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INVARIANT CURVES AND FOCAL POINTS IN A LYNESS ITERATIVE PROCESS

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We investigate the properties of recurrence of the type $x_{n+1} = (a+\sum_{i=0}^{k-2} x_{n-i})/x_{n-(k-1)}$, known as Lyness iterations from [Lyness, 1942, 1945, 1961] and recently analyzed by several authors in the case $a > 0$, see e.g. [Kocic *et al.*, 1993; Csornyei & Laczkovich, 2000]. We reconsider Lyness recurrences at the light of some recent results on iterated maps with denominator, given in [Bischi *et al.*, 1999a], where new kinds of singularities, such as focal points and prefocal curves, have been defined. In this paper, in particular, we give an answer to one of the open problems proposed in [Kocic & Ladas, 1993, pp. 141] concerning the dynamic behavior of Lyness recurrences for $a < 0$. We also give some new results in the case $a > 0$, and we improve a previous result on Lyness "periodic recurrences".

Keywords: Recurrences; rational maps; Lyness equations; focal points.

1. Introduction

The dynamic behavior of some recurrence relations of order k ($k \geq 2$) of the type

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k-2} b_i x_{n-i}}{x_{n-(k-1)}} \tag{1}$$

has been recently investigated by several authors (see e.g. [Kocic *et al.*, 1993; Li & Liu, 1999; Csornyei & Laczkovich, 2001]). After the early works by Lyness [1942, 1945, 1961] several results have been established for recurrences of the type (1), mainly in order to detect bounded oscillatory behaviors of positive trajectories for non-negative values of the parameters. A survey of these results can be found in the book by Kocic and Ladas [1993, Chap. 5], where several interesting references are also given. Csornyei and Laczkovich [2001] give some results concerning special parameters' values for which (1) is a "periodic sequence", that is, all the infinite sequences which are solutions of (1) are periodic. The interest in the recurrences (1) is not only theoretical, but is often related to the fact that they are met in several applications, for example in ecologic modeling, see e.g. [Kocic *et al.*, 1993], and references therein. The authors quoted above focused their attention on the existence of a positive equilibrium with oscillatory behavior around it, whereas the question whether positive and bounded trajectories exist when some parameter is negative was left as an open problem. The main purpose of the present paper is to give an answer to this and other questions concerning recurrences (1). We shall make use of some recent results on iterated maps with a denominator which can vanish, given in [Bischi *et al.*, 1999a, 2000], where new kinds of singularities, called *focal points* and *prefocal curves*, have been defined (see also the related works by Bischi and Gardini [1997] and Mira [1999]). As shown throughout the paper, some interesting features of the recurrences (1) can be evidenced at the light of these new concepts, which allow us to stress some global structures of the phase curves on which the sequences generated by (1) are confined.

For sake of simplicity we shall mainly focus on the simplest case, obtained for $k = 2$ in (1). However, many of the techniques and properties that we present in this paper can be generalized to recurrences (1) with higher values of k, i.e. recurrences of order greater than 2. Some possible generalizations are explicitly shown in the paper.

The plane of the work is as follows. In Sec. 2 we present the *simplest case*, we recall some known results given in the literature, and we state the new results concerning the case $a < 0$. Such new results are proved in Sec. 3, where some properties and the terminology introduced in [Bischi *et al.*, 1999a] are briefly described and extensively used. In Sec. 4 we show that the results and methods contained in Secs. 2 and 3 can also be applied to the case of positive parameter, thus confirming and extending some results given in the literature. In Sec. 5 some extensions are given for higher order recurrences (1) with $k \geq 2$. In Sec. 6 we shall consider the particular case with $k = 3$, we prove that an uncountable many periodic solutions exist for any value of the parameter a, and we improve a result concerning *periodic recurrences* given by Lyness (see [Kocic *et al.*, 1993] or [Csornyei & Laczkovich, 2001]).

2. The Simplest Recurrence

Let us consider the recurrence (1) with $k = 2$, i.e.

$$x_{n+1} = \frac{\alpha + bx_n}{x_{n-1}} \tag{2}$$

with $b \neq 0$. By the change of variable $x = by$ we get the recurrence

$$y_{n+1} = \frac{a + y_n}{y_{n-1}}$$

where $a = \alpha/b^2$, so that we are led to the recurrence in one parameter, which we rewrite again as

$$x_{n+1} = \frac{a + x_n}{x_{n-1}} \tag{3}$$

In the following we shall refer to the recurrence (3) as the *simplest recurrence*. For $a > 0$ it has been proved that (see e.g. [Kocic *et al.*, 1993; Kocic & Ladas, 1993]):

(i) the solutions of (3) satisfy the equation $H(x_{n-1}, x_n) = \text{const.}$ where

$$H(x_{n-1}, x_n) = (a + x_{n-1} + x_n)\left(1 + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_n}\right) \tag{4}$$

(ii) if the initial conditions (i.c. henceforth) are positive, then the solution of the recurrence (3) is given by a positive and bounded sequence, i.e. there exist $m > 0$ and $M > m$ such that $m < x_n < M \ \forall n$;

(iii) a unique positive equilibrium exists and all the positive solutions are oscillating around the positive equilibrium.

Of particular interest are the cases called *periodic recurrences*, in which *all the trajectories are periodic*. The following result holds:

(iv) every positive solution of (3) is periodic of period 5 iff $a = 1$; every positive solution of (3) is periodic of period 6 iff $a = 0$.

Moreover, in [Csornyei & Laczkovich, 2001] is proved that these are the only possibilities for the recurrence (2) to be a *periodic recurrence*. In other words, the infinitely many solutions of (2) are all periodic iff $a = 0$ or $a = 1$, giving rise to periodic solutions of periods 6 and 5, respectively. Indeed, as we shall see in the following, for the recurrence (2) such values of the parameter a can be seen as bifurcation values when we consider such recurrence from the point of view of the singularities specific to two-dimensional iterated maps of the plane which are not defined in the whole plane due to a vanishing denominator, as given in [Bischi *et al.*, 1999a, 1999b, 2000].

Our first result concerns the properties of the recurrence (3) when a is a negative parameter. As we shall see, the equilibria of (3) exist only for $a > -1/4$, so we are interested in the parameter range $-0.25 < a < 0$. Our first result is expressed by the following Theorem:

Theorem 1. *Let* $-0.25 < a < 0$ *in recurrence (3), then*

(j) *the solutions satisfy* $H(x_{n-1}, x_n) = \text{const.}$, *where* H *is defined in (4)*;

(jj) *there are two fixed points of the recurrence (3) given by* $S^* = (x_s^*, x_s^*)$, *with* $x_s^* = (1 - \sqrt{1+4a})/2$ *and* $P^* = (x_p^*, x_p^*)$ *with* $x_p^* = (1 + \sqrt{1+4a})/2$, *of saddle and centre types respectively, such that a positive region in the plane* (x_{n-1}, x_n) *exists, bounded by a closed invariant curve, locus of homoclinic orbits of the saddle* S^*, *and including the equilibrium* P^*, *inside which all the trajectories are bounded and oscillating around* P^*;

(jjj) *the trajectories of i.c. outside the area described in (jj) belong to invariant phase curves which cross through the four points* $(0, -a)$, $(-1, 0)$, $(-a, 0)$, $(0, -1)$ *and are made up of at least three unbounded branches*;

(jv) *there exists a three-cycle* C_3 *of saddle type, given by*:

$$-1,\ -1,\ (1-a),\ -1,\ -1,\ (1-a), \ldots$$

(v) *in the plane* (x_{n-1}, x_n), *the three lines* (r_i), $i = 1, 2, 3$, *of equation* $x_{n-1} + 1 = 0$, $x_{n-1} + x_n + a = 0$, *and* $x_n + 1 = 0$ *respectively, constitute the stable set of the three-cycle* C_3 *given in (jv), and their points generate, through (3), sequences for which an explicit analytic expression exists, given by*:

$$-1,\ u_0,\ -(u_0 + a),\ -1,\ u_1,\ -(u_1 + a), \ldots$$

where

$$u_n = \frac{(a-1)^n(u_0 - 1 + a) - (a-1)(u_0+1)}{-(a-1)^n(u_0 - 1 + a) + (u_0 + 1)}$$

$\forall\, u_0 \neq -a$ *and* $n \geq 0$

In order to prove this theorem we shall use the terminology and some properties of the iterated maps of the plane having a vanishing denominator. In the next section we first recall such properties and then we give a proof of Theorem 1.

3. Proof of Theorem 1

In order to prove the Theorem stated in Sec. 2, let us rewrite the second-order recurrence in (3) as a two-dimensional system of the first order, i.e. an iterated map of the plane. As usual, this is obtained by letting $(x_{n-1}, x_n) = (x, y)$ so that (3) can be written as $T : (x_{n-1}, x_n) \to (x_n, x_{n+1})$, where $T(x, y) \to (x', y')$ is the two-dimensional map given by

$$T : \begin{cases} x' = y \\ y' = \dfrac{a+y}{x} \end{cases} \tag{5}$$

3.1. *Properties of the equivalent two-dimensional map*

The map T is not defined on the line δ_s of equation $x = 0$, so the iteration of T generates uninterrupted sequences provided that the initial condition (x_0, y_0) (i.c. henceforth) belongs to the set E given by

$$E = \mathbb{R}^2 \backslash \Lambda \tag{6}$$

where Λ is the union of the preimages of any rank of the line δ_s

$$\Lambda = \bigcup_{k=0}^{\infty} T^{-k}(\delta_s)$$

In other words, the sequences generated by the recurrence (3) can be obtained by the iteration of the two-dimensional map $T : E \to E$ which is not defined in the whole plane.[1] Following the terminology introduced in [Bischi *et al.*, 1999a] the line δ_s will be called *set of nondefinition*, and its point $Q = (0, -a)$, where the second component of the map T assumes the form 0/0 constituting a *simple focal point* of T, and the associated prefocal set $\delta(Q)$ is the line of equation $x = -a$. This means that a one-to-one correspondence exists between the slopes m of arcs γ through the focal point Q and the points $(-a, y)$ where their images $T(\gamma)$ cross the prefocal set $\delta(Q)$. In this case such correspondence is very simple, given by

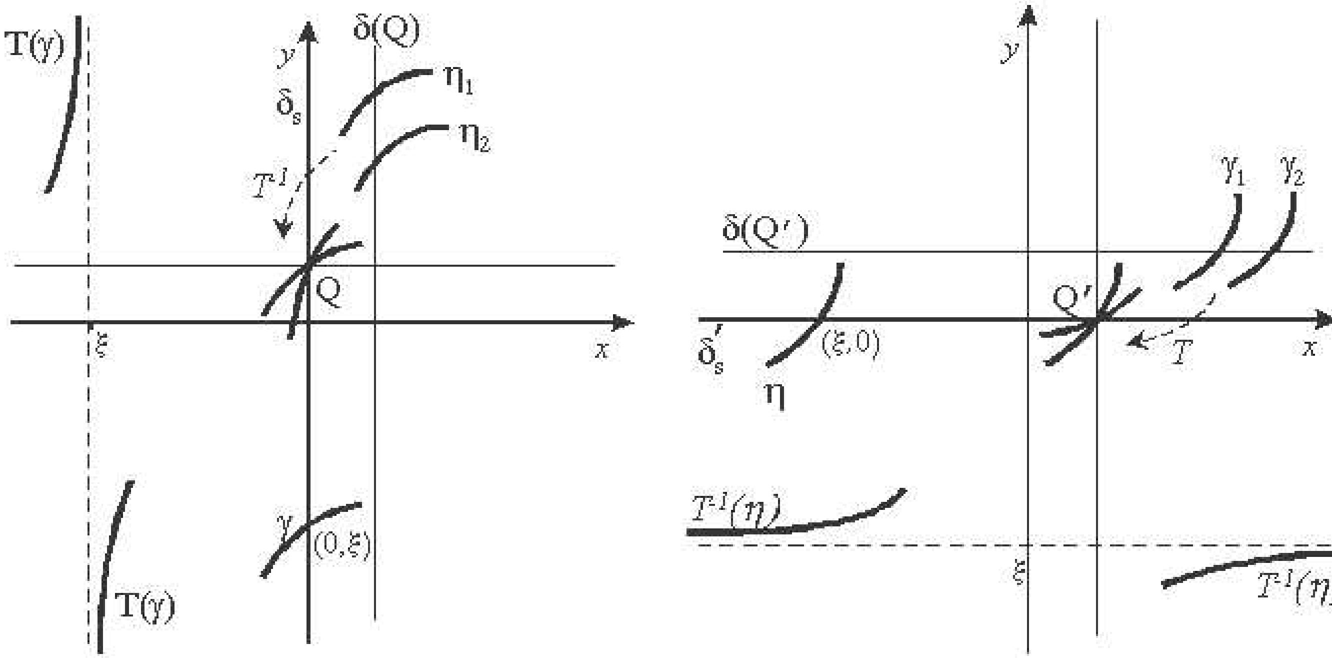
$$m \longleftrightarrow y(m) = m$$

This implies that the image by T of an arc γ crossing through Q with a slope m in Q, is an arc $T(\gamma)$ which crosses the prefocal line at the point $(-a, m)$, and, conversely, the preimage $T^{-1}(\eta)$ of an arbitrary arc η which crosses the line $x = -a$ at a point $(-a, y)$ is an arc which crosses through Q with a slope $m = y$ in Q, where the inverse map is given by

$$T^{-1}(x, y) = \left(\frac{a+x}{y},\ x\right) \tag{7}$$

The latter property justifies the terms *focal point* and *prefocal set*, since all the segments crossing

[1]Notice that the set Λ of points excluded from the phase space of the iterated map has zero Lebesgue measure in $\mathbb{R}^2$.



(a)

(b)

Fig. 1.

the prefocal line are "focalized" by T^{-1} into arcs through Q [see the qualitative pictures in Fig. 1(a)]. Instead, if an arc γ crosses the set of nondefinition δ_s at a nonfocal point $(0,\, \xi)$, then its image by T is an unbounded arc, doubly asymptotic to the line of equation $x = \xi$ (see [Bischi *et al.*, 1999a, 1999b]).

The property that infinitely many invariant curves of the map may cross through focal points of T or of its inverse T^{-1}, although such points are not fixed points or cycles of the map T, is now well known. We shall see that this also occurs in this example, and the properties of focal points and prefocal sets of T and T^{-1} will help us to understand the peculiar properties of the recurrence (3).

3.2. *Proof*

Let us first consider the invariant curves of the map (5). We already know the existence of a constant function $H(x,\, y)$ in the case $a > 0$, and it is straightforward to see that the same function plays the same role also in the case $a < 0$. That is, the phase curves of T satisfy $H(x,\, y) = \text{const.}$ where

$$H(x,\, y) = (a + x + y)\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \qquad (8)$$

being $H(T(x,\, y)) = H(x,\, y)$. Moreover, the map T is invertible and its inverse, given by (7), has invariant phase curves which satisfy the same equation and such curves are symmetric with respect to the line $x = y$. In fact, the two maps, T and T^{-1}, are topologically conjugate by the symmetry $s(x,\, y) = (y,\, x)$, with

$$s \circ T \circ s = T^{-1}\,.$$

So, both T and T^{-1} have the same phase curves. Moreover, looking at the explicit definition (7) of T^{-1}, it is easy to see that the line $y = 0$ is the set of nondefinition for T^{-1}, we denote it by δ_s', and its point $Q' = (-a,\, 0)$ is a focal point for T, the related prefocal set $\delta(Q')$ being the line of equation $y = -a$. This means that whenever an arc, say γ, crosses through the line $y = -a$, then $T^{-1}(T^{-1}(\gamma)) = T(\gamma)$ is an arc which crosses through the focal point Q'. And if an arc crosses the set of nondefinition of T^{-1}, $y = 0$, in a nonfocal point $(\xi,\, 0)$, $\xi \neq 0$, then its image by T^{-1} is an unbounded arc, doubly asymptotic to the line of equation $y = \xi$ [see Fig. 1(b)].

All these facts lead us to the properties of the phase curves of T described in Theorem 1. By elementary calculus we can see that for $-0.25 < a < 0$ there exist two fixed points of T, both in the positive half-plane, a centre P^* and a saddle S^* :

$$P^* = (x_p^*, x_p^*), \quad x_p^* = \frac{1+\sqrt{1+4a}}{2}$$

$$S^* = (x_s^*, x_s^*), \quad x_s^* = \frac{1-\sqrt{1+4a}}{2}.$$

The two eigenvalues, and related eigenvectors, of the Jacobian matrix of T computed in the saddle S^* are given by

$$\lambda_1 = \frac{1-\sqrt{1-4x_s^{*2}}}{2x_s^*}, \quad r_1 = (1, \lambda_1)$$

$$\lambda_2 = \frac{1}{\lambda_1}, \quad r_2 = (1, \lambda_2)$$

Two branches of the invariant curves issuing from the saddle S^* must merge into a closed invariant curve, locus of homoclinic orbits of S^*, in the positive half-plane. This is a consequence of the fact that the phase curves, defined by $H(x, y) = \text{const.}$, are symmetric with respect to the line $x = y$. In fact, considering $H(x, y) = H(x_s^*, x_s^*)$ we show that the curve must cross the line $x = y$ at a point (p_s, p_s), with $x_s^* < x_p^* < p_s$. This can be proved by analyzing the properties of the one-dimensional restriction of H to the line $x = y$, given by the function $F(x) = H(x, x)$, which reads

$$F(x) = (a+2x)\left(1+\frac{1}{x}\right)^2$$

The qualitative graph of $F(x)$ (for $-0.25 < a < 0$) is shown in Fig. 2, and from the properties of its local extrema we prove the existence of p_s, which means that the phase curves of T through the saddle S^* have homoclinic orbits on a closed curve through the fixed point [see Fig. 3(a)]. But in the graph of $F(x)$, shown in Fig. 2, besides the existence of the saddle S^* (the local maximum in x_s^*) and of the center P^* (the local minimum in x_p^*), another critical point of the function $H(x, y)$ can be seen, given by the local maximum at the point $x = -1$. This implies that the two-dimensional T must have a saddle also at the point $(-1, -1)$. However, that point is not a fixed one of T, but a point of a three-cycle of T of saddle type, as described below.

So, we have proved that homoclinic orbits of S^* exist on a closed curve through S^*, which crosses the line $x = y$ at the point (p_s, p_s) and consequently bounds a finite region in the positive quadrant of the plane. And it is clear that any i.c. inside this area belongs to a phase curve which is a closed curve

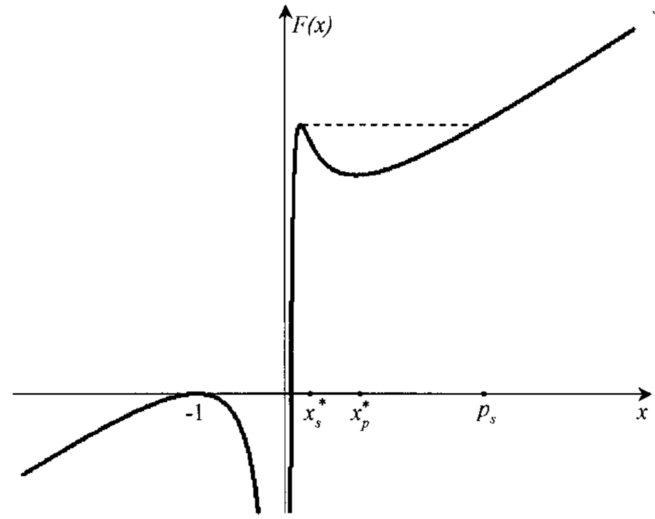


Fig. 2.

around the center P^*. This concludes the proof of parts (j) and (jj) of Theorem 1.

To prove part (jjj) let us consider the two other branches of phase curve issuing from the saddle S^*. Being $x_s^* > -a$ the saddle S^* is located above the two prefocal lines $\delta(Q)$ and $\delta(Q')$ of T and T^{-1} respectively. Thus, one branch issuing from S^* must cross $\delta(Q')$ in a point [see the arc γ in Fig. 3(a)] and consequently its image $T(\gamma)$ must cross Q', its image $T^2(\gamma)$ must cross through $T(Q') = (0, -1) \in \delta_s$ and, as a consequence, $T^3(\gamma)$ must be an unbounded arc, doubly asymptotic to the line of equation $y = -1$. A similar reasoning applies to the other branch of phase curve issuing from S^* (symmetric of the previous ones). That branch includes an arc η which crosses $\delta(Q)$, so that $T^{-1}(\eta)$ crosses through the focal point Q, $T^{-2}(\eta)$ crosses the point $T^{-1}(Q) = (-1, 0) \in \delta_s'$, which means that $T^{-3}(\eta)$ is unbounded and doubly asymptotic to the line of equation $x = -1$. These unbounded branches of the phase curve through S^* are shown in Fig. 3(b).

Of course, other phase curves must cross the lines $\delta(Q)$ and $\delta(Q')$, thus giving those with unbounded arcs the same properties as the phase curves through S^* as described above. This is true for the sequences generated starting from any point of the positive half-plane external to the closed curve of homoclinic orbits of S^*. For example, let us consider an initial condition taken at the points (q, q) with $q > p_s$; an invariant curve through (q, q) must cross the prefocal $\delta(Q')$ at a point, say $(\zeta, -a)$, and the prefocal $\delta(Q)$ in the symmetric point $(-a, \zeta)$. Indeed, such curves fill up the plane

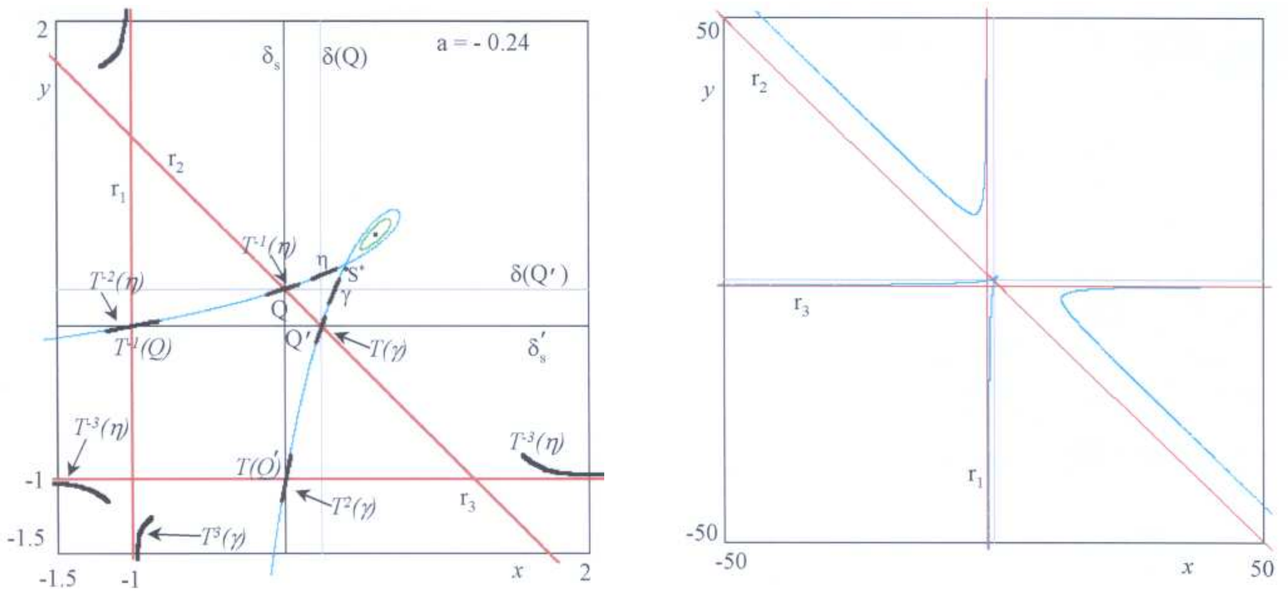


Fig. 3.

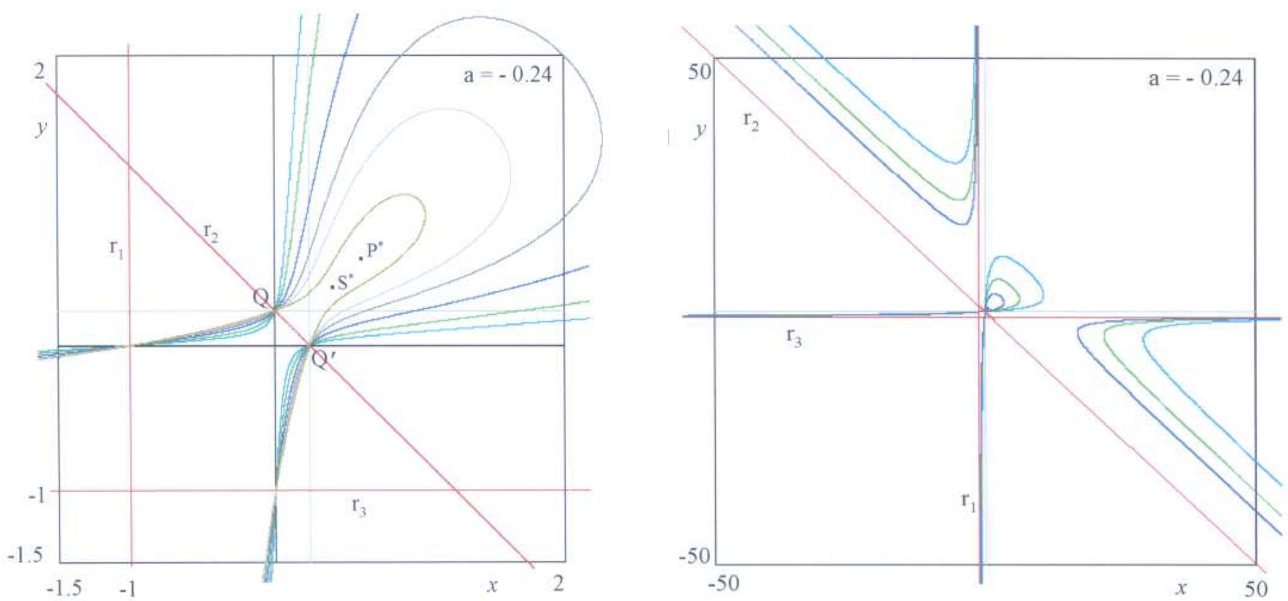


Fig. 4.

(x, y) (just a few of them are shown in Fig. 4). By the above arguments we have proved that all these phase curves cross through the four points

$$Q = (0, -a)\,, \quad T^{-1}(Q) = (-1, 0)\,, \tag{9}$$

$$Q' = (-a, 0)\,, \quad T(Q') = (0, -1) \tag{10}$$

and are asymptotic to the lines of equation $x = -1$ and $y = -1$. But more, we now show that these phase curves are also asymptotic to another line, of equation $x + y + c = 0$, where $c \leq a$, so that these invariant curves must include at least three unbounded branches, thus proving part (jjj) of Theorem 1. This is a consequence of the fact that each of the following lines:

$$(r_1):\ x = -1\,,$$
$$(r_2):\ x + y + a = 0\,, \tag{11}$$
$$(r_3):\ y = -1$$

is mapped into itself by the third iterate of the map, T^3. In fact, let $(-1,\, u)$ be a point belonging to (r_1), then $T(-1,\, u) = (u,\, -a-u) \in (r_2)$, $T^2(-1,\, u) = (-a-u,\, -1) \in (r_3)$, and $T^3(-1,\, u) = [-1,\, ((1-a)/(a+u))] \in (r_1)$. Moreover, the restriction of T^3 to any one of these lines is represented by the one-dimensional map:

$$u' = f(u)\,, \quad f(u) = \frac{1-a}{a+u} \tag{12}$$

The function $f(u)$ has very simple properties (the graph of $f(u)$ is an hyperbola with vertical asymptote in $u = -a$). There are two fixed points of $f(u)$, $u = -1$ (attracting for $a < 0$ and repelling for $a > 0$, with $f'(-1) = 1/(a-1)$), and $u = 1-a$ (attracting for $a > 0$, repelling for $a < 0$, with $f'(1-a) = a-1$). The stable fixed point is globally attracting for $f(u)$. Moreover, by using the function $h(z) = (z+1)/(z-(1-a))$ we have that $f(u)$ is conjugate to the function $g(y)$, where

$$g(y) = h \circ f \circ h^{-1}(y) = \left(\frac{1}{a-1}\right) y$$

is linear, so that we can easily write the analytical solutions of the iterates of $g(y)$, and thus of $f(u)$, obtaining, $\forall\, u_0 \neq -a$, the following trajectory:

$$u_n = f^n(u_0) = \frac{(a-1)^n(u_0-1+a)-(a-1)(u_0+1)}{-(a-1)^n(u_0-1+a)+(u_0+1)}\,.$$

It is also straightforward to see that the fixed points of $f(u)$ belong to a cycle of period 3 of T, say C_3, given by:

$$(-1,\, 1-a)\,, \quad (1-a,\, -1)\,, \quad (-1,\, -1) \tag{13}$$

and the eigenvalues of T^3 on each periodic point are given by

$$\lambda_1 = (a-1)\,, \quad \lambda_2 = \frac{1}{\lambda_1}$$

so that the three-cycle of T, C_3, is of saddle type, its stable set being constituted by the three lines, cyclical for T (and invariant for T^3), given in (11). These three lines constitute the phase curve associated with $H(-1,\, -1)$ (or with any other point on the lines), so that *they cannot be crossed by any other invariant phase curve* of the plane, except at the points of nondefinition of T and T^{-1}, that is, at the intersections of the lines δ_s, of equation $x = 0$, and δ'_s, of equation $y = 0$, with the three cyclical lines in (11), i.e. the four points given in (9). As a consequence, the unbounded arcs which we have proved to be asymptotic to the lines $x = -1$ and $y = -1$ are also doubly asymptotic to $x + y + a = 0$

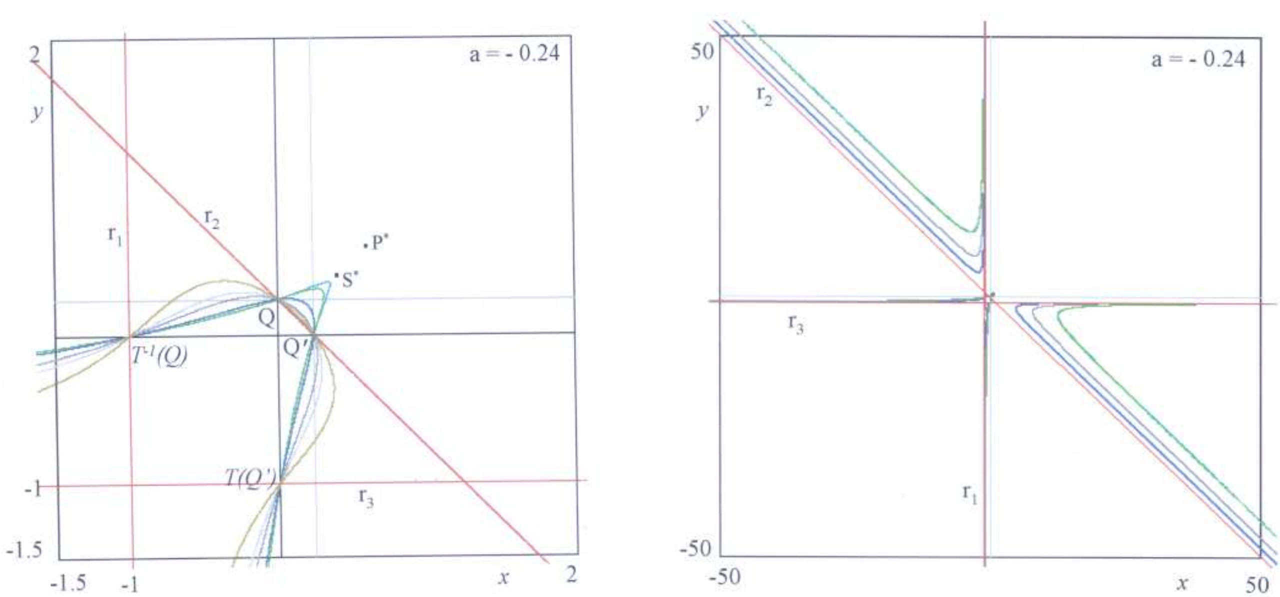


Fig. 5.

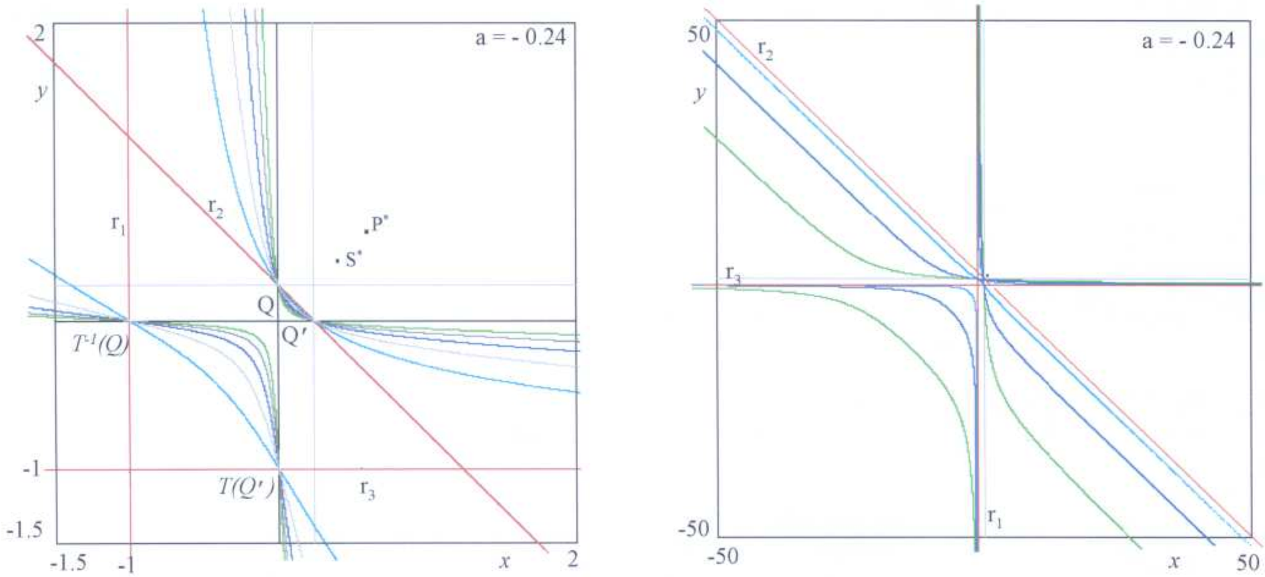


Fig. 6.

or to one line parallel to it, equation $x + y + c = 0$, with $c \leq a$.

In order to see the qualitative shape of all the phase curves outside the homoclinic orbit of S^*, it is enough to consider an i.c. (x, x) on varying x ($x < x_s^*$ and $x > p_s$) or i.c. $(-a, \xi)$ on $\delta(Q)$, or $(\xi, -a)$ on $\delta(Q')$, and varying ξ. As these invariant curves must intersect the prefocal lines $\delta(Q)$ and $\delta(Q')$, the same reasoning as above apply, and we can conclude that they all cross the four points given in (9), are unbounded, and made up of at least three pieces, doubly asymptotic to the lines $x = -1$, $y = -1$ and $x + y + c = 0$ for some $c \leq a$ (their qualitative shape is shown in Figs. 5 and 6). This completes the proof of Theorem 1. ■

We close this section by stressing that the sequences generated by points belonging to unbounded phase curves are *unbounded but not diverging*. This means that if we consider an i.c. $\{x_0, y_0\}$ outside the homoclinic loop of S^*, then the sequence $\{x_n, n \geq 0\}$, is such that $\sup\{|x_n|, n \geq 0\} = +\infty$ but $\lim_{n\to\infty} |x_n|$ does not exist. In other words, these sequences arbitrarily approach the Poincaré Equator (i.e. the points at infinite distance) along the cyclical lines given in (11), but further iteration of T gives points which are at a smaller distance from the origin, a kind of motion which is frequently observed in iterated maps with denominator, as shown in [Bischi *et al.*, 2000].

4. Some Results on the Simplest Recurrence with $a > 0$

In the previous sections we have shown that for $-0.25 < a < 0$ the phase curves of (3) are invariant closed curves around P^* only in a suitable region, and we can refer to this region by using the points (x, x), with $x \in (x_s^*, p_s)$, as defined in the previous section. As the parameter $a \to 0$, the fixed point S^* (as well as Q and Q') approaches the origin, so that the portion of the positive orthant made up of trajectories oscillating around the center (bounded by the homoclinic orbit of S^*) become wider and wider as $a \to 0$. A qualitative picture is shown in Fig. 7.

It is easy to see that the properties of T, and of the phase curves, as described in Sec. 3 are not due to the negativity of the parameter a. That is, those properties continue to hold also for $a > 0$, with obvious changes in part (jj) of Theorem 1. So, starting from our Theorem 1 we can extend the results given in [Kocic *et al.*, 1993; Kocic & Ladas, 1993], in the sense that we can assume the statements (ii) and (iii) that were recalled at the beginning of Sec. 2 are not only sufficient but also necessary. Indeed, we have the following

Theorem 2. *Consider* (5) *with* $a > 0$, *then*

(j) *an* i.c. (x, y) *gives rise to a positive and bounded trajectory* (*oscillating around* P^*) iff $(x, y) \in R_+^2$.

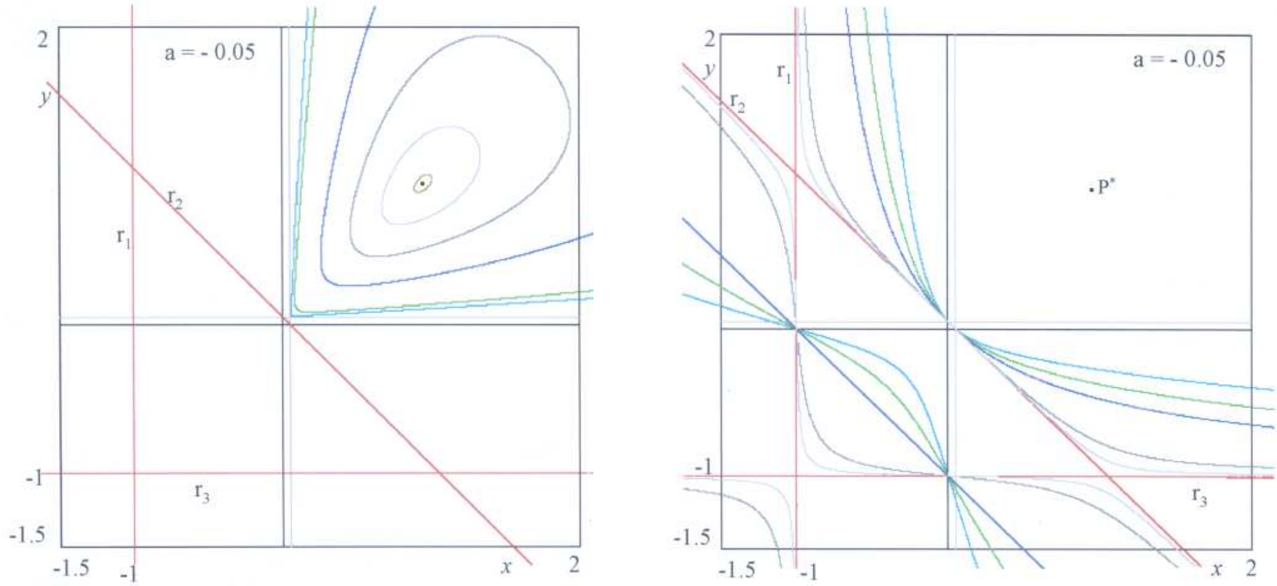


Fig. 7.

(jj) *the trajectory of an i.c.* $(x, y) \notin R_+^2$ *belongs to an invariant phase curve which crosses through the four points*

$$Q = (0, -a), \quad T^{-1}(Q) = (-1, 0),$$
$$Q' = (-a, 0), \quad T(Q') = (0, -1)$$

and includes at least three unbounded branches doubly asymptotic to the lines

$$x = -1; \quad x + y + c = 0 \ (c \le a); \quad y = -1$$

(jjj) *for each* $a \neq 1$ *there exists a three-cycle* C_3 *of saddle type, given by*:

$$-1, -1, (1-a), -1, -1, (1-a), \ldots$$

(jv) *in the plane* (x_{n-1}, x_n), *the three lines* (r_i), $i = 1, 2, 3$, *of equation* $x_{n-1} + 1 = 0$, $x_{n-1} + x_n + a = 0$, *and* $x_n + 1 = 0$ *respectively, constitute the stable set of the three-cycle* C_3 *given in Theorem 1, and their points generate, through (3), sequences for which an explicit analytic expression exists, given by*:

$$-1, u_0, -(u_0 + a), -1, u_1, -(u_1 + a), \ldots \quad (14)$$

where

$$u_n = \frac{(a-1)^n(u_0 - 1 + a) - (a-1)(u_0 + 1)}{-(a-1)^n(u_0 - 1 + a) + (u_0 + 1)},$$

$\forall\, u_0 \neq -a$ *and* $n \geq 0$.

From the structure of the phase curves and the properties seen above and in the previous sections, it follows that the cases $a = 0$ and $a = 1$, which according to [Kocic *et al.*, 1993; Csornyei & Laczkovich, 2001] are associated with *periodic recurrences*, correspond to bifurcation cases related to contacts between singularities of different nature of the map T.

In fact, for $a = 0$ the focal points Q and Q' merge, and both merge with the saddle fixed point S^*. Moreover, the lines of nondefinition of T and T^{-1}, given by δ_s and δ_s' respectively, merge with the two prefocal lines $\delta(Q)$ and $\delta(Q')$, respectively. The related changes in the shape of the phase curves can be seen by comparing Fig. 7 (obtained for $a < 0$) and Fig. 8 (obtained for $a > 0$).

Instead, for $a = 1$ the focal points Q and Q' merge with the periodic points of the "degenerated" three-cycle (14), and the prefocal lines $\delta(Q)$ and $\delta(Q')$ merge with the two "degenerated" cyclical lines given in (11). We note that for $a = 1$ the i.c. which belong to the lines (11) are not included in the set E defined in (6) whose points generate uninterrupted sequences, because such points are mapped into the point $Q = (0, -1)$ after a finite number of iterations of T. So, these points do not generate periodic solutions of period 5. We also note that in [Kocic *et al.*, 1993; Kocic & Ladas, 1993] it was proved that for $a = 0$ *the recurrence is periodic*, because all the i.c. in the set E generate *periodic*

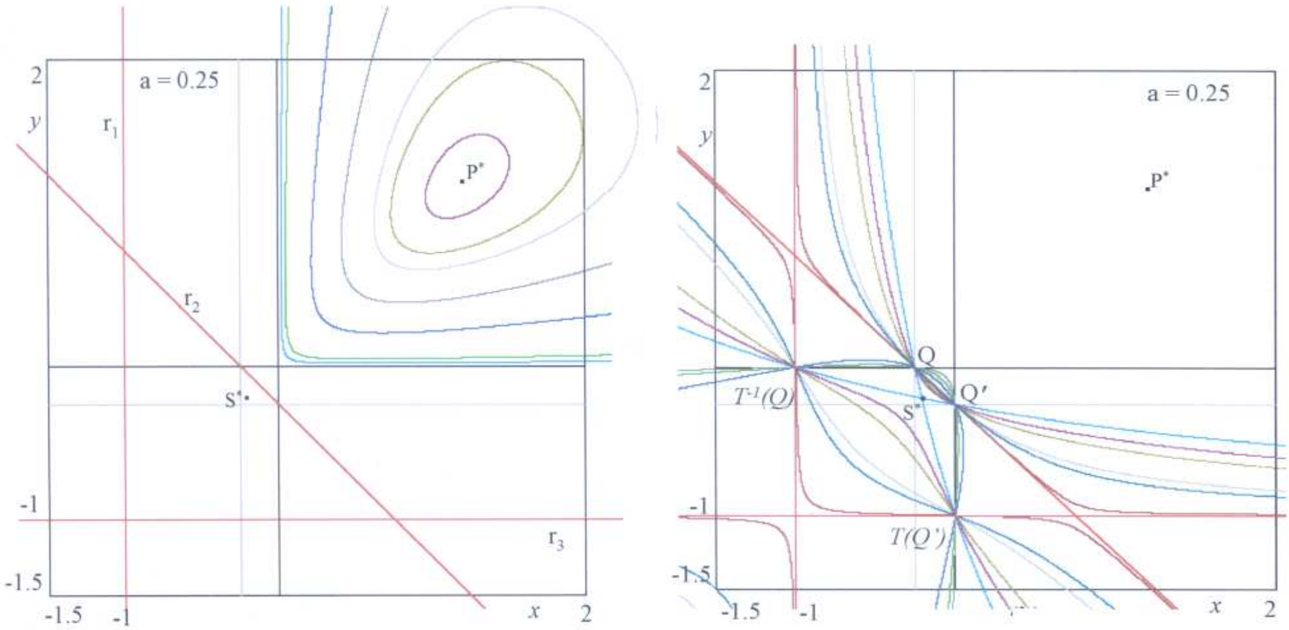


Fig. 8.

sequences of period 6. However, in order to avoid misunderstanding, it is better to specify that this statement was not to be intended as "minimum period 6", because, as we have seen in (14), a cycle of period 3 exists, given by −1, −1, 1, −1, −1, 1, . . .

5. Extensions

In the previous sections we have considered the simplest case of the family of recurrences given in (1), obtained for $k = 2$. However, as stated in the introduction, it is natural to conjecture that similar properties continue to hold, with obvious changes, even for any $k > 2$. Let us consider the recurrence (1) of order k when all the parameters b_i are equal, say $b = b_0 = b_1 = \cdots = b_{k-2}$, $b \neq 0$, so that, after a change of variable, the recurrence may be rewritten as

$$x_{n+1} = \frac{a + x_{n-(k-2)} + \cdots + x_{n-1} + x_n}{x_{n-(k-1)}} \tag{15}$$

where $a = \alpha/b^2$. Clearly the recurrence of order k in (15) is equivalent to a system of k equations of the first order. Identifying $(x_{n-(k-1)}, \ldots, x_{n-1}, x_n) = (y_1, \ldots, y_{k-1}, y_k)$ we get a k-dimensional map $T_{(k)} : \mathbb{R}^k \to \mathbb{R}^k$, $(y'_1, \ldots, y'_{k-1}, y'_k) = T_{(k)}(y_1, \ldots, y_{k-1}, y_k)$ where:

$$\begin{aligned} y'_1 &= y_2 \\ &\vdots \\ y'_{k-1} &= y_k \\ y'_k &= \frac{a + y_2 + \cdots + y_k}{y_1} \end{aligned} \tag{16}$$

Some general results on the generic recurrence (15) are already given in [Kocic *et al.*, 1993; Kocic & Ladas, 1993] for the case $a > 0$, such as:

- the function

$$H(x_{n-(k-1)}, \ldots, x_{n-1}, x_n) = \left(a + \sum_{j=0}^{k-1} x_{n-j}\right) \prod_{j=0}^{k-1} \left(1 + \frac{1}{x_{n-j}}\right) \tag{17}$$

  is constant along the solutions of (15);
- every positive solution of (15) is oscillating about the (unique) positive equilibrium P^*.

Moreover, concerning the special case of "periodic recurrences", in [Kocic & Ladas, 1993] it is shown that for $k = 3$ and $a = 1$ *the recurrence (15) is periodic of period* 8, and in [Csornyei & Laczkovich, 2001] it is proved that for $k = 3$ the same recurrence is periodic iff $a = 1$ whereas for $k > 3$ it cannot be periodic.

But more can be proved on the global dynamics of the recurrence (15) in the regions of the phase space having some negative components, or when the parameter a is negative. We state the following Theorem, which holds for the recurrence (15) independent of the sign of a, and which will allow us to obtain some new insight into the special cases of periodic recurrences associated with $k = 3$.

Theorem 3. *Consider the recurrence (15), or equivalently the map $T_{(k)}$ in (16), for any $k \geq 2$. Then*

(a) *The function given in (17) is constant along the solutions of the recurrence;*

(b) *for any $a \neq k-1$ there exists a periodic solution of (15) of period $(k+1)$ given by*

$$\underbrace{-1, \quad \ldots, -1}_{(k-times)}, \quad (k-1-a), \ldots$$
$$\ldots \underbrace{-1, \quad \ldots, -1}_{(k-times)}, \quad (k-1-a), \ldots \tag{18}$$

(c) *there exist $(k+1)$ cyclical hyperplanes Π_i of $\mathbb{R}^k$, mapped one into the other by the map $T_{(k)}$, invariant for the map $T_{(k)}^{k+1}$, given, in their cyclical order, by:*

$$\begin{aligned}
(\Pi_1): &\quad y_1 + 1 = 0 \\
(\Pi_2): &\quad a + y_1 + \cdots + y_k = 0 \\
(\Pi_3): &\quad y_k + 1 = 0 \\
(\Pi_4): &\quad y_{k-1} + 1 = 0 \\
&\quad \vdots \\
(\Pi_{k+1}): &\quad y_2 + 1 = 0
\end{aligned}$$

Proof. The statement in (a) is straightforward, being $H(T_{(k)}(y_1, \ldots, y_{k-1}, y_k)) = H(y_1, \ldots, y_{k-1}, y_k)$. To prove the existence of the periodic orbit of period $(k+1)$ let us start with the i.c. made up of k times the value $-1 : (x_{n-(k-1)}, \ldots, x_{n-1}, x_n) = (-1, \ldots, -1)$. Then

$$x_{n+1} = \frac{a-(k-1)}{-1} = k-1-a$$

and

$$x_{n+1+j} = \frac{a+(k-1-a)-(k-1-j)-(j-1)}{-1} = -1$$

for $j = 1, \ldots, k$, so that

$$x_{n+1+(k+1)} = k-1-a$$

and we have the $(k+1)$-cycle in (18).

To prove (c) let us consider a point $p \in \Pi_1$,

$$p = (-1, z_2, z_3, \ldots, z_k) \in \Pi_1$$

then

$$T_{(k)}(p) = (z_2, z_3, \ldots, z_k, -(a+z_2+z_3+\cdots+z_k)) \in \Pi_2$$

its image

$$T_{(k)}^{1+1}(p) = (z_3, \ldots, z_k, -(a+z_2+z_3+\cdots+z_k), -1) \in \Pi_3$$

has a "-1" in the last position, so that $T_{(k)}^{1+j}(p)$ has a "-1" in the position $(k-(j-1))$, and

$$T_{(k)}^{1+k}(p) = (-1, \quad \ldots \quad) \in \Pi_1$$

This complete the proof of Theorem 3. ■

We observe that from the construction given above it is possible to also get the explicit formulation of the $(k-1)$-dimensional map which represents the restriction of $T_{(k)}^{1+k}$ to each of the cyclical hyperplanes. The case $k = 2$ was already given in the previous sections, with $T = T_{(2)}$, so, as a further example, let us consider now the case $k = 3$.

6. The Recurrence with $k = 3$ and the Equivalent 3-D Map

In this section we consider the Lyness recurrence (1) with $k = 3$, which reads

$$x_{n+1} = \frac{a + x_{n-1} + x_n}{x_{n-2}} \tag{19}$$

and can be rewritten as an iterated three-dimensional map $T_{(3)} : (x_{n-2}, x_{n-1}, x_n) \to (x_{n-1}, x_n, x_{n+1})$, where $T_{(3)}(y_1, y_2, y_3) \to (y_1', y_2', y_3')$ is given by:

$$T_{(3)} : \begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = \dfrac{a + y_2 + y_3}{y_1} \end{cases} \tag{20}$$

where, as usual, the variables (y_1, y_2, y_3) are defined by $(y_1, y_2, y_3) = (x_{n-2}, x_{n-1}, x_n)$. As stated in Theorem 3, four cyclical planes of $\mathbb{R}^3$ exist on which the fourth-iterate of the map (20), $T_{(3)}^4$, can be identified with a two-dimensional map. After a straightforward computation, it is easy to see that the explicit formulation of this two-dimensional map $Z : (u, v) \to (u', v') = Z(u, v)$ is given by:

$$Z : \begin{cases} u' = -\dfrac{1+u+v}{v} \\ v' = \dfrac{v(1-a)+(1+u+v)}{v(a+u+v)} \end{cases} \qquad (21)$$

This explicit relation allows us to see that the periodic points of $T_{(3)}$ of period 4 are much more than those described in Theorem 3 (b). Indeed, the two-dimensional map Z in (21) has a line of fixed points, which are necessarily periodic points of period 4 for $T_{(3)}$. They are given, $\forall\, u \neq 1-a$, by:

$$(-1, u, -1)\,, \quad (u, -1, 1-a-u)\,,$$
$$(-1, 1-a-u, -1)\,, \quad (1-a-u, -1, u)$$

Stated with other words, for any value $u \neq 1-a$ the sequence

$$-1, u, -1, 1-a-u\,,$$
$$-1, u, -1, 1-a-u, \ldots$$

is one of the infinitely many (uncountable) solutions of period 4 of the recurrence (15) with $k = 3$. Notice that this holds for any value of a, in particular for $a = 1$, at which the recurrence (19) *is periodic with period* 8, as proved in [Kocic & Ladas, 1993; Csornyei & Laczkovich, 2001]. So, like in the previous section, our results help to clarify that the term "periodic of period 8" is not to be intended as "minimum period 8" since infinitely many periodic sequences of period 4 exist. This confirms our previous remark given at the end of Sec. 4.

We close this section by stating the results given above in the form of the following Theorem:

Theorem 4. *Consider the recurrence (19). For any value of a infinitely many periodic sequences of period* 4 *can be generated, given by*

$$-1, u, -1, 1-a-u, -1, u, -1, 1-a-u, -1, \ldots$$

for each $u \neq 1-a$.

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