BORDER COLLISION BIFURCATIONS IN 1D PWL MAP WITH ONE DISCONTINUITY AND NEGATIVE JUMP.
USE OF THE FIRST RETURN MAP

LAURA GARDINI* and FABIO TRAMONTANA†

Department of Economics and Quantitative Methods,
University of Urbino, Italy

*Laura.gardini@uniurb.it
†f.tramontana@univpm.it

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The aim of this work is to study discontinuous one-dimensional maps in the case of slopes and offsets having opposite signs. Such models represent the dynamics of applied systems in several disciplines. We analyze in particular attracting cycles, their border collision bifurcations and the properties of the periodicity regions in the parameter space. The peculiarity of this family is that we can make use of the technical instrument of the first return map. With this, we can rigorously prove properties which were known numerically, as well as prove new ones, giving a complete characterization of the overlapping periodicity regions.

Keywords: Discontinuous piecewise linear 1D map; increasing–decreasing map; border collision bifurcations; increment bifurcation structure.

1. Introduction

It is well-known that several models are described by piecewise smooth dynamical systems (PWS for short). The essential feature in PWS systems (continuous or discontinuous) is the presence of a change of definition in the functions defining the map under study, when a suitable border is met or crossed. This leads to the existence of border collision bifurcations (BCB henceforth). The first works on this subject date back to [Leonov, 1959, 1962] and [Mira, 1978, 1987], and have been studied more recently by [Nusse & Yorke, 1992, 1995; Maistrenko et al., 1993; Maistrenko et al., 1995; Maistrenko et al., 1998]. In the last few years, many papers have been published which deal with PWS systems, due to their wide use in the applied context. We recall, for example, the books [Banerjee & Verghese, 2001; Zhuzubaliyev & Mosekilde, 2003; di Bernardo et al., 2008]. Besides the works cited above, piecewise smooth systems are applied in power electronic circuits [Habe et al., 2003; Banerjee et al., 2000], impacting systems ([Nusse et al., 1994; Ing et al., 2008; Sharan & Banerjee, 2008] to cite a few), piecewise smooth nonlinear oscillators [Pavlovskaia et al., 2004; Pavlovskaia & Wiercigroch, 2007] and in many other applications [Banerjee & Grebogi, 1999; di Bernardo et al., 1999; Sushko et al., 2005, 2006].

The observed dynamics (and BCB properties) differ depending on continuous or discontinuous models. Here, we are interested in discontinuous ones, which have also been studied in some particular cases in recent works (see, for example, [Kollar et al., 2004; Avrutin & Schanz, 2005, 2006, 2008; Avrutin et al., 2010c]). The characteristic features are the particular bifurcation structures which follow the so-called period adding and period increment scheme. The results, however, are particularly important with respect to the bifurcations occurring to the chaotic intervals as studied in [Avrutin et al., 2007, 2008a, 2008b, 2009; Avrutin et al., 2010c].
In this paper, we consider the one-dimensional canonic form suitable for studying the bifurcations occurring in discontinuous one-dimensional maps already introduced by Leonov, and considered in [Gardini et al., 2010] and [Avrutin et al., 2010b], with particular restrictions on the parameters, as the analysis of the bifurcations occurring in all the possible cases is still far from complete. Here we contribute to the study of another particular situation in the parameter space of this one-dimensional map in canonic form, related to the cases of slopes and offsets having opposite signs. We are interested in the bifurcations occurring in the periodicity regions of attracting cycles. The occurrence of the period increment scheme is well-known, but the properties related to this bifurcation structure have not been fully described up to now. This is the object of the present work. Let us consider the one-dimensional piecewise linear map in canonical form:

\[ x' = f(x) = \begin{cases} f_L(x) = a_L x + \mu_L, & \text{if } x < 0 \\ f_R(x) = a_R x + \mu_R, & \text{if } x > 0 \end{cases} \]  

(1)

in which the slopes and the offsets have opposite signs. That is, we consider the parameter space with:

\[ a a_{R}, L < 0, \ \mu_L \mu_R < 0 \]  

(2)

We recall that (as already remarked by Leonov) although by rescaling the state variable one parameter may be set to a constant value, we keep the notation in full because this simplifies the expressions in the formulas of the BCB curves.

In order to study the dynamic behaviors of \( f(x) \), the considered restriction in the parameters, given in (2), allows for a suitable use of the technique of the first return map. With the first return map we have found a simplified expression for some BCB curves of the stable cycles, and particular properties associated with the points of intersection of the BCB curves and the flip bifurcation curves. In particular, we shall recover and rigorously prove the properties of the parameter regions which are associated with the period increment behaviors. As is well-known, in that region at most two attracting cycles can coexist. Moreover, we shall see that, as long as the map is invertible in the invariant absorbing interval, when only one stable cycle exists, then at most one unstable cycle can coexist (and clearly no chaotic set). When the map is noninvertible in the invariant absorbing interval, chaos may occur in attracting cyclic chaotic intervals, and is robust (following [Banerjee et al., 1998]), as persistent under parameter perturbations, or chaos may occur only in a chaotic repellor. In the parameter region associated with chaotic dynamics, regions (or islands) of stable cycles can exist but not a pair of attractors.

The work is organized as follows. Some general properties are recalled in Sec. 2, while in Sec. 3, we shall consider the particular case \( a_L a_R = -1 \), showing that we can have coexistent cycles also with four different periods. The particular case \( a_L a_R = 0 \) is considered in Sec. 4, showing that only one attracting cycle can exist. The analysis of the generic cases in the parameters (2) is performed in Sec. 5. Here in different subsections, by using the first return map, we shall prove peculiar properties of the overlapping (in pair) stability regions, also giving a simplified expression of the BCB curves already determined in other works ([Gardini et al., 2010; Avrutin et al., 2010b], and references therein), as well as the proof of the properties listed above.

2. General Properties

Let us consider the map in the generic form given in (1), assuming \( a_L > 0 \) and \( a_R < 0 \), when the shape of the function is increasing for \( x < 0 \), let us say on the \( L \) side, and decreasing for \( x > 0 \), let us say on the \( R \) side. Then the dynamics associated with the signs \( \mu_L < 0 \) and \( \mu_R > 0 \) are very simple, because a fixed point always exists on the \( R \) side, \( Q^* - \mu_R/(1 - a_R) > 0 \) (stable or unstable, depending on the slope \( a_R \)) while on the \( L \) side \( P^* = \mu_L/(1 - a_L) < 0 \) exists only if it is stable, and in both cases the related dynamics are quickly seen.

The interesting cases are associated with \( \mu_L > 0 \) and \( \mu_R < 0 \), when the shape of the function is as shown in Fig. 1(a). Then inverting the signs of the two slopes the reasoning is the same, and the maps are topologically conjugated. In fact, the map in Fig. 1(b) is conjugated with that in Fig. 1(a) via the following symmetry property of \( f(x) \):

\[ f(x, a_R, a_L, \mu_L, \mu_R) = -f(-x, a_L, a_R, -\mu_R, -\mu_L) \]  

(3)

It follows that we can restrict our analysis to the case \( a_L > 0 \) and \( a_R < 0 \) with \( \mu_L > 0 \) and \( \mu_R < 0 \) (whose shape for \( a_L > 1 \) is shown in Fig. 1(a)).

It is clear that considering a point on the \( R \) side, in one iteration it is mapped to the other (\( L \) side), from where either the trajectory is divergent or the map \( f_L \) is applied until the iterated point enters the
positive side again. Under our assumptions there is no fixed point on the \( R \) side, and as long as \( 0 < a_L \leq 1 \) there is also no fixed point on the \( L \) side. In this case (\( 0 < a_L \leq 1 \)) all the trajectories of the map will enter in the invariant absorbing interval:

\[
I = [a_{RPL} + \mu_R, \mu_L]
\]  

in a finite number of iterations (and its basin of attraction will be the entire real line). The fixed point \( P^* \) on the \( L \) side exists when it is unstable, for \( a_L > 1 \), and then all the points below \( P^* \) have a trajectory which is divergent to \( -\infty \), while the negative points above \( P^* \) (i.e. for \( \mu_L/(1 - a_L) < x < 0 \)) have a trajectory which enters the absorbing interval \( I \), as long as the following inequality holds:

\[
P^* = \frac{\mu_L}{1 - a_L} < a_{RPL} + \mu_R
\]  

which guarantees that the invariant interval \( I \) has no contact with its basin of attraction, so that it is absorbing. In this case, the basin of divergent trajectories is given by

\[
B(\infty) = ] -\infty, P^* [ \cup [ P^*_{1+}, +\infty]
\]  

where \( P^*_{1+} \) is the preimage of the fixed point on the \( R \) side, given by the solution of the equation \( P^* = a_R^* + \mu_R \), that is:

\[
P^*_{1+} = \frac{\mu_L - \mu_R + \mu_R a_L}{a_R(1 - a_L)}
\]  

Here the denominators in \( P^* \) and \( P^*_{1+} \) are assumed different from zero, as the particular cases with \( a_L = 1 \) or with \( a_R = 0 \) are considered in separate sections. Clearly the complementary set of \( B(\infty) \) (given in (6)) in the real line, gives us the basin of attraction of the absorbing interval \( I \):

\[
B(I) = ]P^*, P^*_{1+} [ = \frac{\mu_L}{1 - a_L} - \mu_R + \mu_R a_L
\]  

as we exclude the frontier, which is itself an invariant set. As remarked above, the equality in (5) denotes a bifurcation. When the condition

\[
\chi_{div} = \frac{\mu_L}{1 - a_L} - a_{RPL} + \mu_R
\]

holds, a contact of the invariant interval \( I \) with its basin of attraction occurs, and this denotes a "final bifurcation" because it is followed by a dynamic made up of almost all divergent trajectories when \( P^* > a_{RPL} + \mu_R \).

Thus we are interested in parameters which satisfy the condition in (5) for which we can restrict the study of the map inside the absorbing interval \( I \). Moreover, as noticed above, we can fix an initial condition on the \( R \) side, \( 0 < x < \mu_L \). That is, let \( J_R = [0, \mu_L] \cap [x \geq 0] \), then we can simply construct the "first return map \( F_R \)" in \( J_R \), given by:

\[
F_R: x \in J_R \rightarrow f^m(x) \in J_R, \quad J_R = [0, \mu_L]
\]  

where \( m \) is the first integer for which \( f^m(x) \in J_R \). This map \( F_R \) describes all the BCB occurring to the map \( f \) in \( I \). We remark that in general there is not a unique integer \( m \) for all the points of \( J_R \).

The analysis of the particular cases performed in the
next Secs. 3 and 4 provides examples of the power of the investigation of the dynamics using the first return map $F_R$.

3. Particular Case $a_L = 1$ and $a_R = -1$

In this particular case, the map is only a function of the offsets $\mu_R (<0)$ and $\mu_L (>0)$:

$$x' = f(x) = \begin{cases} f_L(x) = x + \mu_L, & \text{if } x < 0 \\ f_R(x) = -x + \mu_R, & \text{if } x > 0 \end{cases} \quad (11)$$

It is easy to see that we cannot have fixed points of $f$, and the dynamics are trapped in the invariant absorbing interval

$$I = [-\mu_L + \mu_R, \mu_L] \quad (12)$$

as any other point is mapped into $I$ in a finite number of steps. Depending on the parameters, it is possible to have stable (but not attracting) periodic orbits of any period, as shown in the bifurcation diagram in Fig. 2, as a function of $\mu_L$. Such periodic orbits are not attracting because of the particular case, and all the eigenvalues of the existing cycles are either $+1$ or $-1$. We shall see that any point of the absorbing interval $I$ is a periodic point, i.e. it belongs to some cycle of a given period, and as a consequence, any point outside $I$ is preperiodic (which means that it is mapped into a periodic point in a finite number of steps).

Clearly the period of the cycle depends on the values of parameters $\mu_R$ and $\mu_L$. Moreover, at any fixed value of the parameters there is no unique period, but two and even four different periods.

To see this and to describe the bifurcations leading to the changes in the periods of the cycles we make use of the first return map $F_R$.

We know that the discontinuity point behaves as a critical point, in determining the BCB (as a collision occurs due to a merging with the discontinuity point). A particular role is also played by the trajectories associated with the two values in the discontinuity point: $f_R(0) = \mu_R$ and $f_L(0) = \mu_L$. Here, we have a unique critical trajectory, as it can be immediately seen that

$$f_R(0) = \mu_R = f_L \circ f_R(\mu_L) = f_L \circ f_R \circ f_L(0) \quad (13)$$

and a particular integer determines the first return of the critical orbit in the interval $J_R$ for the return map $F_R$. In fact, let $d = F_R(\mu_L)$, then this point $d$ separates the interval of definition of the return map ($J_R = [0, \mu_L]$) in two different subintervals, having different properties. Let $k \geq 1$ be the integer defining the number of iterations (with $f_L$) to apply to $\mu_R$ in order to get a positive point, say $d$, that is:

$$d = f_L^k(\mu_R) = f_L^{k+1} \circ f_R(\mu_L) > 0 \quad (14)$$

and in explicit form:

$$d = \mu_R + k\mu_L \quad (15)$$

We shall see that when the critical point $d$ is positive, it separates the interval $J_R$ into two pieces:

$$J_R = C(J' \cup J''), \quad J' = [0, d], \quad J'' = [d, \mu_L] \quad (16)$$

(16)

(16)

where $C_l$ denotes the closure) such that all the points in $J'$ are periodic of period $p' = 2(k+1)$.
and all the points in $J''$ are periodic of period $p'' = 2(k + 2)$. We arrive at a BCB whenever we have an integer $k$ such that $d = \mu_R + k\mu_L = 0$. For now, we can consider the subintervals in (16) without the critical points defining their boundaries. As we shall see below, however, we can also define them as closed intervals, specifying the behavior of the suitable choice at the discontinuity point.

We notice that the point $d$ can also be determined as the first integer $k$ giving a preimage (of rank $(k+1)$) of the discontinuity point $x=0$ in the interval $J_R$ (i.e. the first positive preimage). That is, the first integer $k$ such that $d = f_R^{-1} \circ f_L^{-k}(0) > 0$. In fact, given this definition, then we have $f_R^{-1} \circ f_L^{-k}(d) = 0$ which implies $-d + \mu_R + k\mu_L = 0$, from which (15) is recovered. From this, we have that each point $x < d$ takes $(k + 1)$ iterations to return to the positive side, while a point $x > d$ takes $(k + 2)$ iterations, and, as before, $d = 0$ denotes a BCB.

In the case shown in Fig. 3(a) we have $k = 4$, thus all the points in the interval $J'$ have prime period $p' = 10$ except for the point in the middle of the interval, which has prime period 5; while all the points in the interval $J''$ have prime period $p'' = 12$ except for the point in the middle of the interval, which has prime period 6. Figure 4(a) shows the related first return map. Let us prove the following:

**Property 1.** Let $k$ be the first integer such that $d = \mu_R + k\mu_L \geq 0$. When $d > 0$ then the first return map $F_R$ is made up of two pieces with slopes $-1$, defined in $J'$ and in $J''$ given in (16), separated by the discontinuity point $d$. The intersection points with the diagonal (i.e. two fixed points of the return map) are $(0, 0)$ and $(d, d)$.
map $F_R$ are periodic points of $f$ with prime period $(k + 1)$ in $J'$ and $(k + 2)$ in $J''$, while all the other points are periodic points of $f$ with even periods: $2(k + 1)$ in $J'$ and $2(k + 2)$ in $J''$.

To prove this property, consider any point $x \in J'$ (and thus $d - x > 0$) then we have:

$$f_R(x) = -x + \mu_R \in [-\mu_L + \mu_R; \mu_R]$$

$$f_R^2 \circ f_R(x) = -x + \mu_R + k\mu_L$$

$$f_R \circ f_R^2 \circ f_R(x) = -x + d + \mu_R$$

which confirms the proposition for points in $J'$. Similarly, let us consider any point $x \in J''$ (and thus $\mu_L - x > 0$) then we have:

$$f_R(x) = -x + \mu_R \in [-\mu_L + \mu_R; \mu_R]$$

$$f_R^2 \circ f_R(x) = -x + \mu_R + (k + 1)\mu_L$$

$$f_R \circ f_R^2 \circ f_R(x) = -x + d - \mu_L + \mu_R$$

which confirms the proposition for points in $J''$. As all the points in $J'$ and in $J''$ so proved have even periods $2(k + 1)$ and $2(k + 2)$ and each periodic orbit has two points in the intervals $J'$ and $J''$, respectively, it follows that the middle points of these intervals have prime periods $(k + 1)$ and $(k + 2)$, respectively. Or also, analytically, let $x = d/2$ then

$$f_R \left(\frac{d}{2}\right) = \frac{d}{2} + \mu_R$$

$$f_R^2 \circ f_R \left(\frac{d}{2}\right) = \frac{d}{2} + \mu_R + k\mu_L = -\frac{d}{2} + d = \frac{d}{2}$$

while considering $x^* = (\mu_L + d)/2$ then we have

$$f_R(x^*) = -x^* + \mu_R$$

$$f_R^2 \circ f_R(x^*) = -x^* + d + \mu_L = x^*$$

which ends the proof.

Regarding the critical point $d$, we notice that if the point $d$ is considered belonging to the first interval $J'$, defining $f(0) = f_R(0) = \mu_R$ then $d$ is periodic of period $2(k + 1)$, in fact:

$$f_R^2(d) = -d + \mu_R$$

$$f_R^2 \circ f_R(d) = -d + \mu_R + k\mu_L = 0$$

$$(f_R \circ f_R^2)(d) = f_R^2 \circ f_R(0) = f_R^2(\mu_R) = \mu_R + k\mu_L = d$$

while if the point $d$ is considered belonging to the second interval $J''$, defining $f(0) = f_R(0) = \mu_L$, then $d$ is periodic of period $2(k + 2)$, in fact:

$$f_R(d) = -d + \mu_R$$

$$(f_R \circ f_R)(d) = -d + \mu_R + (k + 1)\mu_L = \mu_L$$

$$(f_R \circ f_R)(d) = f_R(\mu_L) = f_R^{k+1}(-\mu_L + \mu_R) = -\mu_L + \mu_R + (k + 1)\mu_L = d$$

As stated above, a bifurcation in the periods of the first return map are associated with the trajectory of the critical point ending in the discontinuity point $x = 0$. That is, when $d = 0$ occurs. Let us prove the following:

**Property 2.** Let $k$ be the first integer such that $d = \mu_R + k\mu_L \geq 0$. When $d = 0$ then the first return map $F_R$ is made up of one piece with slope $-1$, defined in $J''$. The intersection point with the diagonal (i.e. the fixed point of the return map $F_R$) is a periodic point of $f$ with prime period $(k + 2)$, while all the other points are periodic points of $f$ with even period $2(k + 2)$.

An example is shown in Fig. 3(b), where $k = 2$, and the related first return map is shown in Fig. 4(b): All the positive points in $J''$ are of period $8 = 2(k + 2)$ and the point in the middle of the interval has prime period $4 = (k + 2)$. To prove Property 2 consider the point $x = \mu_L/2$, then $f_R^2 \circ f_R(\mu_L/2) = \mu_L/2$, as

$$f_R \left(\frac{\mu_L}{2}\right) = -\frac{\mu_L}{2} + \mu_R$$

$$f_R^2 \circ f_R \left(\frac{\mu_L}{2}\right) = -\frac{\mu_L}{2} + \mu_R + (k + 1)\mu_L$$

$$= -\frac{\mu_L}{2} + \mu_R + \frac{\mu_L}{2}$$

while for any other point $x$ in $(0, \mu_L)$ we have $(f_R^2 \circ f_R)(x) = x$, in fact,

$$f_R(x) = -x + \mu_R \in [-\mu_L + \mu_R; \mu_R]$$

$$f_R^2 \circ f_R(x) = -x + \mu_R + (k + 1)\mu_L$$
\[ -x + \mu_L > 0 \in J' \]

\[ f_R \circ f^{k+1}_L \circ f_R(x) = x - \mu_L + \mu_R \]

\[ (f^{k+1}_L \circ f_R)^2(x) = x - \mu_L + \mu_R + (k + 1)\mu_L = x \]

which ends the proof.

Regarding the critical points when we have \( d = 0 \), notice that defining \( f(0) = f_R(0) = \mu_R \), then we have

\[ f^k_L \circ f_R(0) = 0 \]

and thus \( d = 0 \) is of period \( (k + 1) \), and \( \mu_L \) is preperiodic, while defining \( f(0) = f_L(0) = \mu_L \), then we have

\[ f^{k+1}_L \circ f_R \circ f_L(0) = 0 \]

and thus in such a case we have that \( d = 0 \) is of period \( (k + 3) \), and the trajectory includes all the critical points.

It is interesting to see that all the cycles of any period can exist, and keeping fixed \( \mu_R \) the period tends to infinity as \( \mu_L \) tends to 0. Moreover, as suggested by Fig. 2(a), after a specific value of \( \mu_L \) the bifurcations in the periods no longer occur. In fact, we have seen that the period decreases as \( \mu_L \) increases, however the value of \( k \) cannot be lower than 1, and \( k = 1 \) occurs whenever \( \mu_L > -\mu_R \). It follows that the last BCB takes place when \( \mu_L = -\mu_R \) giving \( d = 0 \) for \( k = 1 \) [see an example in Fig. 5(a)]. Then for any higher value of \( \mu_L \), we always have that the image of \( \mu_R, f_L(\mu_R) \), enters \( J_R \) in one iteration, and thus \( k = 1 \). This means that

![Graphs](image-url)

Fig. 5. (a) Graph of the map \( f \) at \( \mu_L = -\mu_R \) and \( \mu_R = -1 \). (b) Graph of the map \( f \) at \( \mu_R = -1 \) and \( \mu_L = 1.4 \) for which \( k = 1 \). (c) The related first return map is shown.
the simplest situation consists of points belonging to 4-cycles in $J'$ (a 2-cycle in the middle point), and points of 6-cycles in $J''$ (a 3-cycle in the middle point). A qualitative picture of this simplest case is shown in Fig. 5(b) and the related first return map in $J_R$ is shown in Fig. 5(c).

As well, we note that all the preperiodic points of the cycles can be easily obtained by taking the preimages $f$ of the intervals $J'$ and $J''$.

Summarizing, we have so proved the following theorem.

**Theorem 1.** Let $a_R = -1$, $a_L = 1$, $\mu_R < 0$ and $\mu_L > 0$, $I = (-\mu_L + \mu_R, \mu_L]$, $K = (-\mu_R/\mu_L)$. If $K$ is not an integer, let $k = \lceil K \rceil + 1$, then $d = \mu_R + k/\mu_L > 0$. $J' \cup J'' = [0, d[ \cup ]d, \mu_L]$ and Property 1 holds (all the points in $J'$ are $2(k+1)$-cycles and the middle point $d/2$ is a $(k+1)$-cycle, while all the points in $J''$ are $2(k+2)$-cycles and the middle point $x^* = (\mu_L + d)/2$ is a $(k+2)$-cycle). When $k = K$ is an integer then $d = \mu_R + k\mu_L = 0$, a BCB occurs and Property 2 holds (all the points in $J''$ are $2(k+2)$-cycles and the middle point $x^* = (\mu_L + d)/2$ is a $(k+2)$-cycle). The critical points and the periodic points cover the whole interval $I$. On the real line, the preperiodic points of the cycles in $I$ are obtained by the preimages of the points in $I$.

**4. Particular Case $a_{RL}L = 0$**

Here, we consider the particular case in which one of the functions defining the map is constant. Let us assume $a_R = 0$, as the other case comes from the property in (3). In this section, we consider the map

$$x' = f(x) = \begin{cases} f_L(x) = a_Lx + \mu_L, & \text{if } x < 0 \\ f_R(x) = \mu_R, & \text{if } x > 0 \end{cases}$$

where $\mu_R < 0$ and $\mu_L > 0$. As any point on the right-hand side is mapped into a unique point ($\mu_R < 0$), it follows that we only have to consider the trajectory of this point. Obviously, when not divergent, i.e. when (5) holds, $\mu_L/(1 - a_L) < \mu_R$, this trajectory is periodic and the only question is of which period. This information is given immediately by the first return map, which completely explains this case, and the related BCB curves, which can be seen in Fig. 6.

In Fig. 6 the region with negative values of $a_L$ is not represented because, as we shall see in the next section, that region only includes a stable 2-cycle or divergence. The integer that gives the period of the trajectories is obtained by the trajectory of the critical point $\mu_R$, since $f_L(x) = \mu_R$ for any $x > 0$, we apply the map $f_L$ to the point $\mu_R$ as long as we have a positive point again (no matter which one it is), thus from

$$f^{\mu}_L \circ f_R(0) = f^{\mu}_L(\mu_R) = \frac{1}{a_L} - \frac{\mu_L}{\mu_R}$$

if $a_L \neq 1$

$$f^{\mu}_L \circ f_R(0) = f^{\mu}_L(\mu_R) = \mu_R + k\mu_L$$

if $a_L = 1$

\[\text{(18)}\]

\[\text{(17)}\]

![Fig. 6](image-url).

(a) Two-dimensional bifurcation diagram at $a_R = 0$ and $\mu_R = -1$. (b) The analytical BCB curves that bound the stability regions of the cycles with symbol sequence $RL^k$, for $k = 1, \ldots, 20$.\[\text{Figure 6}\]
let \( k \geq 1 \) be the first integer such that \( f^L_k(\mu_R) > 0 \), then we have a \((k+1)\)-cycle. The curve of equation \( f^L_k(\mu_R) = 0 \), that is:

\[
a^L_R \mu_R + \mu_L \frac{1 - a^L_L}{1 - a^L_R} = 0, \quad \text{if} \quad a^L_L \neq 1
\]

\[
\mu_R + k \mu_L = 0, \quad \text{if} \quad a^L_L = 1
\]

(19)

denotes a BCB from the region of a \((k+1)\)-cycle to that of a \((k+2)\)-cycle. In the region in which \( f^L_k(\mu_R) > 0 \) the dynamics of the map converge to a \((k+1)\)-cycle. In Fig. 6(b) we show a few BCB curves (for \( k = 1, \ldots, 20 \)). The limit set of these BCB curves, as \( k \to \infty \), is the equation

\[
\mu_L = \mu_R (1 - a^L_L)
\]

(20)

which corresponds to the contact bifurcation given in (9) after which (for \( \mu_L < \mu_R (1 - a^L_L) \)) we have a region associated with divergent dynamics for \( f \).

5. Generic Case

In this section, we consider the generic case which is the main object of this work. In order to describe the BCB curves, let us first show a figure that summarizes the properties of our map. In Fig. 7, we show the regions associated with attracting periodic orbits in the parameter plane \((a_R, a_L)\). As the figure refers to the case where \( \mu_R = -1 \) and \( \mu_L = 1 \) are the bifurcation curves and the colored regions (denoting attracting cycles of different periods) are completely symmetric with respect to the main diagonal. When an asymmetry appears it refers to the coexistence of stable cycles, i.e. overlapping of periodicity regions (as the initial condition is kept as the same point \( x_0 = -0.001 \)). The similar figures at different values of \( \mu_R \) and \( \mu_L \) clearly do not have this symmetric shape, and the regions are slightly deformed, keeping in any case the same main qualitative properties, in particular, those of the points denoted by \( P_n \), although the overlapping regions may change their shape.

In each portion of the parameter plane of Fig. 7(a) we simply illustrate the qualitative shape of the map in the related quadrant.

5.1. Positive quadrant

We recall that in the positive quadrant as long as \( 0 < a_R < 1 \) and \( 0 < a_L < 1 \), we have stable cycles of any period, associated with the well-known period adding rule (examples of some particular maps can be found in [Avrutin & Schanz, 2006, 2008]), and the related BCB curves, characterizing the appearance/disappearance of cycles, are given in analytic form for several levels of complexity in [Gardini et al., 2010; Avrutin et al., 2010b], and the formulas there given can also be applied for higher complexity levels, which explains the BCB curves in that portion of phase plane. For each fixed constellation of the parameters only one stable cycle can exist, or none (when quasiperiodic trajectories occur). As long as the parameters satisfy the condition \( \mu_L (1 - a^R_R) - \mu_R (1 - a^R_L) > 0 \), none of the periodicity regions ever overlap. In Fig. 7, this inequality

![Fig. 7](image-url)

(a) Two-dimensional bifurcation diagram numerically obtained, with stable periodicity regions in different colors.

(b) A few BCB curves are drawn, the equations for which are given in the text.
corresponds to the region of the positive quadrant below the set \( S \) of equation

\[
(S): \mu_L (1 - a_R) - \mu_R (1 - a_L) = 0 \tag{21}
\]

Crossing this set, the map turns from the invariant absorbing interval \([\mu_R, \mu_L] \) into nonuniquely invertible. The region below the set \( S \) is called the stability region in [Gardini et al., 2010], where it is also proved that each pair of the infinitely many BCB curves associated with each periodicity region intersect in points of the set \( S \) where a stability change occurs. For parameters above \( S \), infinitely many unstable cycles exist, and the dynamics are chaotic, with robust chaos in cyclic intervals. In that region the BCB are responsible for the changes in the cyclical chaotic intervals, also called chaotic band and band merging bifurcations, as described in [Avrutin et al., 2007, 2008a, 2008b, 2009; Avrutin et al., 2010c].

A bounded chaotic attractor exists up to the final bifurcation leading to divergence of the generic trajectory. This occurs when the parameters cross the bifurcation curves associated with the contact of the invariant interval with an unstable fixed point of \( f \).

The boundaries of the BCB curves of the periodicity regions of first level of complexity associated with the so-called maximal cycles or principal cycles, will be recalled below, as they also intersect the region of interest here, the one with \( a_L < 0 \).

As we can see from Fig. 7, the sets \( a_R = 0 \) and \( a_L = 0 \) of the parameter space we have only contiguous regions of stable cycles of increasing period, one after the other. This reflects the property proved in the previous section.

5.2. Stability region of the 2-cycle

We describe here, the wide periodicity region associated with the 2-cycle, whose periodic points \( x_0^* > 0 \) and \( x_1^* < 0 \) are given by

\[
x_0^* = \frac{\mu_R + \mu_L}{1 - a_R a_L} > 0, \quad x_1^* = \frac{\mu_R + \mu_L}{1 - a_R a_L} < 0 \tag{22}
\]

The two BCB curves, due to the merging of the periodic points with the discontinuity point in \( x = 0 \), are given by:

\[
\xi_{RL}: a_R a_L + \mu_L = 0, \quad \xi_{RL}: a_R a_L + \mu_R = 0 \tag{23}
\]

(the straight lines of equation \( a_L = -(\mu_L/\mu_R) \) and \( a_R = -(\mu_R/\mu_L) \) in Fig. 7). The 2-cycle is stable as long as \( |a_R a_L| < 1 \) and becomes unstable via a degenerate flip-bifurcation when the parameters satisfy the equation

\[
\theta_{RL}: a_R a_L = -1 \tag{24}
\]

which gives the two portions of hyperbola in red in Fig. 7, and a portion of the existence region is associated with an unstable 2-cycle. Moreover, the boundary of the stability region of the 2-cycle in the lower left portion of the parameter plane \((a_R < 0 \) and \( a_L < 0)\) is the curve of equation

\[
\chi_{RL}: a_R a_L = 1 \tag{25}
\]

which corresponds to the merging of the 2-cycle with a point on the Poincaré Equator, at infinity. In fact, as we can see from (22), the coordinates of the periodic points of the 2-cycle tend to infinity as \( a_R a_L \) tends to 1, after which (for \( a_R a_L > 1 \)) no cycle exists in the real line, and the dynamics are all divergent (similar bifurcations are described in [Avrutin et al., 2010c]).

We recall that all the flip bifurcations occurring in piecewise linear maps to a \( k \)-cycle are degenerate (see also [Sushko & Gardini, 2010]) which means that at the bifurcation value a segment of the real line exists in which all the points are \( 2k \)-cycles, with the bifurcating \( k \)-cycle inside. After the flip bifurcation, all such \( 2k \)-cycles disappear, leaving only one unstable \( k \)-cycle. The cycle which bounds the interval at the bifurcation value also undergoes a border collision. The dynamics occurring after the degenerate flip bifurcation depend on the shape of the map in the other part of the segment of cycles.

5.3. Region with \( a_R a_L < 0 \)

In Sec. 2 we have already remarked that contact bifurcations leading to almost all divergent trajectories occur when the invariant interval \( I = [a_R a_L + \mu_R, \mu_R a_L] \) has a contact with the unstable fixed point \( P^* \). The curve given in (9) associated with the “final bifurcation” is also drawn in Fig. 7(b) (green arc of curve), bounding the regions of mainly divergent dynamics in the upper left portion of the phase plane. This bifurcation is the border collision bifurcation of the critical point (lower boundary of \( I \)) with the unstable fixed point \( P^* \) (also homoclinic bifurcation of \( P^* \)).

From the same picture it is clear that the infinitely many periodicity regions existing for \( 0 < a_R < 2 \) and \( 0 < a_L < 2 \) issue from particular points on the axes \( a_R = 0 \) and \( a_L = 0 \), which are...
also points of intersection of two periodicity regions of maximal cycles. Crossing such axes and entering the region \(a_R a_L < 0\), the periodicity regions of the maximal cycles intersect in pair, that is, for any \(k \geq 2\) the periodicity region of the \(k\)-cycle intersects the region of the \((k+1)\)-cycle, and for each cycle there exists a region in which it is the only stable one, without overlapping other stable periodicity regions. This was already determined for particular families of maps in [Avrutin et al., 2006]. Here, we consider the generic map in standard form, in order to give an analytical proof of this and other properties by using the first return map \(F_R\), which will be considered in the following subsections.

5.3.1. BCB curves

The reason why we limit our interest here only to the regions of cycles of first complexity level is that they are the only cycles which may be stable when \(a_R a_L < 0\). Let us recall that in the particular cases \(a_R a_L = 0\), as we have seen in Sec. 4, only these cycles exist, and their boundaries represent intersection points of two different BCB curves, denoting the intersection of periodicity regions crossing from \(a_R a_L > 0\) to \(a_R a_L < 0\). The periodicity regions associated with these stable periodic orbits of increasing periods are bounded by BCB curves whose analytic expression is easy to obtain. Such cycles have the simplest structure, obtained when we apply in order the maps with symbol \(\cdot \cdot \cdot L\). This is a simple case because in this way we can simply order the periodic points, and easily perform the related computations. For a cycle of period \((n + 1)\) let us call the related periodic points as \(0 < x_0^* < \mu_L\) and \(x_1^* < \cdot \cdot \cdot < x_n^* < 0\). Then the periodic points of the cycle are fixed points of the iterated map \(f^{n+1}(x)\) and \(x_0^*\) is the fixed point of the linear function \(f_1^n \circ f_R(x)\), which is a periodic point for \(f(x)\) as long as

\[
0 \leq x_0^* \leq \mu_L
\]

and the conditions \(x_0^* = 0\) and \(x_0^* = \mu_L\) (this one corresponds to \(x_n^* = 0\)) determine the BCB curves. Then from:

\[
\begin{align*}
\left( f_1^n \circ f_R(x) \right) &= (a^2_R a_R)x \\
&\quad + \left( \mu a^2_R + \mu_L \frac{1-a^2_R}{1-a_L} \right) \quad \text{if } a_L \neq 1 \\
\left( f_1^n \circ f_R(x) \right) &= a_R x + \mu_R + n \mu_L \quad \text{if } a_L = 1
\end{align*}
\]

by using the equality \(x_0^* = f_1^n \circ f_R(x_0^*)\) we obtain the periodic point:

\[
0 \leq x_0^* = a^2_R (\mu_R + \mu_L \phi^L_n) \frac{1}{1-a^2_R \mu_L} \leq \mu_L
\]

where

\[
\phi^L_n = \frac{1-a^2_R}{(1-a_L) a_L} \quad \text{if } a_L \neq 1,
\]

\[
\phi^L_n = n \quad \text{if } a_L = 1.
\]

Notice that in the region we are interested in (\(a_R < 0\) and \(a_L > 0\)), the denominator in (26) is always positive. We denote by \(\xi_{RL}^L\) (resp. \(\xi_{RL}^R\)) the BCB curve obtained due to the merging of a periodic point of the orbit with the discontinuity point \(x = 0\) from the right (resp. left) side. That is, the BCB curve \(\xi_{RL}^L\) is the BCB curve obtained due to the merging \(x_0^* = 0\), while \(\xi_{RL}^R\) is the BCB curve obtained due to the merging \(x_n^* = 0\) which also corresponds with \(x_0^* = \mu_L\). Thus the two BCB curves are obtained due to the merging of the periodic point \(x_n^*\) with the boundaries of its existence interval. From \(0 \leq x_0^* \leq \mu_L\) we get:

\[
a^2_R (\mu_R + \mu_L \phi^L_n) - \mu_L (1 - a^2_R a_R) \leq 0 \quad \text{\(a^2_R (\mu_R + \mu_L \phi^L_n) \geq 0\)}
\]

so that the BCB curves are:

\[
\begin{align*}
\xi_{RL}^L:: & \quad a^2_R (\mu_R + \mu_L \phi^L_n) - \mu_L (1 - a^2_R a_R) = 0 \quad (28) \\
\xi_{RL}^R:: & \quad \mu_R + \mu_L \phi^L_n = 0 \quad (29)
\end{align*}
\]

It is plain that the eigenvalue of a cycle is given by the product of the slopes of the linear maps which are applied in the cycle. So this \((n + 1)\)-cycle becomes unstable via degenerate flip when

\[
\theta^L_{RL}:: a^2_R a_R = -1
\]

denoting the flip-bifurcation curves. In Fig. 7(b) we have drawn a few BCB curves \(\xi_{RL}^L\) (horizontal lines in that projection) and \(\xi_{RL}^R\) in black, together with a few flip-curves \(\theta^R_{RL}\) in red. In the upper left part, it can be seen that all three involve (intersect), a particular blue curve there reported, and also shown in Fig. 7(a). This curve represents the transition of the map \(f\) from invertible in the absorbing interval \([a_R a_L + \mu_L]\) to noninvertible in that interval. The map \(f(x)\) is invertible as long as \(f_1(a_R a_L + \mu_L) > \mu_R\) so that the set \((C_\theta)\) — crossing which the map becomes non-uniquely invertible — is given by \(f_1(a_R a_L + \mu_R) = \mu_R\) that is:

\[
C_\theta:: \mu_L (1 + a_R a_R) + \mu_R (a_L - 1) = 0 \quad (31)
\]
The similar transition when the slopes are positive is given by the set \( (S) \) defined in (21). As already remarked, in that region the crossing of the set \( (S) \) represents a drastic transition from only stability to only robust chaos. This is not the case when the slopes satisfy \( A_{RL} < 0 \). In fact, it is easy to see that in Fig. 7(a) above the set \( C_N \) there is a white region (associated with chaotic dynamics) followed by a small portion of stability of the 4-cycle.

It is worth noticing also that the set \( C_N \) intersects all the BCB curves \( \xi_{RL} \) exactly at the points belonging to the flip bifurcation curves of cycles with the symbolic sequence \( RL^{n+1} \) existing above that curve. The BCB curve \( \xi_{RL^{n+1}} \) also crosses at this point. This is not numerical evidence, and the following Property will be used later:

**Property 3.** Let \( P_n \in \xi_{RL} \cap \theta_{RL^{n+1}} \) for \( n \geq 1 \), then \( P_n \in C_N \) and \( P_n \in \xi_{RL^{n+1}} \).

In fact, from (30) we have \( \alpha_n^2 = -1/(a_L a_R) \), and from (29) and (27), substituting:

\[
\mu_R + \mu_L \frac{a_L a_R + 1}{a_L - 1} = 0
\]

from which the expression in (31) is recovered. After some algebra it can be seen that also the equation of \( \xi_{RL^{n+1}} \) is satisfied. The computations with the formulas given in (28) are not easy, but in the next subsection we shall give a different expression of the same bifurcation curve, with which follows now.

Another peculiarity of the flip-bifurcation curves is that those associated with the cycles of period 3 and 4 intersect each other and exchange at a very particular point, when \( a_L = 1 \) and \( a_R = -1 \). This is immediate from their equations in (30). Indeed the same property holds for any flip-curve, but for the other cycles it is not dynamically relevant because the flip-curves cannot exit from the existence regions of the cycles of periods 2, 3, 4 only. The property which holds iteratively for all the cycles existing for \( a_L > 1 \) is the one noted in Property 3, that is, a flip-bifurcation curve \( \theta_{RL^{n+1}} \) that starts from the BCB curve \( \xi_{RL^{n+1}} \) crosses through \( \xi_{RL} \) at the point \( P_n \) and ends in the BCB curve \( \xi_{RL^{n+1}} \).

**5.3.2. Use of the first return map**

We describe here a different (and useful) approach to study the bifurcations in the case of slopes with opposite sign, for which a first return map \( F_R \) is well defined. As we have seen, when an invariant absorbing interval \( I = [a_{RL} L + \mu_R, RL] \) exists, it is not possible to apply the function on the right side for two consecutive iterations. In fact, whenever we consider a positive point \( x > 0 \), then we have \( f_R(x) < 0 \). This allows us to a correct definition of the first return map. Let \( J_R = [0, \mu_L] I \cap [x > 0] \) then we can simply construct the first return map in \( J_R \), as

\[
F_R: x \in J_R \rightarrow f^{m}(x) \in J_R
\]

where \( m \) is the first integer for which \( f^{m}(x) \in J_R \), and this map describes all the bifurcations (including the BCB) of the map \( f \) in \( I \). As \( f \) is discontinuous with a discontinuity point in \( x = 0 \), which means that points in a neighborhood of \( x = 0 \) are subject to different functions, a similar fate will occur also to points on opposite sides of the preimages of the discontinuity point \( x = 0 \), depending on the side to which they belong. The preimages of the discontinuity point \( x = 0 \) must necessarily be taken first with the left side, i.e. with \( f_R^{-1} \) as long as we obtain a point in the range of \( f_R \) to which we can apply \( f_R^{-1} \) reaching the side \( x > 0 \). Thus, as seen in Sec. 3, an important integer is the first, necessarily \( k \geq 1 \), such that

\[
d_k = f_R^0 \circ f_R^{j}(0) > 0
\]

and clearly, once a first preimage \( d_k \) exists, then also

\[
d_{k+j} = f_R^{-1} \circ f_R^{j+k}(0) > 0, \quad \forall j \geq 1
\]

exist as preimages of the origin on the \( R \) side, as the preimages \( f_R^{j+k}(0) \) tend to the unstable fixed point \( P^* \) as \( j \) tends to \( \infty \), when \( P^* \) exists (i.e. for \( a_L > 1 \)), otherwise, the preimages tend to \( -\infty \). To understand the dynamics of \( f(x) \), however, we are only interested in the points \( d_{k+j} \) belonging to (or entering inside) the interval domain of \( F_R: J_R = [0, \mu_L] \).

It is plain that for the first return map \( F_R \), \( d_k \) must be a discontinuity point, and points in \( 0 < x < d_k \) will take \( (k+1) \) iterations in order to be positive again, while points in a right neighborhood of \( d_k \), between \( d_k \) and \( d_{k+1} \), will take \( (k+2) \) iterations. Thus it is easy to see that an important bifurcation occurs whenever we have \( d_k = \mu_L \) as this denotes that the discontinuity point of the first return map \( F_R \) is entering the interval \( J_R \) [see Figs. 8(a) and 8(b)], and a bifurcation also occurs whenever we have \( d_k = 0 \), as this denotes that the
discontinuity point of the first return map $F_R$ is exiting from the interval $J_R$ [see Fig. 8(c)]. Clearly both these bifurcations cause a change in the definition of the first return map, as the number of iterations to be applied to $f_*^{L-1}$ in order to get the first preimage of the origin on the positive side is changed.

From the definition of $d_k$ in (32) we also have $f_0(x) = f_0^L(x)$, thus considering the function $f_0(x) = a_0^L(x) + a_0^L(\mu + \mu L\phi_k^L)$, from $f_0^L(0) = 0$ we obtain:

$$d_k = \frac{\mu_R + \mu L\phi_k^L}{\mu_L}$$  \hspace{1cm} (34)

So that defining $T_0(x) = f_0^L(x)$ then $T_0(x)$ represents the first return map for $0 < x < d_k$, and we have to consider $T_1(x) = f_0^L(x)$ as the first return function for $d_k < x < d_{k+1}$, thus, when $0 < d_k < \mu L < d_{k+1}$ we have a well defined map [see Fig. 8(b)]:

$$F_R(x) = \begin{cases} T_0(x) = f_0^L(x) & \text{if } 0 \leq x < d_k \\ T_1(x) = f_0^L(x) & \text{if } d_k < x \leq \mu L \end{cases}$$

The components are both decreasing functions, as the slopes of $T_0$ and $T_1$ are negative, and the fixed points of $F_R$ are periodic points of $f_0(x)$, i.e. when $0 < d_k < \mu L < d_{k+1}$, but bifurcations occur whenever the intervals defined in (39) and (40) reduce to a point, that is:

(i) when $d_k = 0$, as the cycle of period $(k+1)$ with symbol sequence $RL^k$ disappears/appears

$$x_0^k = \frac{\phi_k^L(\mu + \mu L\phi_k^L)}{1 - a_0^L a_R}$$

$$x_1^k = \frac{\phi_k^L(\mu + \mu L\phi_{k+1}^L)}{1 - a_0^L a_R}$$

$$\phi_{k+1}^L = \phi_k^L + \frac{1}{a_0^L}$$

$$d_{k+1} = d_k + \frac{\mu L}{-a_0^L a_R}$$

Notice that the ranges of the two functions defined in (35) are given by:

$$T_0(0, d_k) = [\phi_k^L(\mu + \mu L\phi_k^L), 0] = [-a_0^L a_R d_k, 0]$$

As already noted, the two functions $T_0$ and $T_1$ given above in general define a first return map, i.e. when $0 < d_k < \mu L < d_{k+1}$, but bifurcations occur whenever the intervals defined in (39) and (40) reduce to a point, that is:

(i) when $d_k = 0$, as the cycle of period $(k+1)$ with symbol sequence $RL^k$ disappears/appears
from $J_R$, which gives the BCB curves $\zeta_{RL}$ of equation $\mu_R + \mu L \phi^L_k = 0$ given in (29), and

$$(ii)\text{ when } f_L^{k+1} \circ f_R(\mu_R) = \mu_R, \text{ as the cycle of period } (k + 2) \text{ with symbol sequence } RL^{k+1} \text{ appears/disappears from } J_R, \text{ which gives }$$

$$\zeta'_{RL^{k+1}}: a_L^{k+1} + (a_R \mu L + \mu_R + \mu L \phi^L_{k+1}) - \mu_R = 0$$

corresponding to the BCB curves $\zeta'_{RL^{k+1}}$ given in (28). But note that $f_L^{k+1} \circ f_R(\mu_L) = \mu_L$ means $f_L(f_L^{k} \circ f_R(\mu_L)) = \mu_L$ and this occurs iff $f_L^k \circ f_R(\mu_L) = 0$ (that is iff $d_k = \mu_L$). Thus the BCB curves $\zeta'_{RL^{k+1}}$ also have a simpler equation (considering $f_L^k \circ f_R(\mu_L)$ from (35)), given by:

$$\zeta'_{RL^{k+1}}: a_R \mu L + \mu_R + \mu L \phi^L_k = 0 \quad (41)$$

To be also more explicit the proof is as follows:

$$a_L^{k+1}(a_R \mu L + \mu_R + \mu L \phi^L_{k+1}) - \mu_R = 0$$

$$a_R = \mu_R - a_L^{k+1}(\mu_R + \mu L \phi^L_{k+1})$$

$$a_R = \frac{1}{a_L^{k+1}} \frac{\mu_R}{\mu_L} \phi^L_{k+1}$$

$$a_R = \frac{\mu_R}{\mu_L} \phi^L_k$$

$$\mu_R + \mu L \phi^L_k + a_R \mu L = 0$$

We have so proved the following.

**Theorem 2.** Let $a_R < 0, a_L > 0, \mu_L < 0$ and $\mu_R > 0$. Let $d_k = f_L^{k+1} \circ f_R^k(0) = (\mu_R + \mu L \phi^L_k) - a_R$ be the first preimage of the discontinuity point $x = 0$ on the positive side $R$. Then the BCB curves of the cycles of first complexity level are given by

$$c_{RL^{k+1}}: d_k = \mu_R \quad (i.e. \mu_R + \mu L \phi^L_k + a_R \mu L = 0)$$

$$c'_{RL^{k+1}}: d_k = 0 \quad (i.e. \mu_R + \mu L \phi^L_k = 0) \quad (42)$$

where $\phi^L_k$ is defined in (27):

$$\phi^L_k = \frac{1}{1 - a_L^{k+1}} \quad \text{if } a_L \neq 1,$$

$$\phi^L_k = k \quad \text{if } a_L = 1.$$

We recall that by using the equations given in (42) the algebraic steps to prove Property 3 are quickly derived.

Let us also notice that the two different dynamic behaviors observed in Fig. 7 crossing the point $(\alpha_R, \alpha_L) = (-1,1)$ have a clear interpretation in the first return map $F_R$. In fact, from $\lambda_{k+2} = a_L^{k+1} a_R = a_L \lambda_{k+1}$ we immediately have the following property characterizing the flip bifurcations.

**Property 5**

(j) if $\alpha_L < 1$ then $\lambda_{k+1} < \lambda_{k+2} = a_L \lambda_{k+1}$ and thus for a transition in which $\alpha_R$ is decreasing, the degenerate flip bifurcation of the cycle $RL^k$ of lower period occurs first (i.e. before that of cycle $RL^{k+1}$) [Fig. 9(a)];

(jj) if $\alpha_L > 1$ and $-1 < a_R < 0$ then $\lambda_{k+2} = a_L \lambda_{k+1} < \lambda_{k+1}$ and thus for decreasing $a_R$ the degenerate flip bifurcation of the cycle $RL^{k+1}$ of higher period occurs before that of the cycle $RL^k$. (Fig. 9(b), corresponding to point A in Fig. 10);

(jjj) the degenerate flip bifurcations of two coexisting cycles occur simultaneously iff $\alpha_L = 1$ and $a_R = -1$.

![Fig. 9. Qualitative shapes of the first return map $F_R$. The case shown in (b) corresponds to point (A) in Fig. 10, while the one in (c) corresponds to point (C) in Fig. 10.](image-url)
Fig. 10. Qualitative behavior of the first return map in the different regions bounded by the bifurcations curves. The light blue region qualitatively represents the corresponding light blue region for \( k = 2 \) in the parameter plane in Fig. 7(b).

The case in (jj), due to \( \lambda_{k+2} = \lambda_{k+2} = -1 \), corresponds to the particular case fully considered in Sec. 3.

The following properties characterize the crossing of a parameter point through a BCB curve \( \xi_{RLk}^+ \).

Property 6. Let the parameters \((a_R, a_L, \mu_R, \mu_L)\) satisfy \( d_{k+1} = 0 \) then

\[
d_{k+1} = \mu_L - \frac{\mu_L}{\lambda_{k+2}}
\]

\[
T_1(\mu_L) = \mu_L(1 - |\lambda_{k+2}|)
\]

In fact, at a bifurcation value due to \( d_k = 0 \) from (38) we have immediately the expression of \( d_{k+1} \), which proves (43). While from (35) by using (37) we have:

\[
T_1(\mu_L) = a_{k+1}^+ (a_R \mu_L + \mu_R + \mu_L \phi_{k+1}^L)
\]

\[
= a_{k+1}^+ a_R \mu_L + a_{k+1}^+ \mu_R + \mu_L \phi_{k+1}^L
\]

\[
= a_{k+1}^+ a_R \mu_L - a_{k+1}^+ \mu_R d_k + \mu_L
\]

\[
= \mu_L (1 - |\lambda_{k+2}|)
\]

which proves (44).

The particular expressions so determined are used to prove the following.

Property 7. Let the parameters \((a_R, a_L, \mu_R, \mu_L)\) satisfy \( d_{k} = 0 \) (and thus belonging to the BCB curve \( \xi_{RL}^+ \) associated with the cycle \( RL_k^+ \)) then:

(i) If the cycle \( RL_{k+1}^+ \) is stable then we have \( T_1(\mu_L) > 0 \) and \( d_{k+1} = \mu_L/(|\lambda_{k+2}|) > \mu_L \), which means that there are no discontinuity points inside the interval \( J_R \); thus the map \( FR \) (and \( f \)) is invertible: the existing cycle \( RL_{k+1}^+ \) is globally attracting, and the return map \( FR \) is continuous (see Fig. 8(c), corresponding to point B in Fig. 10);

(ii) if the cycle \( RL_{k+1}^+ \) is degenerate, \( \lambda_{k+2} = a_{k+1}^+ a_R = -1 \), so that \( d_{k+1} = \mu_L \), then
Then the return map $F$ is continuous and invertible being the diagonal with slope $-1$ in the whole interval $J_R$, thus also $f$ is invertible, all the points in $J_R$ are cycles of period $2(k+2)$ except for the fixed point (the graph of $F_R$ in the point $P_1$ is shown in Fig. 10).

(iii) if the cycle $RL^{k+1}$ is unstable then $T_1(\mu_L) < 0$, which also means that the discontinuity point $d_{k+1} = \mu_L/(\lambda_{k+1}) < \mu_L$ is inside the interval $J_R$ and the first return map is defined by $T_1(x)$ for $0 < x < d_{k+1}$ with range $T_1([0,d_{k+1}]) = [\mu_L,0]$ and by $T_2(x)$ for $d_{k+1} < x < \mu_L$, which means that in the definition in (35) the integer $k$ is increased by 1, but also that the return map $F_R$ (and thus $f$) is noninvertible (Fig. 9(c), corresponding to point $C$ in Fig. 10).

For $a_L > 1$ and decreasing $a_R$, $-1 < a_R < 0$, we know that the flip bifurcation of the cycle $RL^{k+1}$ of higher period occurs first, and at the bifurcation value, when $\lambda_{k+2} = a_L^{k+1}/a_R = -1$, we have $T_1(\mu_L) = a_{k+1}^{k+1}/a_R - a_{k+1}^{k+1}a_R + \mu_L = \mu_L$ (as in fact the first return map $T_1(x)$ has slope $-1$), and the cycle $RL^k$ is still stable, in fact $0 > \lambda_{k+1} = a_L^a a_R > -1$ and $T_1(0) = a_L^a (\mu_R + a_L^a) = (d_k/a_L) < d_k$, then the map $F_R$ (and thus $f$) is invertible. To summarize, we have proved the following.

**Theorem 3.** Consider $a_L > 1$, $-1 < a_R < 0$ and the region of invertibility of $f$ (below the set $C_N$). Then the return map $F_R$ is either continuous in $J_R$ (region (b) in Fig. 10), in which case there exists a unique globally attracting cycle, or there exist at most one discontinuity point (regions (a) and (c) in Fig. 10), and thus with two cycles, one of which is necessarily stable.

(i) In the overlapping region below the flip bifurcation curves (region (a) in Fig. 10), two stable cycles with symbol sequence $RL^k$ and $RL^{k+1}$ coexist, and no unstable cycle can exist. The points of the interval $(0,d_k)$ converge to the cycle with symbol sequence $RL^k$ while the points of the interval $(d_k,\mu_L)$ converge to $RL^{k+1}$.

(ii) In the region between two flip bifurcation curves there exists a unique stable cycle (region (c) in Fig. 10), attracting all the points except those of the coexistent unstable cycle.

Figure 10 illustrates the qualitative shape of the first return map $F_R$ and related properties of the map $f$ close to each of the infinitely many points $P_k$ with $k > 1$ as shown in Fig. 7.

Notice that crossing through the particular points $P_k$, we can have a direct transition from regular regime to chaos, and thus infinitely many BCB curves must necessarily issue from such points.

While for $a_L < 1$ and decreasing $a_R$, we know that the flip bifurcation of the cycle $RL^k$ of lower period occurs first, however, as already remarked, only cycles of low periods (2,3,4) can be attracting. We shall see two examples in the following subsection.

### 5.4. Examples

As illustrative examples we show two bifurcation diagrams of increasing $a_L$, fixing $-1 < a_R < 0$
and $a_R < -1$. Let us consider first the example in which $a_R = -0.8$ and we increase the parameter $a_L$ from zero [Figs. 11 and 12(a)]. From the two-dimensional bifurcation diagram in Fig. 7 we know that we start with a stable 2-cycle, which is the only existing attractor as long as we cross the bifurcation curve $\xi_{RL}^2$ and an attracting 3-cycles coexists: in the first return map we have a discontinuity point $d_1$ [Fig. 11(a)].

The one-dimensional bifurcation diagram as $a_L$ increases is shown in Fig. 12(a). The 2-cycle coexists with the 3-cycle up to the disappearance of the 3-cycle via BCB, crossing the BCB curve $\xi_{RL}^2$ ($a_L \mu_R + \mu_L = 0$), leaving a unique 3-cycle up to the appearance of the 4-cycle as $a_L$ crosses the BCB curve $\xi_{RL}^3$ (at $a_L = 1.0732$). In Fig. 12(a), we can see that the flip bifurcation of the 4-cycle (at $a_L = 1.0772$) occurs before that of the 3-cycle (at $a_L = 1.118$), after which we only have chaotic dynamics. The first return map $F_R$ in $J_R$ when the 3-cycle is unstable and all the other points in $I$ converge to the stable 3-cycle is shown in Fig. 11(b), where $F_R$ is given by $T_0 = f_2^L \circ f_R$ in $(0, d_1)$, and by $T_1 = f_3^L \circ f_R$ in $(d_1, \mu_L)$. The map becomes noninvertible when the 3-cycle is still stable, and at the flip-bifurcation of the 3-cycle, the dynamics become chaotic. When $T_1(\mu_L) = 0$ a new BCB curve $\xi_{RL}^4$ is crossed and a new discontinuity point $d_2$ appears in the first return map which is so defined:

$$
T_0 = f_2^L \circ f_R \quad \text{in } (0, d_1),
$$

$$
T_1 = f_3^L \circ f_R \quad \text{in } (d_1, d_2),
$$

$$
T_2 = f_4^L \circ f_R \quad \text{in } (d_2, \mu_L).
$$

![Graph of the function $f$ and first return map $F_R$ at $\mu_R = -1$, $\mu_L = 1$ and $a_R = -0.8$. (a) $a_L = 1.4$. (b) $a_L = 1.55$.](image)

![One-dimensional bifurcation diagrams at $\mu_R = -1$ and $\mu_L = 1$, as a function of $a_L$. (a) $a_R = -0.8$. (b) $a_R = -1.8$.](image)
as shown in Fig. 13(a). When $T_2(\mu_L) = 0$, a new BCB curve $b_{BCB}^n$ is crossed and a new discontinuity point $d_3$ appears and the first return map increases by one piece, by $T_3 = f_{L}^3 \circ f_{R}$ in $(d_3, \mu_L)$, and so on. The number of discontinuity points $d_3$ and pieces in the return map tends to infinity as the final contact bifurcation with the unstable fixed point $P^*$ is approached [see Fig. 13(b)]. The integer associated with the first discontinuity point increases by 1 whenever a periodicity region ceases to exist (as we increase the parameter $\mu_L$). In Fig. 12(b), we also show a one-dimensional bifurcation diagram at $\alpha_R = -1.8$ and we increase the parameter $\alpha_L$ from 0. We can see that now the flip bifurcation of the 2-cycle occurs first, followed by the flip bifurcation of the 3-cycle and the last one is that of the 4-cycle.

References


