1,2}| > 1$, which happens for $a > 1$.

Now let us check the stability of the system (9)–(10). Its eigenvalues are $\mu_1 = c$ and $\mu_2 = 1 - r$. As $0 < r < 1$ and $0 < c < 1$, we note that both eigenvalues are positive and less than unity. Accordingly, the system (9)–(10), defined in **Region II**, is a contraction.

The economics of this is that in **Region II** the system tends to equilibrium, with zero capital and zero income. The stability of the map (9)–(10) would be a big problem if the process were not easily mapped *back* into **Region I**, where the fixed point, as we have seen, may be unstable. Then the process can jump back and forth between the Regions any number of times, so being kept going forever. If

$$r < 1 - c,$$

$$(a+c)^2 - 4a < 0,$$

$$a > 1,$$

hold, then, as proved in Appendix A (available on the JEBO website), this jumping between regions occurs in finite numbers of iterations for *any* initial conditions we may care to choose. The first condition states in terms of subject matter that the rate of capital *depreciation* r is *less than* the rate of *saving* $1 - c$. This seems to be a condition fairly likely to be fulfilled in most cases.

2.3. The fixed point bifurcation

As we have seen, only the fixed point of (7) may become unstable. Indeed, at

$$a = 1, \tag{12}$$

the eigenvalues (11) are complex conjugate and have unitary modulus. Thus, the fixed point has a bifurcation analogous to the Neimark bifurcation. At the bifurcation we can write the eigenvalues (11) as $\lambda_{1,2} = \cos\theta \pm i \sin\theta$, where

$$\cos\theta = \tfrac{1}{2}(a+c), \tag{13}$$

and

$$\sin\theta = \frac{1}{2}\sqrt{4a-(a+c)^2}.$$

In the case that there is a rational rotation around the fixed point with a rotation number m/n, the solution $\theta = 2\pi m/n$ holds. Then, using (13) and (12), we get the exact value of the parameter c which corresponds to the rotation number m/n:

$$c = 2\cos\left(\frac{2\pi m}{n}\right) - 1. \tag{14}$$

For instance, at the bifurcation (12) the rotation number m/n, where $m = 1$ and $n = 1,\ldots,6$ holds for $c = 1, -3, -2, -1, (\sqrt{5}-3)/2 < 0$ and 0, respectively. Obviously, these values are out of the admissible parameter range, so $m/n = 1/6$ is the lowest basic resonance that falls into the admissible parameter range $0 < c < 1$, starting at $c = 0, a = 1$. We see the points c according to (14) for $n = 6,\ldots,15$ marked by the ascending sequence of circles on the $a = 1$ line in Fig. 3. The global dynamics of the system (1)–(4) at $a = 1$ is briefly described in Appendix B (also available on the JEBO website).

Our study of this bifurcation was confined to the fixed point of (6)–(7), and the reader may note that then (7) is no different from the original Samuelson–Hicks model. However, globally, the moving floor component, which is related to capital stock, is important, and

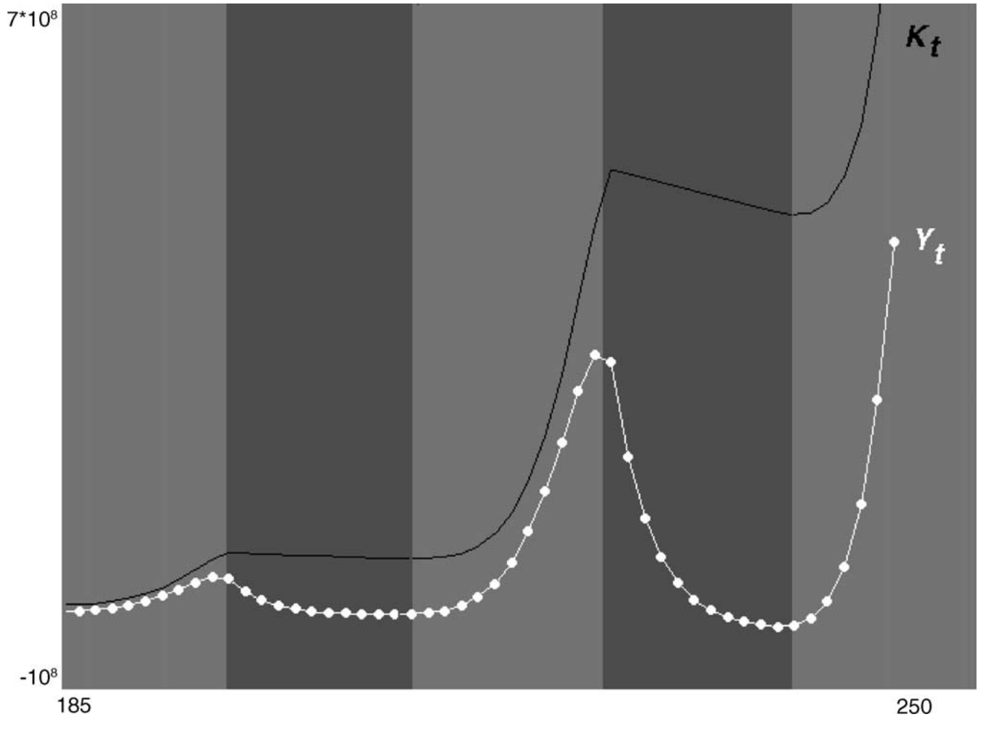

Fig. 1. Time series of oscillating income around a growth trend (white curve), and growing capital stock with recessions (black).

eventually it is the growing capital stock that is responsible for the secular growth created by the proposed model for $a > 1$.

In Fig. 1, we show typical growth paths for capital and income. Income (or rather its deviation from equilibrium) is the white curve, oscillating around the zero value, with increasing height of the peaks and increasing amplitude. The picture was calculated for parameters $a = 2.25$, $c = 0.65$, $r = 0.01$. As we will see below, this case results in a 23-period growth cycle around a growing trend. As the growth rate makes a constant amplitude oscillation, income itself oscillates with increasing amplitude. As for capital, the black curve shows an ever increasing trend with periodic recessions. As we see, the slope of these increases. This is a reflection of the fact that the floor restriction slackens with increasing capital, and larger disinvestment is allowed with growing capital stock. The falling segments occur where the floor is activated. In Fig. 1, we emphasized this through the colour of the vertical strips, the darker shade indicating that the floor is activated (i.e., that the system is in **Region II)**. In the bright strips the system is in **Region I**.

3. Stationary relative dynamics

It is convenient to study the phenomena in the above way only when $a \leq 1$, that is when the zero fixed point for income is stable. If it is not, then there is exponential growth in the model, so all variables eventually explode. If we try to display time series such as Fig. 1 over prolonged periods, we just get horizontal lines first and then oscillations growing so fast that they break any frame for the figure. In the same way, in the phase diagrams we only see spiralling motions that move out from the frame, even if there is something more to see, such as growth cycles, or a quasiperiodic trajectory around a trend. However, we cannot catch these visually, neither can we use standard mathematical analyses for such growing systems.

We would need to find some variable transformations that make the oscillations around the exploding trends stationary periodic, quasiperiodic, or chaotic, whatever they are, but such that they can be studied by standard methods. As the growth trend is not given by any exogenous term growing at a *given* rate, unlike the Gandolfo version, we have to *define some transformed variables within the model* such that they undergo stationary cyclic or other motion.

As a pedagogical device, one of the present authors (Puu, 1963) 40 years ago suggested studying the evolution of new relative variables for the original Samuelson–Hicks model: $Y_t = (c + a)Y_{t-1} - aY_{t-2}$ through defining $y_t := Y_t/Y_{t-1}$. The objective was to avoid complex numbers in the study of second order difference equations through making the iteration one dimensional, though non-linear. In fact the original model becomes $y_t = (c + a) - a/y_{t-1}$. However, this strategy makes, for instance, cyclic variations in the income growth variable really become cyclic. We now have a system with capital as an additional variable, but never mind, we can work out a suitable relative representation for the capital variable as well.

Let us again eliminate investments and consumption in (3)–(4) through substitution (1)–(2) and then define new variables. Let

$$x_t := K_t/Y_{t-1} \tag{15}$$

and

$$y_t := Y_t/Y_{t-1}. \tag{16}$$

These new variables are the actual capital/output ratio, as distinguished from the optimal ratio a, and the relative change of income from one period to the next, quite as in the framework suggested in Puu (1963). Again there is a reduction of dimension for the system, now from 3 to 2. Using these new variables defined in (15)–(16), we can restate the dynamical system as follows. Suppose we have

$$x_{t-1}(a(y_{t-1}-1)+rx_{t-1}) \geq 0.$$

This corresponds to **Region I** in the original model as we see from (5). The reader may wonder about the occurrence of x_{t-1} as a multiplicative factor in this new region definition. The reason for it is that division of the inequalities (5) or (8) through by Y_{t-1}, which can take on a negative sign, may change the sense of the inequality. For this reason we would have to split the two regions in four, but we can avoid this complication if we multiply the left hand sides of the inequalities (5) or (8) through by a variable defined in the new system (15)–(16), which always takes the sign of Y_{t-1}. Such a variable is x_t, as we see from (15), because capital stock K_t is always nonnegative.

Using (15)–(16), the system in **Region I** can then be written

$$x_t = \frac{x_{t-1}}{y_{t-1}} + a\left(1-\frac{1}{y_{t-1}}\right), \tag{17}$$

$$y_t = c + a\left(1-\frac{1}{y_{t-1}}\right). \tag{18}$$

Suppose that, on the contrary,

$$x_{t-1}(a(y_{t-1}-1)+rx_{t-1}) < 0.$$

This obviously corresponds to **Region II** in the original model. Then (17)–(18) are replaced by

$$x_t = (1-r)\frac{x_{t-1}}{y_{t-1}}, \tag{19}$$

$$y_t = c - r\frac{x_{t-1}}{y_{t-1}}. \tag{20}$$

3.1. Fixed growth points

Written as relative dynamics (17)–(20), the new system has fixed points as well. In terms of economics they represent equilibrium growth rates. Consider first **Region I**. Putting $x_t = x_{t-1} = x$, $y_t = y_{t-1} = y$ in (17)–(20), we obtain

$$x = a \tag{21}$$

and

$$y^2 - (a+c)y + a = 0. \tag{22}$$

According to (21) the equilibrium capital/output ratio x equals the optimal one as indicated by the accelerator a, which intuitively seems most reasonable. As for (22), it determines either two real or two complex conjugate equilibrium values for the relative income growth rate

$$y_{1,2} = \tfrac{1}{2}(a+c) \pm \tfrac{1}{2}\sqrt{(a+c)^2 - 4a}. \tag{23}$$

We may observe that (22) is equal in form to the characteristic equation for the original Samuelson–Hicks model, so the growth rate equilibria exist whenever the original multiplier-accelerator model has solutions with two *real* roots.

A saddle-node bifurcation resulting in the appearance of the fixed points (x, y_1) and (x, y_2) obviously occurs when

$$(a+c)^2 = 4a.$$

For $(a+c)^2 < 4a$ no fixed point exists for the income growth rate, whereas for $(a+c)^2 > 4a$ there are two fixed points, one stable and one saddle. This is easy to see from (18), as the derivative (putting $y_{t-1} = y$ at equilibrium) is

$$\frac{\partial y_t}{\partial y_{t-1}} = \frac{a}{y^2}.$$

Using the larger root according to (23), we get

$$\frac{a}{y_1^2} = \frac{a+c-\sqrt{(a+c)^2-4a}}{a+c+\sqrt{(a+c)^2-4a}} < 1,$$

and, using the smaller

$$\frac{a}{y_2^2} = \frac{a+c+\sqrt{(a+c)^2-4a}}{a+c-\sqrt{(a+c)^2-4a}} > 1.$$

So far we only checked the stability of (23). We should complete the discussion by differentiating (17)–(18) and again deleting the index in the right-hand side,

$$\frac{\partial x_t}{\partial x_{t-1}} = \frac{1}{y},$$

so stability for the capital variable at the fixed point depends on the equilibrium value of y. From (23) we see that if the roots are real, then both are positive (due to the minus term under the root sign). Further, whenever $c < 1$, which it must be to make any sense, we have

$1 < y_2 < y_1$ according to (23). Hence,

$$\frac{1}{y_1} < 1$$

and

$$\frac{1}{y_2} < 1.$$

Thus, the fixed point (x, y_1) is a stable node while (x, y_2) is a saddle. This is a somewhat pedestrian way of checking stability, for income and capital separately, but quite as in the original model, the Jacobian matrix of the system (17)–(18) is triangular, making the main diagonal derivatives actually the eigenvalues.

3.2. Fixed decline points

As was the case before the reduction to relative dynamics, the model also has fixed points in **Region II** (i.e., where (19)–(20) hold). Put again $x_t = x_{t-1} = x$, $y_t = y_{t-1} = y$ in (19)–(20). The system obviously has the fixed point

$$x = \frac{1-r}{r}(c + r - 1),$$

and

$$y = 1 - r.$$

If we recall that the new y-variable is the income ratio for one period to the previous, then we realize that the fixed point means income change at the constant rate $(1 - r)$, which is the rate of capital depreciation. The accelerator then gives the same investment as the floor condition, and capital is just depreciating at the same rate. When the system is in this new fixed point, corresponding to a negative constant growth rate $-r$, then the system is on its way towards the zero equilibrium for capital and income according to the original setup in **Region II**. The eigenvalues in the fixed point are 0 and $c/(1 - r)$. It is stable if $r < 1 - c$, but for the same condition this fixed point does not belong to **Region II.** Thus, the system switches between the two definition regions, so the attractivity of the new fixed point does not really matter.

There is an additional fixed point in **Region II**:

$$x = 0$$

and

$$y = c.$$

The eigenvalues are 0 and $(1 - r)/c$, so as the second eigenvalue is the reciprocal of the corresponding one in the previous case, it exchanges stability with the aforementioned

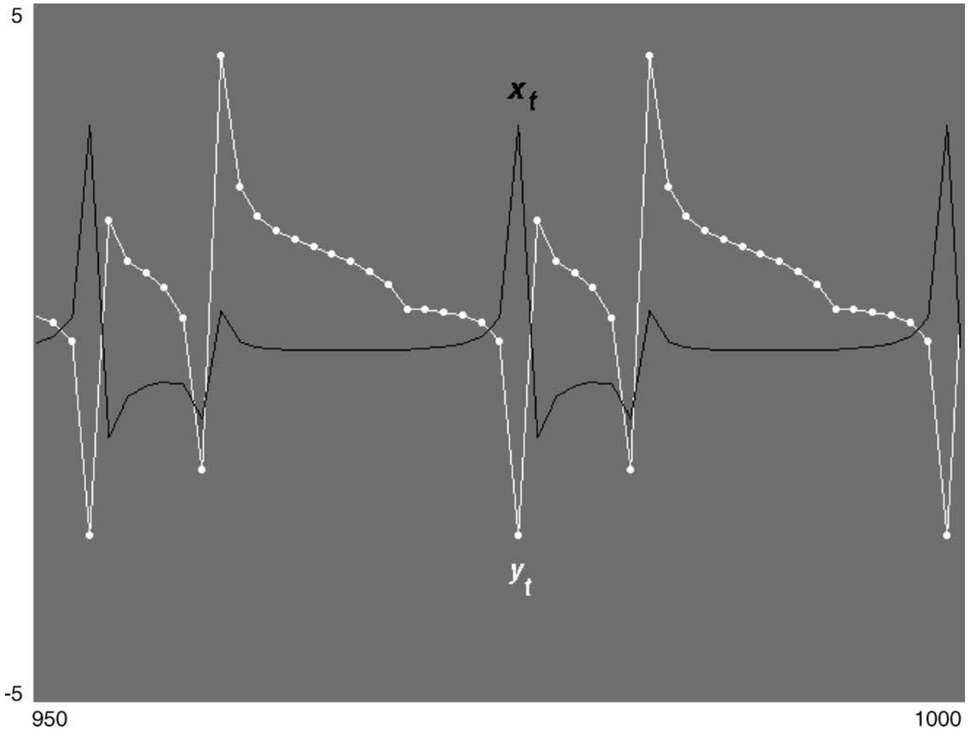

Fig. 2. The 23-period cycle in the relative dynamics system (same as in Fig. 1).

fixed point. The economics of this fixed point is a classical multiplier process converging to equilibrium, with zero capital. In a sense the two fixed points are the same. They both eventually lead to zero income (deviation from equilibrium due to autonomous expenditures) and zero capital. However, the paths of approach are different, and we now focus on growth rates, not the income and capital variables themselves.

3.3. Periodicity

So far we studied the fixed points of the relative dynamics system. Numerical experiment indicates cycles in the growth rates. For instance, as we see in Fig. 2, the parameter combination $a = 2.25$, $c = 0.65$, $r = 0.01$ results in a 23-period growth cycle. The case was illustrated in Fig. 1, though it was difficult to see any regular pattern of oscillation in that picture. On the white income growth trace in Fig. 2 we marked the successive iterates with little circles, so it is easy to identify the periodicity. The black trace is for the capital to income ratio. It is worth noting that the long horizontal sections coincide with the level of the accelerator coefficient, which seems to catch the trajectory for considerable periods of time.

To see some more possibilities, we show a bifurcation diagram in Fig. 3. It represents the parameter plane a (horizontal), c (vertical). The third parameter r is fixed at 0.01. Changing this parameter, the rate of depreciation, changes little in the picture. The changes

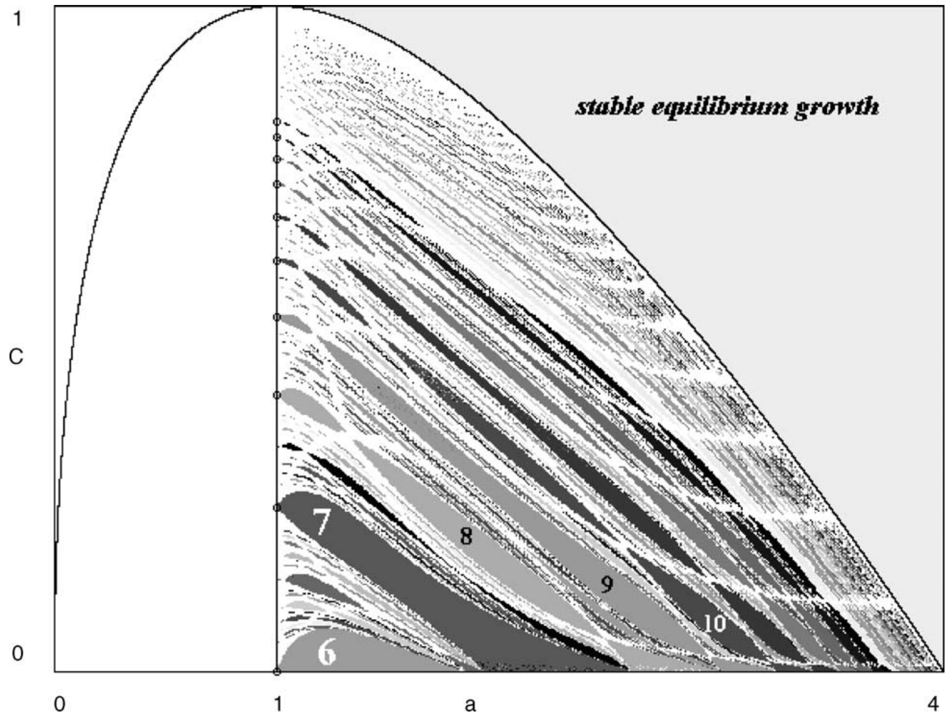

Fig. 3. Bifurcation diagram. Tongues of periodicity and stable equilibrium region.

mainly concern the structure of white streaks that seem to run through the points where the periodicity tongues look twisted.

The tongues were computed for the relative dynamics model (17)–(20), for periodicities 1–45. However, the starting points for the periodic tongues were computed in the first model in (14), as explained above. On the bifurcation line, the original system *is* periodic (for rational rotation numbers), so this is not surprising. To the left of the vertical line at $a = 1$, the zero fixed point of the original system is stable.

In Fig. 3, we also see a parabola turned upside down. It is the locus of points $(a + c)^2 = 4a$, the borderline between complex (below) and real (above) roots to the characteristic equation. To the left of the line $a = 1$, the attracting zero fixed point is a focus below the parabola and a node above it. To the right of the line $a = 1$, we have the tongues of periodicity drawn below the parabola. Above it there is the area where the roots (23) are real, and there is an attractive fixed point for the relative dynamics system. The original system then goes to a stable growth rate at the larger value of (23).

4. Absolute and relative variables

We should now clarify a little more the relation between the 3D map in original absolute variables $(K_{t-1}, Y_{t-1}, Y_{t-2}) \mapsto (K_t, Y_t, Y_{t-1})$ and the derived 2D map in relative variables

$(x_{t-1}, y_{t-1}) \mapsto (x_t, y_t)$. The need for the relative system arises in the case when the accelerator generated motion is explosive (i.e. when $a > 1$ holds). If so, then the absolute variables always explode to infinity. This can occur in two ways: when growth rates $y_{1,2}$ according to (23) are real, then asymptotically the absolute system settles to exponential growth at the higher (stable) rate. If they are conjugate complex, which is the more interesting case, then the relative system settles at oscillatory motion in the growth rates. As we saw in the bifurcation diagram Fig. 3, this oscillatory motion is predominantly periodic. The location of the parameter point that our example displayed in Figs. 1 and 2, is within the 23-period tongue and may be difficult to detect because the tongue is so tiny and stuffed among other tongues. Easier to find are parameter points such as $a = 1.25$, $c = 0.05$, which is located in the six-period tongue, or $a = 1.25$, $c = 0.25$, which is located in the seven-period tongue, both leading to regular motion of low periodicity. Unfortunately these low periodicities correspond to parameter points for which the propensity to consume is unrealistically low, but they may yet be useful for further numerical studies.

The periodicities are a novel feature of the present model when the floor is tied to the stock of capital. In the original Samuelson–Hicks model, $Y_t = (c + a)\,Y_{t-1} - aY_{t-2}$, periodic solutions do *not* occur. The closed form solution is a product of a growth factor ρ^t, where $\rho = \sqrt{a}$, and an oscillatory factor, $\cos(\theta t)$, where $\theta = \arccos((a + c)/(2\sqrt{a}))$. As it is most unlikely that $\theta = 2\pi m/n$ (i.e., that the oscillation frequency is a rational multiple of 2π), oscillatory motion is always *quasi* periodic. In the (a, c)-parameter plane, $\cos(2\pi m/n) = (a + c) / \left(2\sqrt{a}\right)$ for m, n integers results in a set of thin *curves*, not in a set of thick tongues with nonzero area measure.

As we saw, the growth factor for this original model is $\sqrt{a}$. Numerical studies indicate that when the floor is activated in periodic solutions to our model, the growth factors are considerably smaller, even if there still is exponential growth as long as $a > 1$ holds (see below for a numerical example.)

For the contrary case $a < 1$, with a stable zero fixed point for income, we still need the model in absolute variables because the relative system then makes no sense.

However, even with explosive motion we are still primarily interested in the absolute variables, income rather than its growth rate (note that motion of the relative variables is *always bounded*, even when the absolute variables explode.) It is then worth noting that the original system for the evolution of capital and income can always be retrieved from any solution to the relative system. One could arbitrarily put income in the first time period equal to unity and then obtain its evolution as a continued product of the growth factors calculated from the relative model. A different initial first period income would then just scale the entire time series obtained (up or down) in proportion.

If the solution for the growth rates is periodic, then the continued product over a complete cycle takes a given *constant* value, no matter at which observation we start. This constant for our exemplificatory 23-period cycle is for instance 7.400966. . ., which, averaging geometrically over a complete cycle, yields a growth factor of 1.0909. . . per period for the income variable. This can be compared to the growth factor for the original Samuelson–Hicks model, which equals $\sqrt{2.25} = 1.5$ per period, as we had $a = 2.25$, and a growth factor exceeding 11,200 over one complete cycle!

What has been said also means that we can regenerate all further cycles in the time series from the values for just one cycle through multiplication by this continued product over one cycle. Note how this lowering of growth rates in the present model, as compared to the original Samuelson–Hicks model, strengthens the Duesenberry argument, as it is now much less likely that the present moderate growth rates will hit any "ceiling" growing with labour force or the like.

5. Conclusion

Above we suggested a business cycle model, consisting of about half of the bits and pieces proposed by Hicks in his classical work. In particular, the "floor" was retained, but the "ceiling" omitted, in concordance with Duesenberry's argument.

As a new element, the floor was tied to actual capital stock through a fixed depreciation factor. Hence, in the process of growth with capital accumulation, the level of the "floor" changes, thus allowing increasing amplitude oscillations around the growing trends for income and capital. Further, the secular growth trends are created within the model and need not be introduced in terms of exogenous growing expenditures.

In order to analyze the oscillating growth rates around the rising trends, a transformed system of relative dynamics, in terms of the income growth rate and the capital/income ratio, was proposed. This reduced the system from three to two dimensions, though it also introduced new complexity through transforming linear relations to nonlinear with possibly vanishing denominators.

The relative dynamics system and the detailed structure we see in Fig. 3 have some intricacy of a mathematical nature, which the authors intend to study more closely in a coming publication.

Appendix A

Let us rewrite the system of Eqs. (1)–(4) in the form of an iterated map. We introduce the following variables:

$$x_t = Y_{t-1},$$

$$y_t = Y_t,$$

$$z_t = K_t.$$

For convenience we skip the index and use a dash to denote the one period advancement operator. Then the system (1) can be written as a three-dimensional piecewise linear continuous map F given by two linear maps F_1 and F_2 which are defined, respectively, in **Region I** (denoted R_1) and **Region II** (denoted R_2):

$$F : \begin{cases} (x', y',' z') = F_1(x, y, z), & \text{if } (x, y, z) \in R_1, \\ (x', y',' z') = F_2(x, y, z), & \text{if } (x, y, z) \in R_2, \end{cases} \tag{A.1}$$

where

$$F_1 : \begin{cases} x' = y \\ y' = -ax + (c+a)y, \\ z' = a(y-x) + z \end{cases} \quad R_1 = \left\{(x, y, z) : z \geq \frac{a}{r}(x-y),\ z > 0\right\}$$

$$F_2 : \begin{cases} x' = y \\ y' = cy - rz, \\ z' = z(1-r) \end{cases} \quad R_2 = \left\{(x, y, z) : 0 < z < \frac{a}{r}(x-y)\right\}.$$

Here a, c and r are real parameters: $0 < c < 1, a > 0, 0 < r < 1$; x, y and z are real variables: $z > 0$ (one can check that if an initial value of z is positive, then it remains positive under the iterations).

The purpose of this consideration is to give conditions and explain a mechanism of constant shifting between the two regions that keeps the process going for $a > 1$ without converging to zero equilibrium.

To proceed we need to reformulate some results described in Section 2.2.

The fixed point (x^*, y^*, z^*) of F_1 is any point of the z-axes: $(x^*, y^*, z^*) = (0, 0, z^*)$ where $z^* \geq 0$. The eigenvalues of F_1 are

$$\lambda_1 = 1, \quad \lambda_{2,3} = \tfrac{1}{2}(a+c) \pm \tfrac{1}{2}\sqrt{(a+c)^2 - 4a}$$

and the corresponding eigenvectors are

$$v_1 = (0, 0, 1), \quad v_{2,3} = (1, \lambda_{2,3}, a).$$

The fixed point $(0, 0, z^*)$ is stable iff $|\lambda_{2,3}| < 1$, which holds for $a < 1$. It is a stable node for $(a+c)^2 > 4a$ and a stable focus for $(a+c)^2 < 4a$. For $a > 1$ the fixed point $(0, 0, z^*)$ is unstable.

The fixed point (x^*, y^*, z^*) of the map F_2 is at the origin: $(x^*, y^*, z^*) = (0, 0, 0)$. The eigenvalues of F_2 are

$$\mu_1 = 0, \quad \mu_2 = c, \quad \mu_3 = 1 - r$$

and the corresponding eigenvectors are

$$w_1 = (1, 0, 0), \quad w_2 = (1, c, 0), \quad w_3 = (-1, r-1, (c-1+r)(r-1)/r)$$

The fixed point of F_2 is always stable for the parameter range considered.

Thus, for $a < 1$ the whole system is stable.

Let us denote the plane, which separates the two regions R_1 and R_2, by LC_{-1} :

$$\mathrm{LC}_{-1} = \left\{(x, y, z) : z = \frac{a}{r}(x-y)\right\}.$$

If we apply either F_1 or F_2 to this plane, we get a new plane, denoted LC_0, which is the same from *both* applications:

$$LC_0 = F(LC_{-1}) = \left\{(x, y, z) : z = \frac{1-r}{r}(cx - y)\right\}.$$

Its consecutive images by F are

$$LC_i = F(LC_{i-1}), \quad i = 1, \ldots$$

We call these sets *critical planes*, emphasizing the fact that they play an important role, as critical lines, or critical surfaces, for noninvertible maps (Gumowski and Mira, 1980).

Proposition. *Let* $a > 1, (a+c)^2 < 4a, c < 1-r$. *Then any point* $(x_0, y_0, z_0) \in R_1$ *is mapped to* R_2 *in a finite number of iterations while any* $(x_0, y_0, z_0) \in R_2$ *is mapped to* R_1 *also in a finite number of iterations.*

The first part of the Proposition is obvious: for $(a+c)^2 < 4a$, $a > 1$, any point of the z-axis is an unstable focus, thus, any point $(x_0, y_0, z_0) \in R_1$ rotates under iterations by F_1 around the z-axis away from it. Note that in the direction of the eigenvector $v_1 = (0, 0, 1)$ we have $\lambda_1 = 1$; thus, all points $(x_i, y_i, z_i) = F_1(x_{i-1}, y_{i-1}, z_{i-1})$, $i = 1, \ldots, j$, belong to a rotation plane $z = ax + z^*$ corresponding to the eigenvectors v_2 and v_3 and passing through the fixed point $(0, 0, z^*)$. Any such a rotation plane obviously intersects the critical plane LC_{-1}. Thus, after a finite number of iterations the trajectory enters the region R_2: there exists $j > 0$ such that $(x_j, y_j, z_j) \in R_2$.

Let now $(x_0, y_0, z_0) \in R_2$. First note that, as the map F_2 does not depend on x (a fact that causes one eigenvalue to be zero), any point $(x_0, y_0, z_0) \in R_2$ is mapped by F_2 on the plane LC_0 in one step: $(x_1, y_1, z_1) = F_2(x_0, y_0, z_0) \in LC_0$. Obviously, as long as iterated points are in R_2, they all belong to LC_0 and approach the stable zero fixed point of the map F_2 in the eigendirections w_2 and w_3.

For $c < 1-r$ the eigenvalues μ_2, μ_3 of the map F_2 are such that $\mu_2 < \mu_3$ (i.e., the iterated points move more quickly to the zero fixed point in the w_2 direction and asymptotically become tangent to w_3), but for the same condition $c < 1-r$ we have $w_3 \in R_1$. Thus, approaching the zero fixed point, the trajectory necessarily enters the region R_1: there exists $k > 0$ such that $(x_k, y_k, z_k) = F_2(x_{k-1}, y_{k-1}, z_{k-1}) \in R_1$.

Appendix B

It is also interesting to describe the dynamics of the map F given in (A.1) at $a = 1$. It is a bifurcation value for the map F_1 when two of its eigenvalues, being complex-conjugate, are on the unit circle: $|\lambda_{2,3}| = 1$. It means that the fixed point $(0, 0, z^*)$ is a center. Any point in its neighborhood, denoted P, is either periodic with rotation number m/n, or quasiperiodic, depending on the parameters. The value of the parameter c that corresponds to the rotation number m/n is given in (14). What is this neighborhood P? Obviously, $P \in R_1$, and it belongs to the rotation plane $z = ax + z^*$ passing through $(0, 0, z^*)$ corresponding to the eigenvectors v_2 and v_3. Without going into details we just say that in the case of a periodic rotation the set P is a polygon whose boundary is made up by segments of critical lines that are intersections of LC_i, $i = 0, \ldots, m - 1$, with the rotation plane (see Sushko et al. where an analogous consideration is provided in detail for a two-dimensional piecewise linear map). In the case of quasiperiodic rotation the set P is an ellipse, each point of which is tangent to some critical line.

We can consider a union U of such sets P constructed for each fixed point $(0, 0, z^*)$. This union is either a polygonal cone with m sides (in the case of m-period rotation), or a cone (quasiperiodic case), that issue from the origin. Any point $(x, y, z) \in U$ is either m-periodic, or quasiperiodic, while any point $(x, y, z) \notin U$ is mapped to the boundary of U in a finite number of iterations.

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