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Routes to Complexity in a Macroeconomic Model Described by a Noninvertible Triangular Map

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Summary. In this paper a particular discrete-time macroeconomic model is considered, where the savings are proportional to income and the investment demand depends on the difference between the current income and its exogenously assumed equilibrium level, through a nonlinear S-shaped increasing function. The model proposed can be seen as a particular case of a general class of business cycle models, known as Kaldor-type models, which are characterized by the fact that the investment demand also depends on the capital stock (and this assumption is usually considered the main requirement for the occurrence of oscillating behaviour of income and capital). The resulting model is described by a dynamical system in income and capital whose time evolution is given by the iteration a two-dimensional map of triangular type: this means that one of the components of the map, namely the one driving the income evolution, is an independent one-dimensional map. Due to the particular triangular structure of the system, the asymptotic dynamic behaviour and the bifurcations can be completely described starting from the properties of the associated one-dimensional map. The dynamic behaviors of the model are explored under different ranges of the main parameters, such as the firms' speed of adjustment to the excess demand and the propensity to save. Although our exercise shows that the basic dynamic scenario is given by a situation of bi-stability, i.e. coexistence of two stable steady states, the existence of more complex dynamics is proved, for sufficiently high values of the adjustment parameter. The effects of the switching to the regime of noninvertibility of the map on the basins' structure are considered, together with the bifurcations which modify the structure. Moreover, some bifurcations which change the qualitative properties of the attracting sets are analyzed, in particular a global (homoclinic) bifurcation which causes the transition from a situation of bi-stability to a regime characterized by wide chaotic oscillations of income and capital around their exogenously assumed equilibrium levels, i.e. a typical situation of irregular business cycles.

1 Introduction

Since the early attempts, made in the years 1930s-1940s, to describe and understand the *business cycle*, that is, the presence of self-sustained oscillations of the main economic variables (income, capital, inflation rate, ...), several economists became aware that the complexity of these phenomena could only be explained by the use of nonlinear models.

One of the first, and simplest, nonlinear models of the business cycle was proposed by Kaldor [8], who assumed nonlinear investment and savings functions, and explained the economic fluctuations as a consequence of the long term "shifting" of the investment demand curve caused by changes in the capital stock (see also [6]). This kind of dependence of the investment demand on the capital stock, together with the nonlinearity of the investment function or the savings function, has generally been considered, in the literature on Kaldor's model, as the basic structural requirement for the existence of self-sustaining cycles (see [2], [6]).

Our starting point is a particular discrete-time *Kaldor*-type model, proposed in [18] and investigated in its general dynamic behaviour in [5], where the savings are assumed proportional to the income and the investment demand depends both on the income, through a nonlinear *S*-shaped increasing function, and on the capital stock, through a linear decreasing function. Such model, which can ultimately be reduced to a two-dimensional dynamical system in income and capital, is able to generate endogenous fluctuations for certain ranges of the parameters.

In the present paper, we wonder what happens if the main Kaldorian assumption, i. e. the dependence of the investment demand from the capital stock, is neglected. As we shall see, in this case we get a discrete-time dynamical system described by the iteration of a map of the plane of *triangular* type. Its analysis constitutes a pedagogical tour through the properties of *triangular maps*, i.e. two-dimensional maps which have the peculiarity that one of their components is decoupled from the other, so that it is an independent one-dimensional map. As shown in [7], the particular structure of such maps implies that the asymptotic dynamics, as well as the bifurcations, can be deduced from the associated one-dimensional map.

We show that, for economically meaningful values of the parameters, the model has three steady states, and we study the influence of the main parameters, like the *propensity to save* and the firms' *speed of adjustment* to the excess demand, on the local stability of the equilibria. The basic dynamic scenario of the model is given by a situation of *bistability*, where two stable steady states coexist: one characterized by a low level of income and capital (*poverty steady state*) and one by a high level of the dynamic variables (*wealth steady state*), each with its own basin of attraction. In the presence of bistability, the question of the delimitation of the basins of the coexisting attracting sets naturally arises. The triangular map considered, whose iteration gives the time evolution of the model, is invertible or noninvertible according to the values range of the parameters, and we show how the dynamic behaviour of the system is deeply influenced by the switching to the regime of noninvertibility of the map. This leads to an increasing complexity not only in the nature of the attracting sets but also in the structure of their basins of attraction. In fact, although in the literature on nonlinear dynamical systems applied to economic models the term *complexity* is generally related to the structure of the attractors, in this paper we also stress the presence of a different kind of complexity, related to the structure of the basins. This kind of complexity has been rather neglected in the literature, mainly because it requires an analysis of the *global* properties (i.e. not based on the linear approximation) of the dynamical systems and the global bifurcations that change the qualitative structure of the basins are usually detected through geometrical and numerical methods. Here we show how the fact that the map driving the dynamics may be noninvertible for certain ranges of the parameters plays an important role in the creation of complex topological structures of the basins.

Finally, a particular global bifurcation is analyzed, which marks the switching from a situation of bi-stability, where the phase-plane is shared between the basins of two coexisting attracting sets (steady states, periodic orbits or even chaotic attractors) to a regime of more complex asymptotic dynamics, characterized by chaotic oscillatory behaviour. This proves that endogenously driven oscillations can also be obtained without what is considered the main kaldorian assumption, i.e. the dependence of investment on capital stock.

The paper is organized as follows. In section 2 we describe the model and discuss the underlying assumptions. In section 3 we analyze the main properties of the two-dimensional map driving the dynamics, such as its triangular structure, its symmetry, the existence of fixed points and the conditions for their local stability, the conditions for the invertibility or non invertibility of the map. In section 4 we discuss the role of noninvertibility in the creation of complex topological structures of the basins of the attracting sets. In section 5 we analyze the local and global bifurcations causing the transition to more and more complex asymptotic dynamics. Some concluding remarks are contained in section 6.

2 The Model

The model we consider is a particular case of a *Kaldor*-type business cycle model proposed in [18] and investigated in its general dynamic behaviour in [5]. The model studied in [18] and [5] starts from a well known discrete-time version of the Kaldor model (see e.g. [3], [11], [12]):

$$\begin{cases} Y_{t+1} - Y_t = \alpha(I_t - S_t) & \text{(a)} \\ K_{t+1} = (1-\delta)K_t + I_t & \text{(b)} \end{cases} \tag{1}$$

where the dynamic variables Y_t and K_t represent, respectively, the income (or output) value and the capital stock in period t, α ($\alpha > 0$) represents an *adjustment parameter* measuring the firm's reaction to the spread between demand and supply, the parameter δ ($0 < \delta < 1$) represents the *capital stock's depreciation rate*, $I_t = I_t(Y_t, K_t)$ is the investment demand in period t and $S_t = S_t(Y_t)$ represents savings in period t.

In the seminal paper by Kaldor [8], as well as in many other papers which formulated the same model in terms of dynamical systems (see, for instance, [2], [3]) the business cycle is represented by endogenously driven oscillations of income and capital, which are substantially explained by the following mechanism:

(i) at least one of the functions I or S are nonlinear (in particular, sigmoid-shaped);

(ii) I is a decreasing function of K.

Let us discuss the role of assumptions (i) and (ii). As it is well known (see, for instance, [6]), under the simple assumption that the investment demand I_t is independent from the capital stock K_t, i.e. $\partial I_t/\partial K_t = 0$, and that both I_t and S_t are linear increasing functions of Y_t, with $dS/dY > \partial I/\partial Y$, the system is globally asymptotically stable, while by introducing nonlinearities into the investment demand curve, for example by assuming that I_t is a sigmoid-shaped function of Y_t, we may have a situation of bi-stability: Figs. *1a* and *1b* qualitatively represent the income adjustment process in case of disequilibria (captured by eq. (1a)), in the linear and nonlinear cases.

The essential dynamic feature that enables the model to display cyclical behaviour is assumption (ii), which causes the long term shifting of the investment function as a consequence of changes in the capital stock, as qualitatively described by Kaldor in [8]. By assuming that the investment demand curve shifts downwards (resp. upwards) when the income, and consequently the capital stock, increases (resp. decreases), cyclic movements of the level of income and capital may occur, as qualitatively shown in Fig. *2* (for an economic justification of these assumptions, see again [6], pp.122-129).

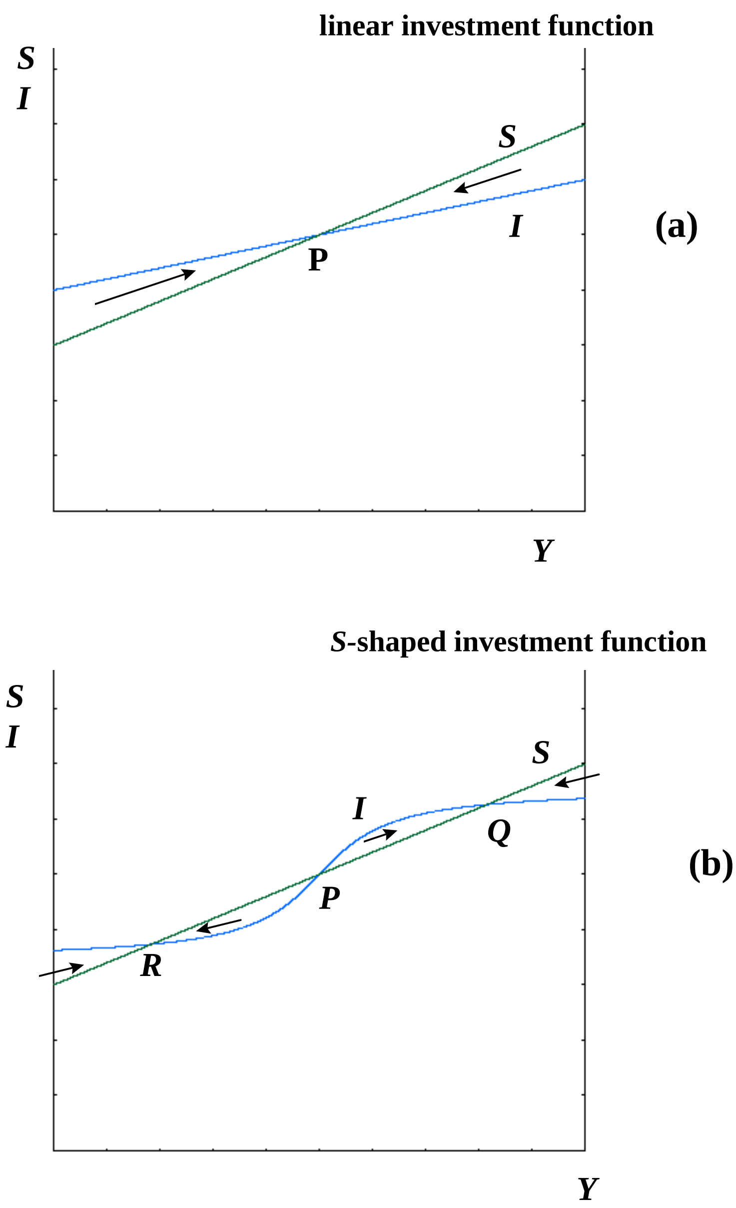


Figure 1: *Effect of the introduction of nonlinearities into the investment function.*

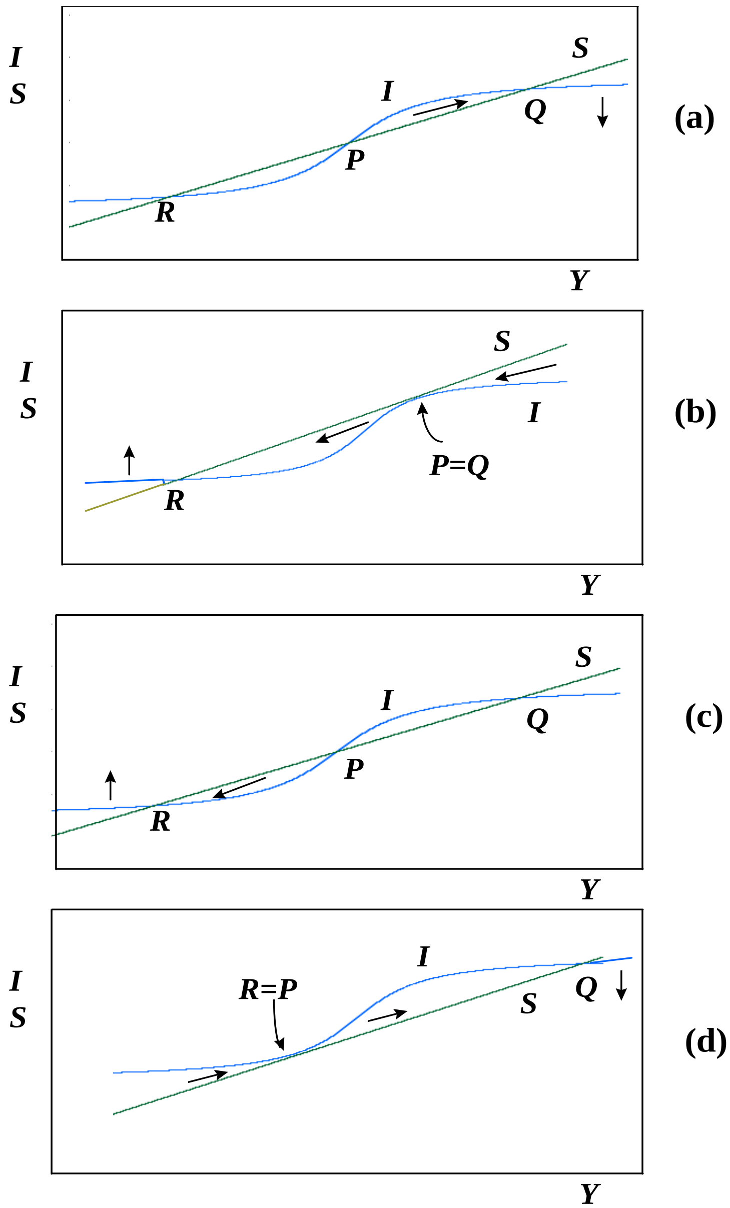


Figure 2: *"Shifting" of the investment function causing endogenous oscillations in income and capital.*

In particular, in the form proposed in [18] savings are assumed, as usual, proportional to income:

$$S_t = \sigma Y_t \ , \tag{2}$$

where the coefficient σ , $0 < \sigma < 1$, represents the *propensity to save.* On the other hand, investment demand is assumed to be an increasing and sigmoid-shaped function of income and a linear decreasing function of capital stock:

$$I_t = \sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K_t\right) + \arctan(Y_t - \mu) \ , \tag{3}$$

where γ is a positive parameter, μ ($\mu > 0$) is the exogenously assumed equilibrium level of income and therefore $\sigma\mu$ represents the equilibrium level of savings (and also of investment demand), while $\sigma\mu/\delta$ is the equilibrium capital stock. As usual in Kaldor business cycle models, one or three steady states may exist: in this latter case, besides the exogenously assumed equilibrium $P = (\mu, \sigma\mu/\delta)$, two more steady states exist, a "wealth" equilibrium Q, characterized by high equilibrium levels of income and capital, and a "poverty" equilibrium R, with low levels of income and capital. As shown in [5], a large variety of dynamic behaviours can occur: in particular the model is able to generate regular *endogenous oscillations* around the equilibrium P.

In the present paper we are interested in what happens if we neglect assumption (ii), by assuming that the investment demand curve is not affected by changes in the capital stock, i.e. $\partial I/\partial K = 0$. This assumption is usually described in the literature as leading to a simple situation of bi-stability. However, as we shall see, complex dynamic phenomena are possible also in this case, and the complexity is related, on one hand, to the asymptotic dynamics, i.e. to the nature of the attracting sets, and, on the other hand, to the structure of their basins of attraction.

By setting $\gamma = 0$ in eq. (3) and substituting into the system (1), the model ultimately reduces to the following two-dimensional dynamical system in income and capital:

$$\begin{cases} Y_{t+1} = Y_t + \alpha\sigma\mu + \alpha \arctan(Y_t - \mu) - \alpha\sigma Y_t & \text{(a)} \\ K_{t+1} = \sigma\mu + \arctan(Y_t - \mu) + (1-\delta)K_t & \text{(b)} \end{cases} \tag{4}$$

The study of the dynamical properties of the system (4) allows us to explore the long-run behaviour of income and capital stock, starting from a given initial condition.

3 Some General Properties

As described at the end of the previous section, the time evolution of income and capital is obtained by the iteration of a two-dimensional nonlinear map $T : (Y_t, K_t) \to (Y_{t+1}, K_{t+1})$ given by:

$$T: \begin{cases} Y' = (1-\alpha\sigma)Y + \alpha\sigma\mu + \alpha \arctan(Y - \mu) & \text{(a)} \\ K' = (1-\delta)K + \sigma\mu + \arctan(Y - \mu) & \text{(b)} \end{cases} \ , \tag{5}$$

where the symbol $'$ denotes the unit time advancement operator, that is, if the right-hand side variables are income and capital at time t,then the left-hand ones represent income and capital at time $t+1$.

We shall now describe some properties of the map T, that is, the *triangular* structure of the map, the particular structure of the second component (5b), the existence of fixed points and their local stability analysis, some symmetry properties and the role of non invertibility of the map. In order to analyze these properties, we briefly recall the meaning of some terms, which will be used in the following.

Let A be a subset of the plane. We say that A is a trapping set of T (or T is trapping on A) if $T(A) \subseteq A$ (that is if A is mapped into itself by T); T is invariant on A (or A is invariant by T) if $T(A) = A$, i.e. if A is trapping and for any $\mathbf{y} \in A$ there exists $\mathbf{x} \in A$ such that $T(\mathbf{x}) = \mathbf{y}$. A p-cycle of T is a periodic orbit of T of least period p, $p \geq 1$. A p-periodic point of T is a point belonging to some p-cycle of T.

In the following, by $DT(Y,K)$ we shall denote the jacobian matrix of the map T, by T^n, $n \geq 1$, the n-th iterated of the map T and by $DT^n(Y,K)$ the jacobian matrix of the map T^n.

3.1 The triangular structure of the map

We can observe that the first component of the map T does not depend on K: the map is therefore a *triangular map*, characterized by the following structure:

$$T : \begin{cases} Y' = F(Y) & \text{(a)} \\ K' = G(Y,K) & \text{(b)} \end{cases} . \tag{6}$$

This means that the dynamics of the income Y are only affected by income itself, being $Y_{t+1} = F(Y_t)$, whereas the time evolution of the capital stock is also influenced by the income, being $K_{t+1} = G(Y_t, K_t)$. By using the terminology of the engineering systems (see e.g. [19]) we may say that the one-dimensional system (6a) is the "driving system" and the capital stock is "driven" by the income dynamics[1]. As a consequence, the dynamics of the map T is deeply influenced by the dynamics of the one-dimensional map $Y' = F(Y)$. In particular, many of its bifurcations are associated to those of the one-dimensional map $Y' = F(Y)$ and all the cycles of T stem from cycles of F. Moreover, since the jacobian matrix of the map T, given by:

$$DT(Y,K) = \begin{bmatrix} 1 + \frac{\alpha}{1+(Y-\mu)^2} - \alpha\sigma & 0 \\ \frac{1}{1+(Y-\mu)^2} & 1-\delta \end{bmatrix} , \tag{7}$$

[1] In the physical and engineering literature triangular maps are often referred to as *skew products*.

is lower triangular, it can't have complex eigenvalues and thus the occurrence of regular oscillations, similar to those usually observed in Kaldor-type models, is ruled out.

Let us briefly recall some useful properties of two-dimensional triangular maps (for a wider discussion see [9], [10], [7]).

Property 1 The eigenvalues of $DT(Y,K)$ are always real, given by $z_1 = F'(Y)$ and $z_2 = G_K(Y,K)$. Any fixed point of T is therefore either a node or a saddle.

Property 2 The eigenvalues of $DT^n(Y,K)$, for any integer $n \geq 1$, are real. Any cycle of T is therefore either a node or a saddle. If $\mathcal{C}_p = \{(Y_i,K_i),\ i = 1,2,...,p\}$ is a p-cycle of T, the eigenvalues of the cycle (i.e. the eigenvalues of the jacobian matrix of T^p in any point of the cycle) are given by: $z_1 = \prod_{i=1}^{p} F'(Y_i)$ and $z_2 = \prod_{i=1}^{p} G_K(Y_i,K_i)$.

Property 3 Let $\mathcal{C}_p = \{(Y_i,K_i),\ i = 1,2,...,p\}$ be a p-cycle of T; then $\{(Y_1,Y_2,...,Y_p\}$ is a periodic orbit of the one-dimensional map F of least period r where r is such that $rm = p$ for some integer $m \geq 1$.

Property 4 Let (Y_i,K_i), $i = 1,2,...,p$, be a point of a p-cycle of T and (Y_i,K) a point on the vertical line $Y = Y_i$; then there exists some integer $m \geq 1$ such that $T^r(Y_i,K)$, where $r = p/m$, is trapping on the line $Y = Y_i$ and may be considered a one-dimensional map of the state variable K.

In particular Property 4 implies that:

I1) no point on the vertical line $Y = Y_i$ can belong to the stable set of some other cycle of T with periodic points all outside that line;

I2) any p-periodic point of T must belong to trapping (for some T^r, with $rm = p$, $m \geq 1$) vertical lines $Y = Y_i$, where Y_i is a r-periodic point of the one-dimensional map F;

I3) if $\mathcal{C}_r = \{(Y_1,Y_2,...,Y_r\}$ is an r-cycle of the map F and a p-cycle $\mathcal{C}_p$ of T exists, on the vertical lines $Y = Y_i$, $i = 1,2,...,r$, of period $p = rm$ for some integer $m \geq 1$, then the eigenvalue z_1 of the p-cycle $\mathcal{C}_p$ is related to that of the r-cycle $\mathcal{C}_r$ of F (let's denote it by τ) as follows: $z_1 = \tau^m$;

I4) if a p-cycle $\mathcal{C}_p$ of the two-dimensional triangular map T is a saddle with $|z_1| = |\prod_{i=1}^{p} F'(Y_i)| > 1$ and $|z_2| = |\prod_{i=1}^{p} G_K(Y_i,K_i)| < 1$, then the points of the local stable set of $\mathcal{C}_p$ belong to the vertical lines through the periodic points.

Property 5 If $\mathcal{C}_r = \{(Y_1,Y_2,...,Y_r\}$ is an r-cycle of the map F, then the restriction of the map T^r to any of the vertical lines $Y = Y_i$, $i = 1,2,...,r$, is trapping on that line. If the r-cycle of F is attracting (resp. repelling) then the vertical lines $Y = Y_i$, $i = 1,2,...,r$, are attracting (resp. repelling) for T^r.

As far as the bifurcations of the map T are concerned, it is easy to see from the above properties that any bifurcation of the one-dimensional map F gives a bifurcation of T. In particular, a fold bifurcation of F creates a couple of cyclical trapping lines of T (one repelling and one attracting). At a flip bifurcation of a cycle of F, trapping cyclical vertical lines from attracting (for T) become repelling and new cyclical attracting lines are created.

Finally, it is well known that if the two-dimensional map T is an endomorphism, with F and G continuously differentiable, the locus LC_{-1} of T is generally given by $\det DT(Y,K)=0$. Therefore, for a triangular map the following property holds.

Property 6

The locus LC_{-1} of the phase plane is made up of curves LC_{-1,a_k} and $LC_{-1,b}$ such that:

(i) LC_{-1,a_k} are vertical lines of equation $Y=c_{-1,a_k}$, where c_{-1,a_k} satisfy $F'(c_{-1,a_k})=0$;

(ii) $LC_{-1,b}$ is the locus $G_K(Y,K)=0$.

The critical curves $LC_{i,a_k}=T^{i+1}(LC_{-1,a_k})$, for $i\geq 0$, belong to vertical lines $x=c_{i,a_k}$ where $c_{i,a_k}=F^{i+1}(c_{-1,a_k})$ are critical points of $F(Y)$.

3.2 The structure of the second component of the map

It can be noticed that the second component of the map (5) is separable with respect to the variables Y and K and linear in K, i.e. it has the following structure:

$$K'=(1-\delta)K+I(Y)\ , \tag{8}$$

where $I(Y)=\sigma\mu+\arctan(Y-\mu)$ is the investment demand function. This implies that also the second component of r-th iterated of the map T^r, $r>1$, has the same structure. More precisely, it is easy to prove by induction that the map T^r $(r\geq 1)$ has the following form:

$$T^r:\ \begin{cases} Y'=F^r(Y) & \text{(a)} \\ K'=(1-\delta)^r K+\sum_{s=1}^{r}(1-\delta)^{r-s}I(F^{s-1}(Y)) & \text{(b)} \end{cases}\ . \tag{9}$$

From the analytical expression (9) of the map T^r we can easily conclude that the fixed points and the cycles of the map (5) can only be *stable nodes* or *saddles*. This is due to the fact that one of the eigenvalues of the Jacobian matrix of the map T^r is constant and equal to $(1-\delta)^r<1$.

Moreover the triangular structure, together with the linear structure of the second component, enables us to formulate the following

Proposition 1 *The stable manifold W^s of a saddle cycle of T is made up of the lines of equation $Y=Y_i$, where Y_i, $i=1,2,...,r$, are the periodic points of the corresponding cycle* *of the one-dimensional map F, and of the lines of equation $Y = Y_{-j}$, where Y_{-j}, $j = 1, 2, ...$, are the preimages of any rank of the periodic points.*

Proof. Since T is a triangular map we know (from the implication I4 of Property 4) that the points of the local stable set of a saddle cycle belong to the vertical lines through the periodic points. We can conversely easily show that, for our particular map, any point on these vertical lines belongs to the stable set. In fact, let (Y^*, K^*) be a point belonging to a saddle cycle for the map T, where Y^* is a r-periodic point of the map F. Since Y^* is a fixed point of the one-dimensional map $F^r(Y)$, i.e. $F^r(Y^*) = Y^*$, we can see from (9) that a point (Y^*, K) is mapped, after r iterations into the point (Y^*, K') of the same vertical line $Y = Y^*$, where:

$$K' = (1-\delta)^r K + \sum_{s=1}^{r} (1-\delta)^{r-s} I(F^{s-1}(Y^*)) \ . \tag{10}$$

This means that the trajectory obtained by iterating the map T^r (which is constrained to move on the trapping vertical line $Y = Y^*$ due to Property 4) is driven by the one-dimensional linear map (10), having the only fixed point:

$$K^* = \frac{\sum_{s=1}^{r} (1-\delta)^{r-s} I(F^{s-1}(Y^*))}{1-(1-\delta)^r} \ ,$$

globally stable. The speed of the dynamics of the map T^r on the vertical line $Y = Y^*$ is affected by the depreciation rate δ, $0 < \delta < 1$: the higher is δ the faster is the convergence to the fixed point.

Finally, the points with coordinates $(Y^*_{-j}, \bullet)$, where Y^*_{-j} is a preimage of rank-j of Y^* are mapped, after j iterations, into a point $(Y^*, \bullet)$ on the vertical line $Y = Y^*$. This completes the proof. $\square$

3.3 Fixed points

The equilibrium points (or steady states) of the map T are the solutions of the algebraic system:

$$\begin{cases} \sigma\mu + \arctan(Y-\mu) - \sigma Y = 0 \\ \sigma\mu + \arctan(Y-\mu) - \delta K = 0 \end{cases} ,$$

obtained by setting $Y' = Y$ and $K' = K$ in (5). The system can be rewritten as:

$$\begin{cases} K = \frac{\sigma}{\delta} Y & \text{(a)} \\ \sigma(Y-\mu) = \arctan(Y-\mu) & \text{(b)} \end{cases} . \tag{11}$$

It is trivial to realize that the steady states are independent from the firms' adjustment parameter α. The first equation says that the fixed points belong to the line $K = \frac{\sigma}{\delta} Y$ in the phase-plane, and from the second equation we have that the Y-values (which are the fixed points of the one-dimensional map F) can be obtained as intersections of the two curves of equation $f(Y) = \sigma(Y - \mu)$ and $g(Y) = \arctan(Y - \mu)$. It follows that if $\sigma \geq 1$, the system (11) admits the point $P = (\mu, \mu\frac{\sigma}{\delta})$ as unique solution, while in the case $0 < \sigma < 1$ three solutions exist, the point P and the points Q and R , which are symmetric with respect to P. Of course, since σ represents the propensity to save and the case $0 < \sigma < 1$ includes the interval of values of interest for us, the case of three fixed points is the only one economically meaningful. The explicit coordinates of the fixed points Q and R cannot be written. We can numerically compute them as $(Y_Q, \frac{\sigma}{\delta}Y_Q)$ and $(Y_R, \frac{\sigma}{\delta}Y_R)$, where Y_Q and Y_R are obtained from the second equation in (11) and $Y_R = 2\mu - Y_Q$ due to the symmetry property, as described in the next section.

3.4 Symmetry property

It is worth noting that the map T is symmetric with respect to the fixed point $P = (\mu, \mu\frac{\sigma}{\delta})$. This means that symmetric points are mapped into symmetric points (with respect to P). Denote by $F(Y)$ and $G(Y, K)$ the two components of the map T:

$$F(Y) = (1 - \alpha\sigma)Y + \alpha\sigma\mu + \alpha \arctan(Y - \mu) \ ,$$
$$G(Y, K) = (1 - \delta)K + \sigma\mu + \arctan(Y - \mu)$$

and observe that the symmetric of the point (Y, K) with respect to P is the point $(2\mu - Y, 2\frac{\sigma\mu}{\delta} - K)$. The above property, which can easily be verified, can be formalized as follows:

$$F(2\mu - Y) = 2\mu - F(Y)$$
$$G(2\mu - Y, 2\frac{\sigma\mu}{\delta} - K) = 2\frac{\sigma\mu}{\delta} - G(Y, K) \ ,$$

or better, by denoting with $S(Y, K) = (2\mu - Y, 2\frac{\sigma\mu}{\delta} - K)$ the symmetry with respect to the fixed point $P = (\mu, \mu\frac{\sigma}{\delta})$:

$$S(T(Y, K)) = T(S(Y, K)) \qquad \forall (Y, K) \ .$$

This implies that a cycle of T is either symmetric with respect to P or admits a symmetric cycle, as stated by the following

Proposition 2 *Let* $\mathcal{C} = \{(Y_1, K_1), ..., (Y_p, K_p)\}$ *be a cycle of* T *of period* $p \geq 1$. *Then*

- *either* $S(\mathcal{C}) = \mathcal{C}$
- *or* $S(\mathcal{C}) = \mathcal{C}' \neq \mathcal{C}$

where $\mathcal{C}' = \{S(Y_1, K_1), ..., S(Y_p, K_p)\}$ *is another different cycle of* T, *of the same period* p, *with periodic points which are symmetric with respect to* P *of the periodic points in* $\mathcal{C}$.

3.5 Local stability analysis of the fixed points

Let us now turn to the local stability of the fixed points $P = (\mu, \mu\frac{\sigma}{\delta})$, $Q = (Y_Q, \frac{\sigma}{\delta}Y_Q)$ and $R = (Y_R, \frac{\sigma}{\delta}Y_R)$. The local stability analysis of a fixed point can be generally carried out by studying the localization of the eigenvalues of the Jacobian matrix in the complex plane, and it is well known that a sufficient condition for the local stability is that both the eigenvalues are inside the unit circle in the complex plane. The triangular structure of the map T simplifies our analysis, since the Jacobian matrix (7) of T has real eigenvalues, located on the main diagonal, given by: $z_1(Y) = 1 + \frac{\alpha}{1+(Y-\mu)^2} - \alpha\sigma$, $z_2 = 1 - \delta$, with $0 < z_2 < 1$.

The first eigenvalue of the fixed point $P = (\mu, \mu\frac{\sigma}{\delta})$, is: $z_1(P) = 1 + \alpha(1-\sigma)$. Since $z_1(P) > 1$ for the ranges of interest of the parameters, we can conclude that the fixed point P is always a *saddle*. The results about the nature of the fixed points Q and R and their local stability analysis (it's enough to consider only one of them, since the symmetry property implies $DT(Q) = DT(R)$) are summarized by the following proposition which defines the stability region in the parameters space $\Omega = \left\{(\alpha, \sigma) \in \mathbb{R}^2 | \alpha > 0, 0 < \sigma < 1\right\}$.

Proposition 3 *The equilibria Q and R are stable nodes for each (α, σ) in the region $\Omega_s(Q)$ defined as:*

$$\Omega_s(Q) = \{(\alpha, \sigma) \in \Omega | \alpha < \alpha_f(\sigma)\} \ ,$$

where: $\alpha_f(\sigma) = 2/\left\{\sigma - [1 + (Y_Q - \mu)^2]^{-1}\right\} > 2$. Outside this region, Q and R are saddles.

Proof. Let us consider the fixed point Q. We can rewrite the first eigenvalue as:

$$z_1(Q) = z_1(R) = 1 - \alpha[\sigma - u(Y_Q)] \ ,$$

where: $u(Y) = [1 + (Y - \mu)^2]^{-1}$.

We already know that the fixed points of the map T are either stable nodes or saddles. Since $0 < z_2 < 1$, Q is stable iff $|z_1(Q)| < 1$, i.e.:

$$\begin{cases} \sigma > u(Y_Q) & \text{(a)} \\ \sigma < 2/\alpha + u(Y_Q) & \text{(b)} \end{cases} \ . \tag{12}$$

Recalling condition (11b), we notice that for a fixed μ the equation:

$$\sigma(Y_Q - \mu) - \arctan(Y_Q - \mu) = 0 \quad (Y_Q > \mu, \quad 0 < \sigma < 1) \tag{13}$$

implicitly defines the income equilibrium level Y_Q as a (differentiable) function of the parameter σ. By denoting with $h(\sigma, Y_Q)$ the left-hand side of (13), from the implicit function theorem we have:

$$\frac{dY_Q}{d\sigma} = -\frac{h_\sigma(\sigma, Y_Q)}{h_{Y_Q}(\sigma, Y_Q)} = -\frac{Y_Q - \mu}{\sigma - u(Y_Q)} \ .$$

Since it can be proved through simple geometrical considerations that the income equilibrium value Y_Q is a strictly decreasing function of σ, it follows that $\sigma - u(Y_Q) > 0$ and therefore condition (12a) is always satisfied.

Condition (12b) can be rewritten as:

$$\alpha < \frac{2}{\sigma - u(Y_Q)} = \alpha_f(\sigma) \ .$$

Since $0 < u(Y_Q) = [1 + (Y_Q - \mu)^2]^{-1} < 1$ and condition (12a) holds, it follows that $0 < \sigma - u(Y_Q) < 1$ which implies: $\alpha_f(\sigma) > 2$. Of course, for $\alpha > \alpha_f(\sigma)$, the fixed point Q is a saddle. □

It is worth noting that the only way the fixed point Q (and thus R) can lose stability is through a bifurcation with $z_1(Q) = -1$, i.e. a *flip* (or *period doubling*) bifurcation, where the stable fixed point becomes unstable (a saddle in our case) giving rise to a stable cycle of period two. The determination of the flip-bifurcation curve of the fixed points Q and R, in the parameters' plane Ω, can only be done through numerical evaluation of the quantity $\alpha_f(\sigma) = 2/[\sigma - u(Y_Q)]$.

This curve and the region of local stability of Q and R in the parameters space Ω are represented in Fig. *3*.

3.6 Invertibility conditions

For some regions of the parameters' space the map T is a noninvertible map of the plane. This means that while starting from some initial values for income and capital stock, say (Y_0, K_0), the iteration of (5) uniquely defines the trajectory $(Y_t \ , K_t) = T^t(Y_0, K_0)$, $t = 1, 2, ...$, the backward iteration of (5) is not uniquely defined. In fact, a point (Y', K') of the plane can have several rank-*1* preimages.

As we already pointed out, many of the properties of the map at study are related to those of the one-dimensional map:

$$Y' = F(Y) = (1 - \alpha\sigma)Y + \alpha\sigma\mu + \alpha \arctan(Y - \mu) \ .$$

It can be immediately proved that the two-dimensional map T is invertible if and only if the one-dimensional map F is. It is worth noting that this property is not simply due to the triangular structure of the map T but also to the fact that the second component of T, i.e. the function:

$$G(Y, K) = (1 - \delta)K + \sigma\mu + \arctan(Y - \mu) \ ,$$

is linear in K.

Turning to the conditions under which the one-dimensional map $F(Y)$ is invertible, it is easy to show that a point Y' has a unique preimage if and only if $\alpha\sigma \leq 1$, while in the opposite case, $\alpha\sigma > 1$, a point may have one, two, or three different preimages. In fact, in the noninvertibility case $\alpha\sigma > 1$, F is a bimodal map with a local minimum point, critical point of rank-0 denoted by $c_{-1,m}$, and a local maximum point, critical point of rank-0 denoted by $c_{-1,M}$, where:

$$c_{-1,m} = \mu - \sqrt{\frac{\alpha}{\alpha\sigma - 1} - 1} \ ; \quad c_{-1,M} = \mu + \sqrt{\frac{\alpha}{\alpha\sigma - 1} - 1} \ . \tag{14}$$

The critical points of rank-1 are given by their images:

$$c_m = F(c_{-1,m}) \ ; \quad c_M = F(c_{-1,M}) \ .$$

Thus the points Y with $Y < c_m$ or $Y > c_M$ have a unique preimage, the points satisfying $c_m < Y < c_M$ have three distinct preimages, each of the points $Y = c_m$ and $Y = c_M$ has two preimages which merge in a critical point together with a second distinct preimage, called *extra-preimage*.

If we consider again the two-dimensional map T, we can immediately see that if Y is a point with three (resp. one, two) preimages for the one-dimensional map F, then the whole vertical line for this point has three (resp. one, two) preimages for the two-dimensional map T. Thus, following the notation used in [15], we have that the map T is, for $\alpha\sigma > 1$, of the so-called type $Z_1 - Z_3 - Z_1$, which means that the phase plane is subdivided in different regions Z_j $(j = 1, 3)$ each point of which has j distinct rank-1 preimages. The critical curves of rank-1, denoted by LC, generally bound such Z_j regions, and are defined as the locus of points having at least two *merging* rank-1 preimages; for the map T, LC is thus given by the two vertical lines:

$$Y = c_m \ , \ Y = c_M \ .$$

The locus of merging rank-1 preimages, which constitutes the critical curve of rank-0, denoted by LC_{-1}, is made up of the two lines:

$$Y = c_{-1,m} \ , \ Y = c_{-1,M} \ .$$

The region of noninvertibility of the map in the space Ω of the parameters α and σ is represented in Fig. *3*.

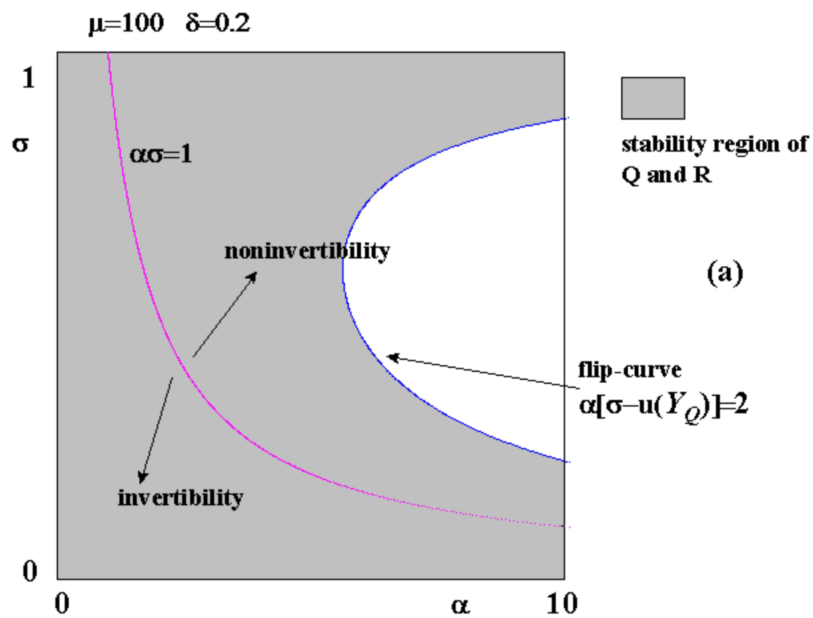


Figure 3: *Region of local stability of the equilibria and region of noninvertibility of the map in the parameters' plane.*

Remark In this section we have obtained the equations of LC and LC_{-1} starting from the critical points of the one-dimensional map F, i.e. we have substantially applied Property 6, which holds for any two-dimensional triangular map.

4 Some Effects of Noninvertibility

In this section we stress that the fact that the map driving the dynamics may be noninvertible plays an important role in the creation of complex topological structures of the basins of attraction. In order to illustrate the effects of the switching to the noninvertibility regime, we shall make use of the analytical properties of the one-dimensional map F driving the income evolution. We will show how the dynamic behaviours of the two dimensional map T can be completely described starting from those of the associated one dimensional map. It is worth noting that this result is not a consequence of the triangular structure only, but of the joint effect of the triangular structure and the linearity, with respect to K, of the second component of T.

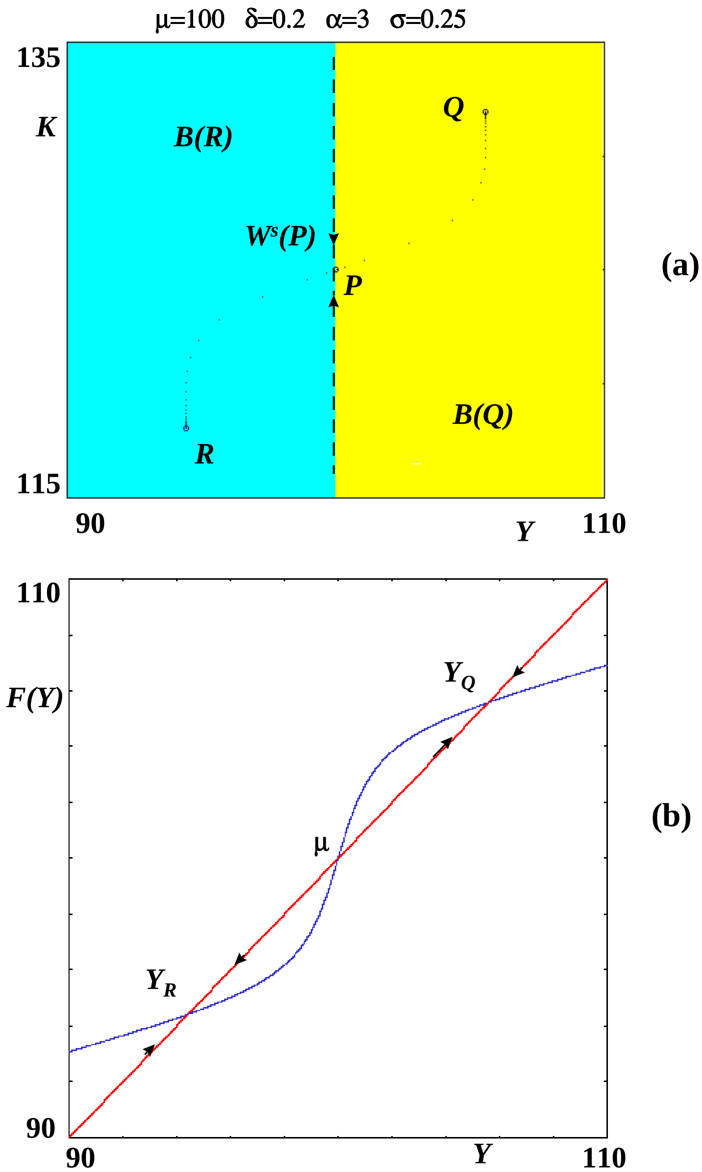


Figure 4: *Situation of bi-stability with a simple basins' structure occurring when the map driving the system is invertible.*

In our numerical explorations we shall fix the exogenous equilibrium level of the income at the value $\mu = 100$, and the depreciation rate of capital at the value $\delta = 0.2$. This latter assumption is without loss of generality, because the stability property of the equilibrium points and of the cycles of the map doesn't depend on δ and the same qualitative dynamics as those commented in this section can be obtained with a different value of δ, $0 < \delta < 1$. In the space Ω of the parameters α and σ we shall follow the particular path obtained by assuming the *propensity to save* σ as fixed at the value $\sigma = 0.25$ and increasing the adjustment parameter α.

In Fig. *4a*, obtained with $\alpha = 3$, we show the basins of attraction $\mathcal{B}(Q)$ and $\mathcal{B}(R)$ of the two stable nodes Q and R, separated by the stable manifold of the saddle P, which, as we know from Proposition 1, is the vertical line of equation $Y = \mu$. Fig. *4b* shows how this situation of bi-stability is related to the shape of the one-dimensional map F: here it is evident that the initial conditions Y_0, with $Y_0 < \mu$, originate trajectories converging to Y_R, while the initial conditions Y_0, with $Y_0 > \mu$, originate trajectories converging to Y_Q. This means that an economic system with a low (resp. high) initial level of income will maintain these characteristics over time and will converge to the "poverty" steady state (resp. "wealth" steady state).

The simple structure of the basins shown in Fig. *4a* changes as soon as the map enters the regime of noninvertibility, i.e. for $\alpha > 1/\sigma = 4$.The basins of the two fixed points Q and R become non connected and structured in vertical strips and the stable manifold of the saddle P, which separates the basins of Q and R, is now made up of several vertical lines (see Fig *5a*). In fact, in the regimes in which the one-dimensional map F is noninvertible, the fixed point P of the map T has three different preimages, P itself and two more points symmetric with respect to P, say $P_{-1,1}$ and $P_{-1,2}$ (as well as μ, $\mu_{-1,1}$, $\mu_{-1,2}$ are the preimages of the fixed point μ of the one-dimensional map F: see Fig. *5b*). Then by Proposition 1 the stable manifold of the saddle point P is made up of the vertical lines of equation $Y = \mu$, $Y = \mu_{-1,1}$, $Y = \mu_{-1,2}$ and of the lines of equation $Y = \mu_{-n,1}$and $Y = \mu_{-n,2}$, $n = 2, 3, ...$, where $\mu_{-n,1}$ and $\mu_{-n,2}$ are the preimages of rank-n of μ (some of which are represented in Fig. *5b*). The basins of the equilibria Q and R in Fig. *5a* are then made up of infinitely many disjoint vertical strips as well as in Fig. *5b* the basin of each equilibrium is given by infinitely many disjoint intervals.

From an economic point of view, the new structure of the basins created by the switching to the noninvertibility regime means that also economies with a high (low) initial level of income may become poor (rich) in the long run, and that this situation may be reversed many times during the transient dynamics.

The noninvertibility of F, i.e. the existence of two local extrema, also causes the appearance of diverging trajectories, due to the appearance of a repelling 2-cycle $\{s_1^*, s_2^*\}$ separating the basin of infinity from the basins of the attracting fixed points. Precisely, we notice that by increasing the speed of adjustment α for a fixed value of the propensity to save σ, two new repelling fixed points of the map $F^2 = F \circ F$ (and thus a repelling 2-cycle $\{s_1^*, s_2^*\}$ of the map F) are created when α crosses the curve $\alpha_c(\sigma) = 2/\sigma$, which is located in the noninvertibility region of the space Ω of the parameters α and σ. To realize this, it is enough to observe that $\lim_{Y\to\mp\infty}\frac{d}{dY}(F^2(Y)) = (1-\alpha\sigma)^2$ and that as soon as this slope becomes greater than one, i.e. for $\alpha > 2/\sigma$, two new intersections of F^2 with the line $\varphi(Y) = Y$ are created, as it is also evident from the comparison of Fig. *5b* (where $\alpha = 7.25 < 8 = 2/\sigma$) with Fig. *6a* (where $\alpha = 9 > 2/\sigma$).

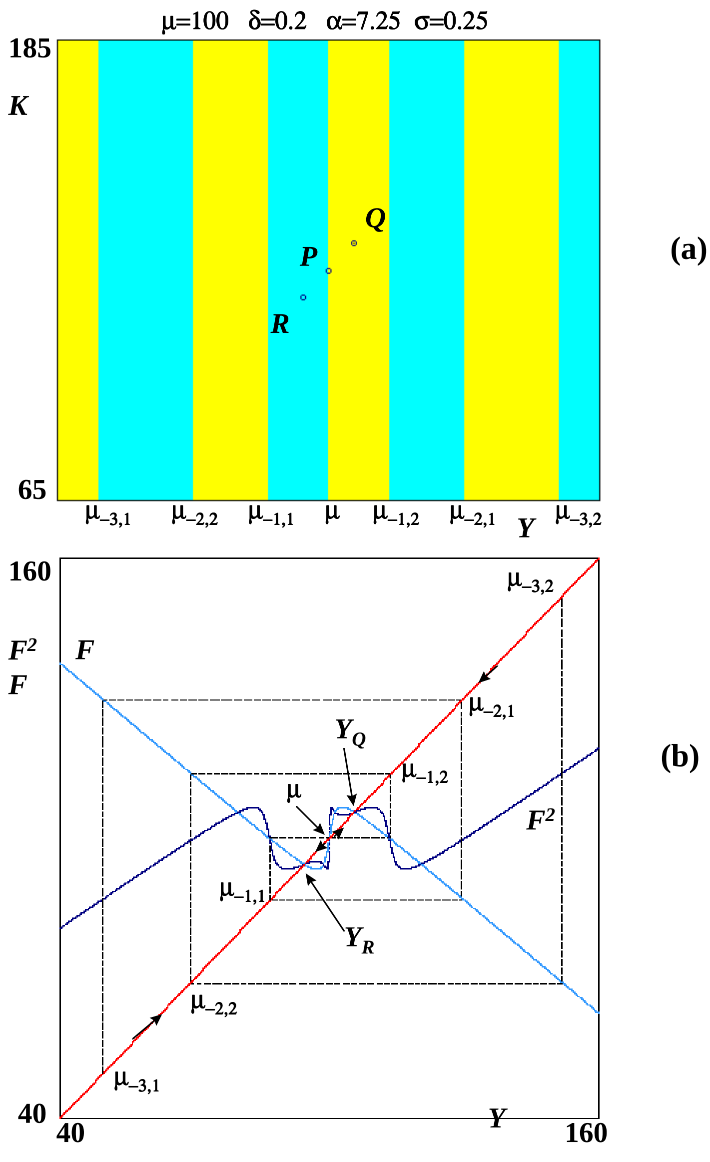


Figure 5: *Effects of the noninvertibility on the structure of the basins of the coexisting equilibria.*

We can see from Fig. *6a* that the points $Y_0 \in [s_1^*, s_2^*]$ have bounded trajectories, while the points $Y_0 \notin [s_1^*, s_2^*]$ have diverging trajectories. It is also evident that when α is further increased the points of the 2-cycle (and therefore the basin of infinity) approach the fixed points P, Q and R.

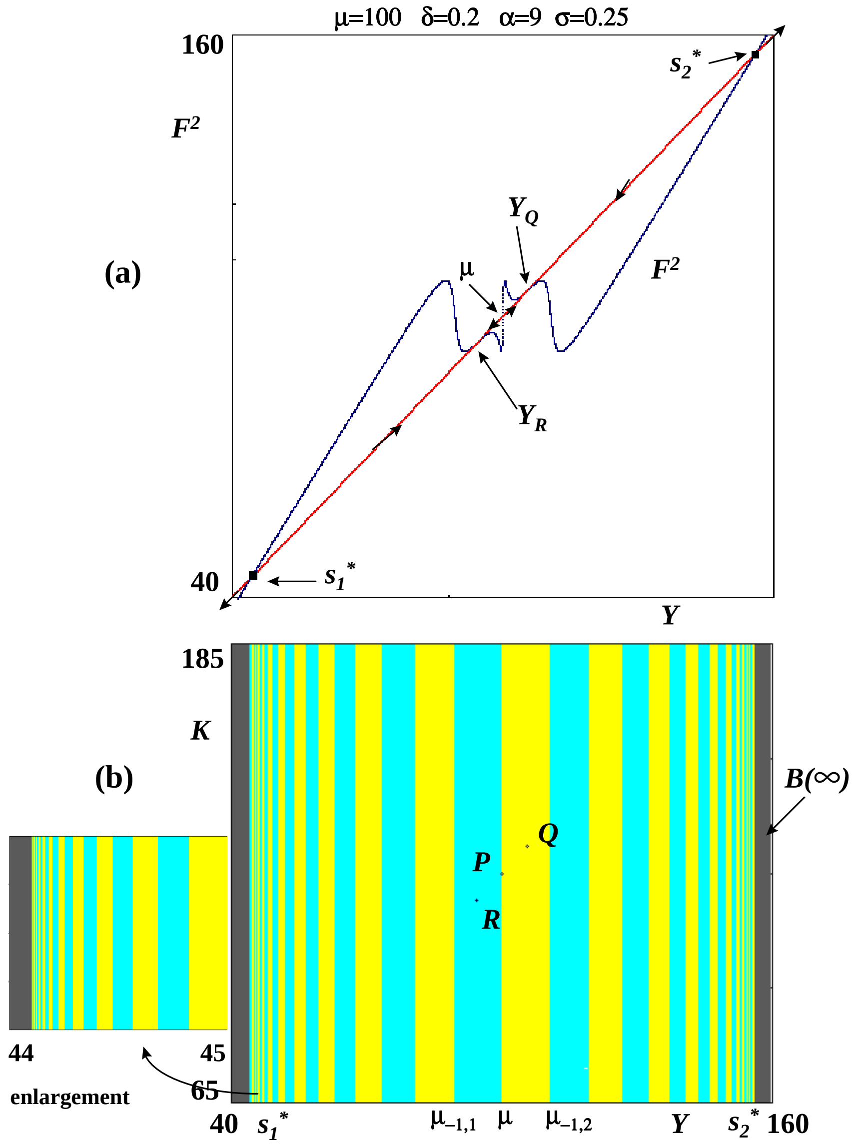


Figure 6: *Appearance of divergent trajectories and increasing complexity in the topological structure of the basins.*

Fig. *6b* and its enlargement show that the vertical strips constituting the basins of Q and R accumulate on the vertical lines $Y = s_1^*$ and $Y = s_2^*$. This is substantially due to the fact that, being these vertical lines repelling sets for the forward iteration of T (see Property 5 of triangular maps in section 3.1), they behave as attracting sets for the iteration of the inverses of T (see, for instance, [1]). From an economic perspective, we could say that such a situation causes a loss of predictability about the long run evolution of the system: if the initial state is near the line $Y = s_1^*$ or $Y = s_2^*$, a slight change in the initial state may give a completely different long run evolution of the economy (the "wealth" equilibrium, the "poverty" equilibrium or even divergent behaviour) if the change causes a crossing of some basin boundary.

5 Local and Global Bifurcations Changing the Structure of the Attracting Sets

In this section we describe the local and global bifurcations which increase the complexity of the asymptotic dynamic behaviour of the system. Again, we shall see that the bifurcations and the dynamic behaviours of the two dimensional map T can be completely described on the basis of those of the one-dimensional map F driving the income evolution, thanks to the triangular structure of T.

To illustrate these bifurcations, we shall fix the parameters μ and δ at the same values of the previous section ($\mu = 100$, $\delta = 0.2$) and we shall follow, in the space Ω of the parameters α and σ, a particular bifurcation-route obtained by fixing the adjustment parameter α at a sufficiently high value, say $\alpha = 8.5$, and increasing the propensity to save σ.

Let us start with $\sigma = 0.25$. In this case the equilibrium points Q and R are stable and the structure of their basins is very similar to the one described Fig. *6b*. We already know from the local stability analysis carried out in section 3.5 that the equilibria Q and R are stable for $\alpha[\sigma - u(Y_Q)] < 2$, where $u(Y_Q) = [1 + (Y_Q - \mu)^2]^{-1}$ (and where the equilibrium value of income Y_Q is a function of σ, implicitly defined by eq. (13)). Numerical computations show that, when $\alpha = 8.5$ and σ is increased, the fixed points lose stability for $\sigma = \sigma_f \simeq 0.2733$: here a flip bifurcation occurs, where the stable nodes Q and R become saddles and two symmetric stable cycles of period two appear. Fig. *7a* represents these cycles, together with their basins of attraction, for $\sigma = 0.3$. We point out that such a local bifurcation simply replaces each stable steady state with an attracting 2-cycle, without modifying the basins of the coexisting attractors (which are given by the vertical strips represented in Fig. *7a*: more precisely, each set of vertical strips is the closure of the basin of the corresponding cycle). What happens by increasing further σ can be easily understood by observing the bifurcation diagram of the map F represented in Fig. *7b*. This diagram shows that F

undergoes a typical sequence of flip bifurcations leading to chaotic dynamics.

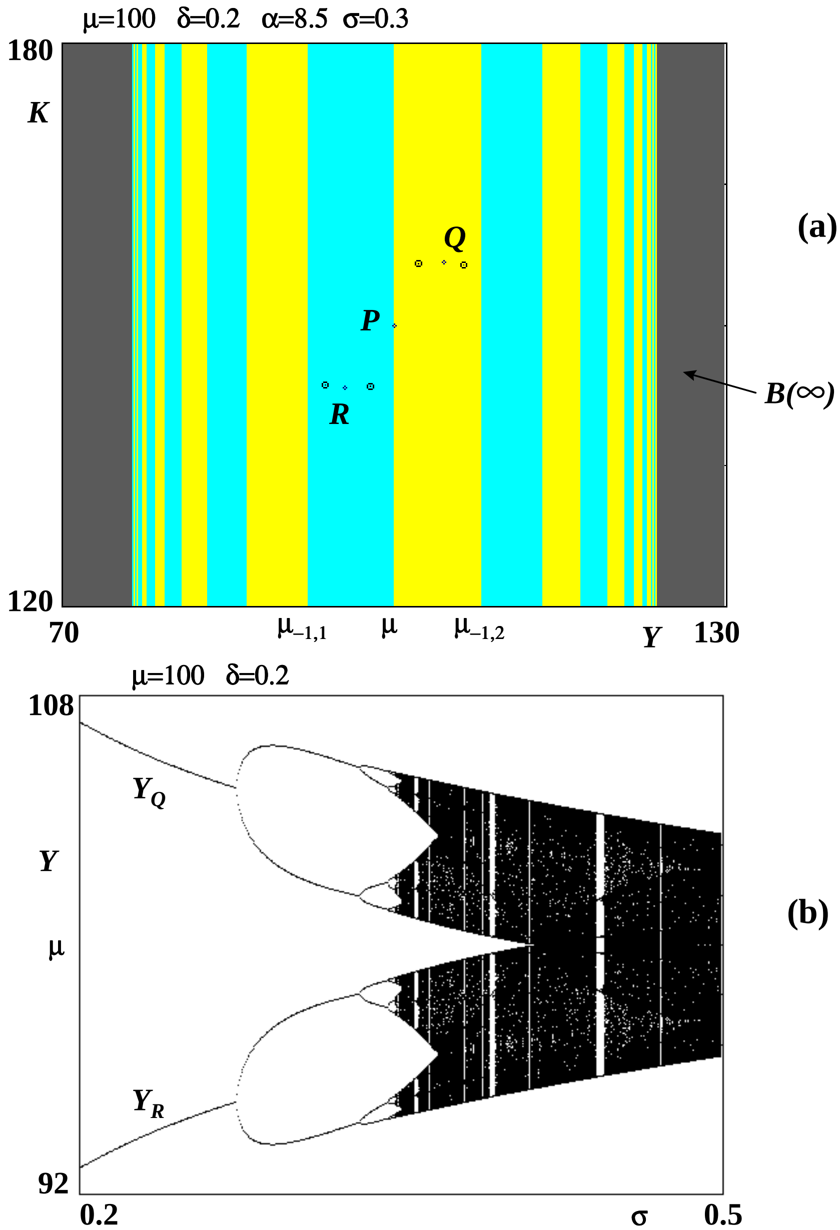


Figure 7: *Period-doubling bifurcation of the equilibria (a) and transition to complex dynamics due to a flip-bifurcation sequence (b).*

Figs. *8* and *9* show the changes in the structure of the attractors sharing the phase plane when σ is increased (for high values of α). In Fig. *8a* we observe two symmetric chaotic attractors, each one made up of two pieces, which increase in size as σ increases and then merge giving rise to two disjoint one-piece chaotic attractors (Fig. *8b*). By increasing further σ, the attracting sets of Fig. *8b* in turn merge into the attractor shown in Fig. *9a*, whose shape is symmetric with respect to the saddle P.

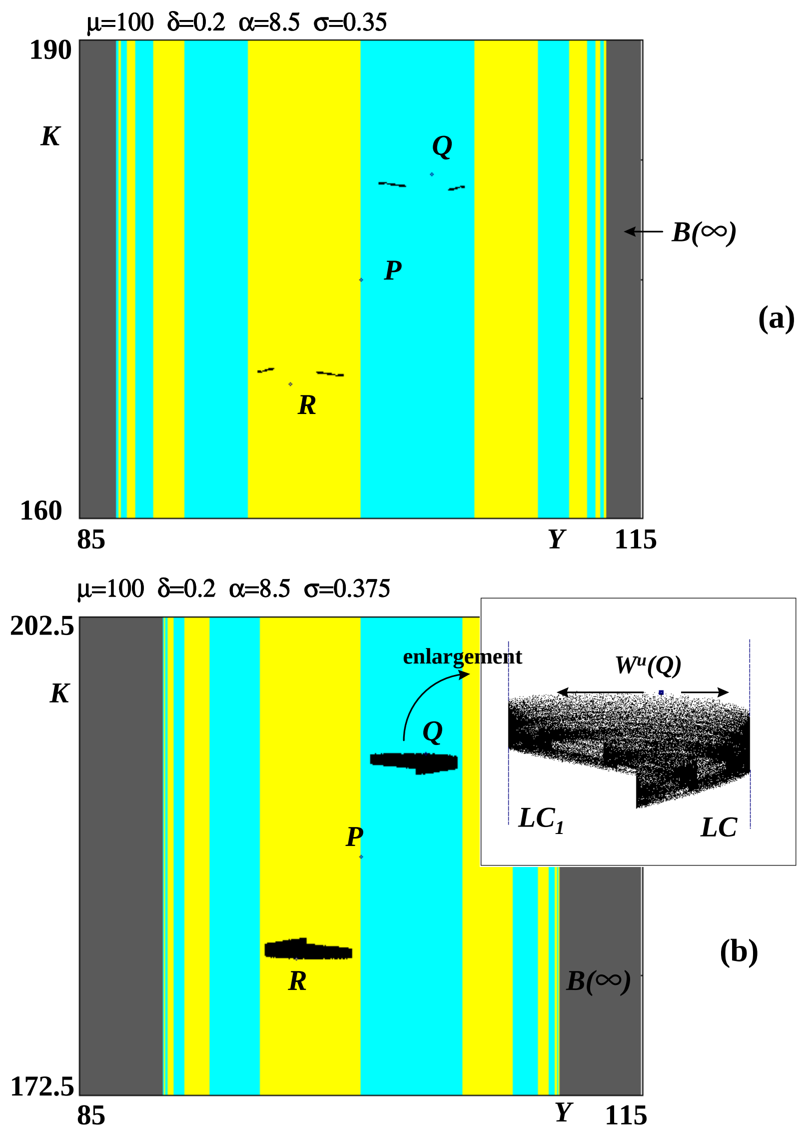


Figure 8: *Coexistence of two symmetric chaotic attractors.*

As it is evident from the bifurcation diagram in Fig. *8b*, in this chaotic regime also *periodic windows* exist, that is intervals of the parameter σ in which the attracting sets are periodic orbits, but the existence of chaos is revealed by the transient part of the trajectories (Fig. *9b*).

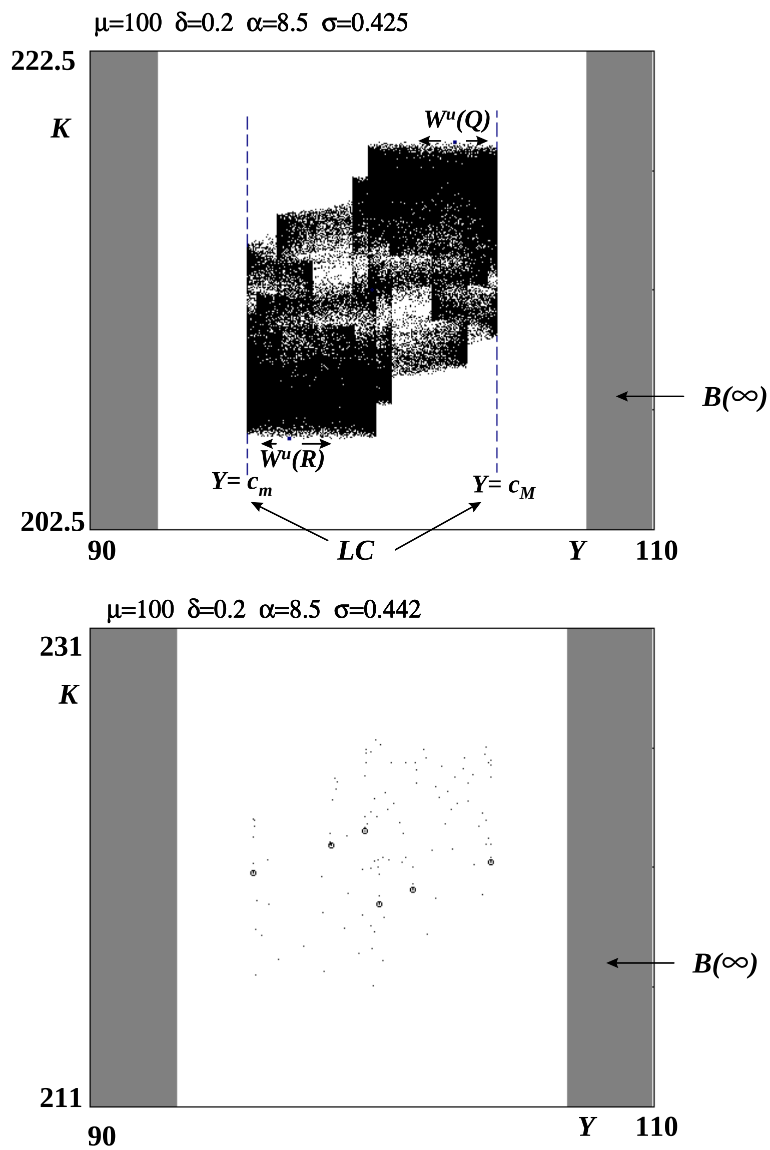
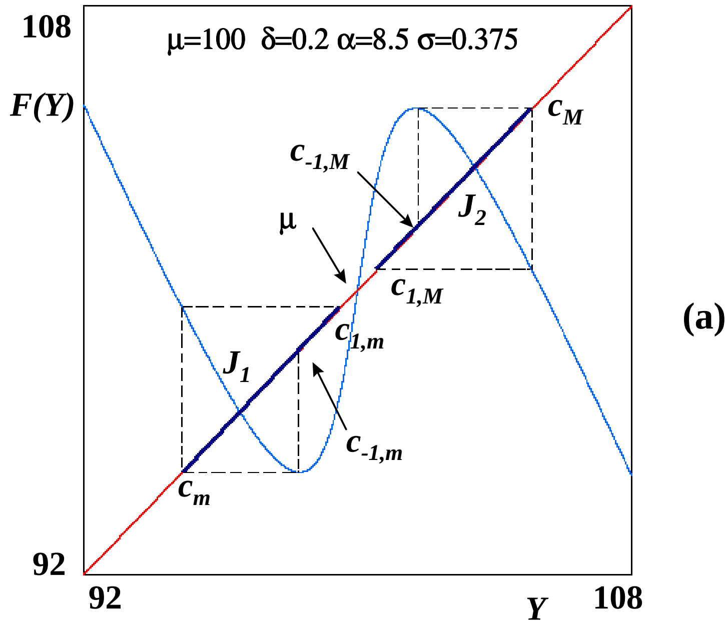


Figure 9: *Attracting sets existing after the homoclinic bifurcation of the saddle P: chaotic attractor (a) and periodic orbit with a chaotic transient (b).*

The merging of the two attractors into the unique attractor represented in Fig. *9a* is due to an *homoclinic bifurcation* of the saddle P. Let's consider Fig. *10a* (where $c_{1,m}$ and $c_{1,M}$ denote the images of c_m and c_M, respectively). Before the homoclinic bifurcation value $\sigma_h \simeq 0.4135$ (Fig. *10a*) the intervals $J_1 = [c_m, c_{1,m}]$ and $J_2 = [c_{1,M}, c_M]$ are *invariant*. We remark that, as it is also evident from the bifurcation diagram of Fig. *7b*, the map F undergoes, for each of the two disjoint invariant intervals $J_1$and J_2, the same sequence of bifurcations which characterize the well known logistic map $f(x) = ax(1-x)$ when the parameter a ranges between the values 3 and 4 (see, for instance, [14], [4]). After the homoclinic bifurcation we have a big qualitative change in the trajectories of the system: the intervals J_1 and J_2 are no longer *invariant* intervals, but so is their union $J = J_1 \cup J_2$ or, better, the interval $[c_m, c_M]$ (Fig. *10b*). Comparing again the dynamics of F with those of the one-dimensional logistic map, we can notice that the described homoclinic bifurcation is equivalent to the one occurring in the logistic map when the parameter a crosses the value 4. The difference arises from the fact that, in the case of the logistic map, after this bifurcation the points located in a neighborhood of the critical point, as well as their preimages of any rank, belong to the basin of infinity, so that the generic trajectory becomes divergent, while here the trajectories which escape remain bounded and span the whole invariant interval $J = [c_m, c_M]$.

We remark also that the existence of an homoclinic orbit, as already pointed out by Poincaré a century ago, implies very complicated dynamics (for an extensive mathematical treatment, see [16]): in fact, it implies the existence of an invariant Cantor set on which the restriction of the map is chaotic in the sense of Li and Yorke. Sometimes, it indicates the existence of the so called "invisible chaos": for example, chaotic trajectories certainly exist in the situation shown in Fig. *9b*, even if the generic numerically observed trajectory converges to a cycle of period-6.

The homoclinic bifurcation of the saddle P described above, produces a remarkable qualitative change in the asymptotic behaviour of the system, marking the switching from a regime of bi-stability (where the attractors may be fixed points, cycles or even chaotic attractors) to a more complex regime characterized by oscillations, although not regular but chaotic, around the saddle point P. This can be seen from Figs. *11a* and *11b*, which represent the *versus time* trajectories of the income Y before and after the bifurcation ($\sigma = 0.375$ and $\sigma = 0.425$, respectively). More precisely, from Fig. *11b* two different kinds of fluctuations can be observed: short-term chaotic oscillations around the "wealth" steady state Y_Q, or the "poverty" steady state Y_R, and long-term wider oscillations from "wealth" to "poverty" and vice-versa. This kind of dynamic behaviour is well known in the literature on chaotic dynamical systems, being very similar to the one observed on the well known Lorenz Attractor (see, for instance, [13], [17] ch.12).



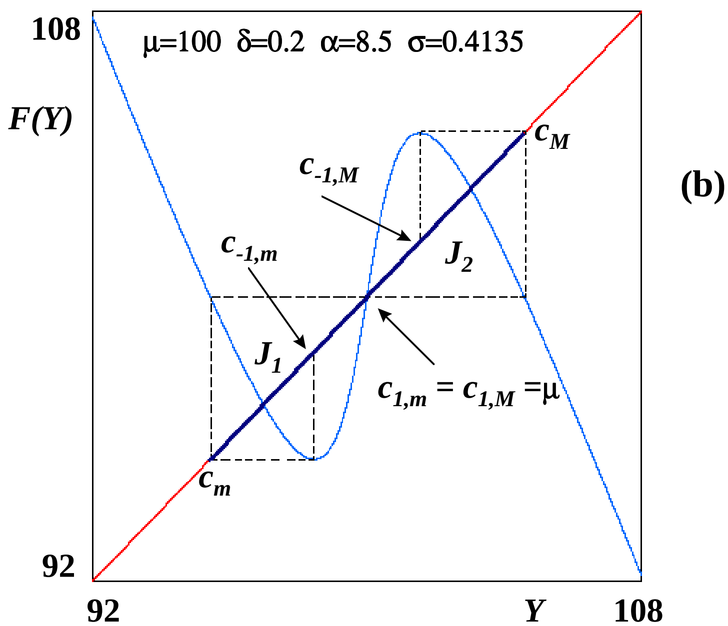


Figure 10: *Loss of the invariancy property of the intervals J_1 and J_2 due to the homoclinic bifurcation of the saddle P.*

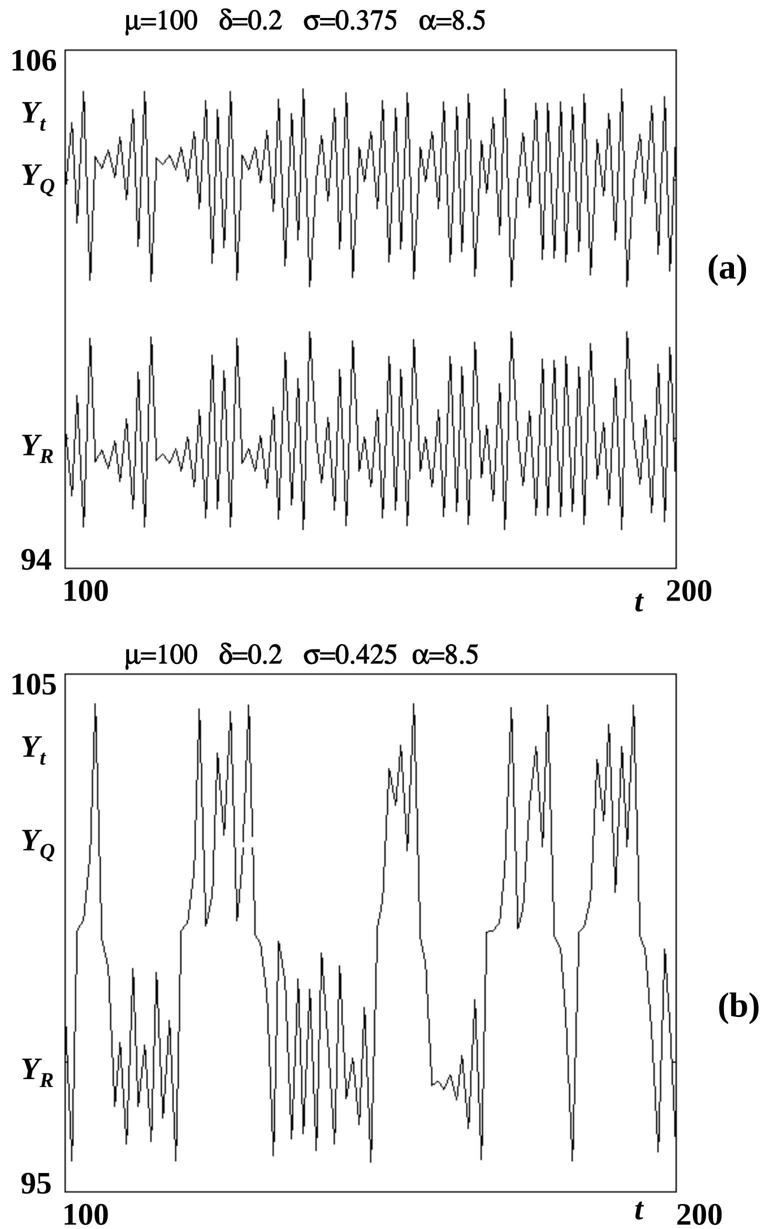


Figure 11: *Representation of the income as a function of time, before (a) and after the homoclinic bifurcation (b).*

We stress that this kind of oscillations, which are observed in particular regions of the parameters space Ω characterized by high values of the adjustment parameter α, occur even if we have assumed that the investment demand is independent from the capital stock, i.e. we have ruled out the possibility of cyclical "shifting" of the investment demand function.

Remark The peculiar shape of the attractors shown in Figs. *8b* (and its enlargement) and *9a* is typical of the so-called *mixed absorbing areas*. An *absorbing area* of *mixed* type is defined as a bounded region $\mathcal{A}$ of the phase-plane such that a neighborhood $U \supset \mathcal{A}$ exists whose points enter $\mathcal{A}$ after a finite number of iterations and never escape, and whose boundary is given by the union of critical curve segments (segments of the critical curve LC and its images) and portions of unstable sets of saddles (see [15], Ch. 4). This is the case, for instance, of the area containing the chaotic attractor represented in the enlargement of Fig. *8b*: here the lateral borders are given by segments the critical lines LC and LC_1 of equation $Y = c_M$ and $Y = c_{1,M}$, respectively, while the area is upper bounded by a portion of the unstable manifold $W^u(Q)$ of the saddle fixed point Q and lower bounded by portions of the unstable sets of some saddle cycles. This feature determines the "fuzzy" shape of the borders of the attractor observed in the enlargement. Similar reasons explain the shape of the chaotic attractor shown in Fig. *9a*.

6 Conclusions

In the present paper, starting from a discrete-time Kaldor-type business cycle model, proposed in [18] and described by a two-dimensional dynamical system in income and capital, we have focused on a particular case obtained by neglecting the dependence of the investment demand from the capital stock. This latter feature has generally been considered as the basic structural requirement for the occurrence of cyclical behaviour of income and capital (see, for instance, [6], [2]). The resulting model has the peculiarity that both the savings and the investment demands only depend on the income level (in particular the investment demand is a sigmoid-shaped increasing function of income): this implies that the dynamics of the system is driven by a two-dimensional map of triangular type, since one of its components, namely the one driving the income evolution, is an independent one-dimensional map. Due to the particular triangular structure of the system, we have been able to fully understand the asymptotic dynamic behaviour and the bifurcations, starting from the properties of the associated one-dimensional map.

We have explored the dynamics of the model under different regimes of the main parameters, as the propensity to save and the firms' speed of adjustment to the excess demand. Our exercise has shown that the basic dynamic scenario is given by a situation of bistability, i.e. coexistence of two attracting sets (which may be fixed points or periodic orbits or even chaotic attractors, for sufficiently high values of the adjustment parameter): one characterized by poverty (low levels of income and capital) and one by wealth (high levels of the dynamic variables), each with its own basin of attraction. We have focused on the question, which naturally arises in the presence of bistability, of the delimitation of the basins of the coexisting attractors, and we have shown that for particular ranges of the parameters the basins may be non-connected sets. Besides the bistability situation, our analysis has shown that different dynamic scenarios are possible, characterized by complex chaotic dynamics, for high values of the adjustment parameter. To summarize we have shown the existence of two different routes to complexity.

(a) Qualitative changes in the asymptotic behaviour of the system, i.e. in the nature of the attracting sets. The main change consists in the transition from a regime of bistability to a situation characterized by wide chaotic fluctuations of income and capital around their exogenously assumed equilibrium levels (a typical business-cycle situation). This change is related to the occurrence of a global (homoclinic) bifurcation and the resulting dynamical behaviour proves that endogenously driven oscillations can also be observed without what is considered the main Kaldorian assumption, i.e. the dependence of investment demand on the capital stock.

(b) Qualitative changes of the topological structure of the basins, which change from connected to non connected, related to the property of noninvertibility of the map. This kind of complexity leads to an interesting economic interpretation. We may say that, in the case of connected basins, the initial situation of the economy (wealth or poverty) is maintained over time, whereas in the presence of non connected basins such a situation may be reversed many times (and even reversed in the long run). Thus, noninvertibility gives more uncertainty about the fate of the system in the long run, whereas invertibility traps the long-run evolution, so that things seem to be determined since the beginning of the process.

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