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Border Collision Bifurcations in a simple oligopoly model with constraints

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ABSTRACT

In this paper, we run through an example of oligopoly, recently introduced in the literature, where firms operate under constraints. The dynamical system describing firms' choices over time assumes the form of a piecewise-smooth map because of such constraints. By carrying on a leading example, we show possible routes to complexity in the model, mainly due to border collision and homoclinic bifurcations.

Keywords: piecewise-smooth dynamical system, border collision bifurcations, Cournot oligopoly.

2000 Mathematics Subject Classification: 37D45, 37N40, 91A20.

1 Introduction

The presence of complex dynamic phenomena in Cournot oligopoly models is well documented in the mathematical economics literature, starting from (Rand, 1978) and (Dana and Montrucchio, 1986). The first microfounded example was provided in (Puu, 1991), where it is shown that a duopoly model with isoelastic demand curve and constant marginal costs can lead to both simple and complex dynamics through the well-known period doubling route to chaos (see also (Kopel, 1996)).

Often, dynamic oligopoly models are formulated under simplified assumptions, so that they are analytically tractable. For instance, it is often postulated that firms can adjust outputs to their desired levels, without constraints on minimum and maximum production. Only few works on the subject relax these assumptions (see for instance (Bischi, Chiarella, Kopel and Szidarovszky, 2010), (Puu and Norin, 2003)). As a matter of fact, with such constraints firms' production strategies over time often assume the form of *piecewise smooth* maps, i.e. discrete dynamical systems whose state space can be partitioned into regions where the functional form of the map changes (see (Mosekilde and Zhusubaliyev, 2003) and (Di Bernardo, Budd, Champneys and Kowalczyk, 2008)).

For piecewise smooth maps, beside the standard bifurcations, well-studied for smooth systems (either local or global), other interesting dynamic phenomena are possible. These are related to the existence of borders (or *switching manifolds*) in the phase space where the functional form defining the map changes, and thus to discontinuous Jacobian. The collision of an invariant set of the piecewise smooth map with such a border may lead to a bifurcation often followed by drastic changes in the dynamics of the system. The bifurcations phenomena related to these border collisions are nowadays called *Border Collision Bifurcations* (BCB for short). This term was first introduced by (Nusse and Yorke, 1992) and then adopted by many authors. The simplest form of investigation consists in the analysis of a fixed point which crosses the boundary of definition of a piecewise smooth map (or piecewise linear). The most complete analysis is reported in (Banerjee, Ranjan and Grebogi, 2000), where the authors show that such a contact may produce any kind of effect (transition to another cycle of any period or to chaos), depending on the eigenvalues of the two Jacobian matrices involved on the two sides with respect to the border. Clearly, in the case of a cycle of period m it is necessary to investigate the fixed points of the related m-th iteration of the map.

In this paper, we run through an extension of Puu's 1991 model, recently presented in (Bischi et al., 2010), where besides isoelastic demand and constant marginal cost, firms are constrained to minimum and maximum production levels as a results of plants' capacity limits or agreements with trade unions, governments, etc. The particular example in (Bischi et al., 2010) is referred to as a *semi-symmetric* oligopoly, since all firms have the same marginal cost but one, which produces at lower expenses. Agents decide their production plans according to the partial adjustment towards best replies, as explained below. All in all, firms' decisions over time can be modelled by means of a piecewise-smooth map. This example is particularly interesting as many dynamic phenomena occur, mostly related to the introduction of production constraints. Besides standard bifurcations, we also detect border collisions without qualitative changes, true border collision bifurcations and homoclinic bifurcations possibly involving the switching manifolds. Even though many global properties of this example were already discussed in (Bischi et al., 2010), we prefer to go through the same steps and add further insights on other interesting dynamic phenomena. In this way, we show a possible approach for studying global bifurcations in piecewise smooth maps, by following, along a given path in the parameters space, attracting sets of the map (and of iterates) and their contacts with critical lines.

The paper is organized as follows. Section 2 recalls the economics of the Cournot game and the corresponding piecewise smooth dynamical system, whose fixed points are studied in terms of existence and stability in Section 3. Section 4 reports the possible routes to complex behavior as some parameters of the game are changed. Section 5 concludes.

2 The dynamic oligopoly with constraints

2.1 Microfoundation of the model

In this Section we recall the set-up of the *semi-symmetric* Cournot oligopoly with production constraints, as proposed in (Bischi et al., 2010).
An homogeneous good is produced by n firms: the first firm currently produces and sells a quantity denoted by x and the remaining $n-1$ are identical firms, each of which currently produces and sells the (same) quantity y. In particular, the difference between the first and the remaining competitors pertains their cost functions. In the following, production costs of the first firm and the remaining $n-1$ ones are linear, given, respectively, by[1]

$$C_1(x) = c_1 x \text{ and } C_2(y) = c_2 y$$

with $c_1 \neq c_2$.
Assuming isoelastic demand function, i.e. that a representative consumer maximizes a Cobb-Douglas utility function, the inverse demand function reads

$$p = \frac{1}{x+(n-1)y},$$

where p is the actual commodity price (see (Puu, 1991)).
Expected profits for the first and one of the remaining $n-1$ firms are then

$$\begin{aligned} \pi_1(x, y^e) &= \left[\frac{1}{x+(n-1)y^e} - c_1\right] x, \\ \pi_2(x^e, y) &= \left[\frac{1}{x^e+(n-1)y} - c_2\right] y, \end{aligned} \tag{2.1}$$

respectively, where the superscript ^{e} denotes expected quantities.
As expected profits are strictly concave functions in own strategies, by first order conditions on (2.1), it is straightforward to obtain the Cournot best reply (or best response) functions, specifying the best strategy to adopt as a function of competitors' expected strategies. Under Cournot expectations, i.e. that firms' expected quantities are equal to the last observed ones ($y^e = y$ and $x^e = x$), best replies for firm 1 and any other remaining firm can be written, respectively, as

$$x_1(y) = \sqrt{\frac{(n-1)y}{c_1}} - (n-1)y, \tag{2.2}$$

$$y_1(x, y) = \sqrt{\frac{x+(n-2)y}{c_2}} - x - (n-2)y. \tag{2.3}$$

Moreover, we assume that each firm can not exceed an equal capacity limit L, i.e. the production of each agent must be chosen in the interval $[0, L]$. This constraint is exogenously given and can be derived from total capacity production or can be imposed by a regulator agent.

[1] The introduction of fixed costs does not change the model set-up and is therefore omitted.

Now we turn to the firms' strategy update rules. At each time period, each firm observes competitors' current quantities, which are taken, by Cournot expectations, as a proxy for next period productions. Then, next period quantities are calculated according to the partial adjustment towards the best replies, i.e. as a weighted average between actual and best reply quantities. All in all, the firms' dynamic choices are expressed by the following map (continuous and piecewise differentiable)

$$\begin{cases} x' = (1-k_1)x + k_1 x_b, \\ y' = (1-k_2)y + k_2 y_b, \end{cases} \tag{2.4}$$

where $'$ denotes the unit-time advancement operator, $0 < k_1, k_2 \leq 1$ are the speeds of adjustment to best reply, and x_b and y_b are the *constrained* best replies taking into account both nonnegativity and capacity constraints, which are computed according to the following scheme:

$$x_b = \begin{cases} 0 & \text{if} \quad x_1 < 0, \\ x_1(y) & \text{if} \quad 0 \leqslant x_1 \leqslant L, \\ L & \text{if} \quad x_1 > L, \end{cases} \tag{2.5}$$

$$y_b = \begin{cases} 0 & \text{if} \quad y_1 < 0, \\ y_1(x,y) & \text{if} \quad 0 \leqslant y_1 \leqslant L, \\ L & \text{if} \quad y_1 > L, \end{cases} \tag{2.6}$$

where $x_1(y)$ and $y_1(x,y)$ are the unconstrained best replies given in (2.2) and (2.3). Notice that the definition of the best replies forces the dynamic variables x, y to stay always in $[0, L]$.

2.2 The piecewise smooth map

Following (Bischi et al., 2010), we fix in (2.2), (2.3), (2.5), (2.6)

$$c_1 = \frac{5}{16},\ c_2 = \frac{3}{8} \text{ and } L = 2,$$

and we focus on changes of the speeds of adjustment k_1 and k_2 and the number of competitors n.

To study this continuous piecewise smooth map we divide the strategy space (x, y) into regions where the map has different definitions. The curves that divide these regions are curves of non differentiability and will be called *critical lines*, following the notation used in Mira et al., 1996, for reasons that will be clear below commenting the routes to complexity in the model.

Notice that $x_1(y) < 0$ for $16(n-1)y < 5(n-1)^2 y^2$ so that we have

$$x_1(y) < 0 \text{ for } y < 0, \text{ or } y > \frac{16}{5(n-1)},$$

and $x_1(y) > 2$ for $16(n-1)y > 5[(n-1)y+2]^2$ so that

$$x_1(y) > 2 \text{ for } 5(n-1)^2 y^2 + 4(n-1)y + 20 < 0.$$

This second inequality is never satisfied, hence the constraint at $x_1 = 2$ is ineffective.

Analogously $y_1(x,y) < 0$ for $8[x+(n-2)y] < 3[x+(n-2)y]^2$, so that we have

$$y_1(x,y) < 0 \text{ for } (x+(n-2)y)[8-3x-3(n-2)y] < 0, \text{ or } y > \frac{8-3x}{3(n-2)},$$

and $y_1(x,y) > 2$ for $8[x+(n-2)y] > 3[x+(n-2)y+2]^2$, thus

$$y_1(x,y) > 2 \text{ for } 3[x+(n-2)y]^2 + 4[x+(n-2)y] + 12 < 0.$$

This last inequality is never satisfied, hence the constraint at $y_1 = 2$ is ineffective.
Thus, the map becomes

$$\begin{cases} x' = (1-k_1)x + k_1 x_b, \\ y' = (1-k_2)y + k_2 y_b, \end{cases}$$

where

$$x_b = \begin{cases} \sqrt{\frac{16(n-1)y}{5}} - (n-1)y & \text{if } y < \overline{y}, \\ 0 & \text{if } y \geqslant \overline{y}, \end{cases}$$

$$y_b = \begin{cases} \sqrt{\frac{8(x+(n-2)y)}{3}} - x - (n-2)y & \text{if } y < \widetilde{y}, \\ 0 & \text{if } y \geqslant \widetilde{y}, \end{cases}$$

where $\overline{y} = \frac{16}{5(n-1)}$ and $\widetilde{y} = \frac{8-3x}{3(n-2)}$.
The lines $y = \overline{y}$ and $y = \widetilde{y}$ divide the plane (x, y) into 4 regions:
In region I, where $y < \overline{y}$ and $y < \widetilde{y}$, we have the map T_1 :

$$T_1 : \begin{cases} x' = (1-k_1)x + k_1 x_1(y), \\ y' = (1-k_2)y + k_2 y_1(x,y). \end{cases} \tag{2.7}$$

In region II, where $y < \overline{y}$ and $y \geqslant \widetilde{y}$, we have the map T_2 :

$$T_2 : \begin{cases} x' = (1-k_1)x + k_1 x_1(y), \\ y' = (1-k_2)y. \end{cases} \tag{2.8}$$

In region III, where $y \geqslant \overline{y}$ and $y \geqslant \widetilde{y}$, we have the map T_3 :

$$T_3 : \begin{cases} x' = (1-k_1)x, \\ y' = (1-k_2)y. \end{cases} \tag{2.9}$$

In region IV, where $y \geqslant \overline{y}$ and $y < \widetilde{y}$, we have the map T_4 :

$$T_4 : \begin{cases} x' = (1-k_1)x, \\ y' = (1-k_2)y + k_2 y_1(x,y). \end{cases} \tag{2.10}$$

We notice that region IV has a small portion in the positive quadrant of interest only for low values of n. In fact, the two lines $y = \overline{y}$ and $y = \widetilde{y}$ which bound the region IV intersect in the point $(\overline{x}, \overline{y})$ (which is the rightmost corner point of region IV) where $\overline{x} = \frac{8}{3} - \frac{16(n-2)}{5(n-1)}$ and it is $\overline{x} < 0$ for any $n > 7$, thus in the case $n > 7$ region IV is in the half-plane $x < 0$ and it is never visited by a trajectory starting in the positive quadrant of interest, so that the dynamic model is described by three different maps, involving only regions I, II and III.

2.3 Critical curves

We recall that the (*rank-1*) critical lines denoted LC of a continuos map T are the locus of points in the plane having at least two coincident *rank-1* preimages, located on the set LC_{-1}, which is called the (*rank-0*) critical line or set of merging preimages (see (Mira, Gardini, Barugola and Cathala, 1996)). In case of a piecewise smooth map the curve along which the map changes its definition, is also called critical line. As it will be clear below, the set LC_{-1} plays a crucial role in explaining BCB and in obtaining the boundary of compact trapping regions, called absorbing areas (see again (Mira et al., 1996)).

In our case, candidates to be critical curves LC_{-1} are:

a) Curves of non differentiability, i.e. the lines $y = \overline{y}$ and $y = \widetilde{y}$;

b) Curves of vanishing Jacobian $DT(x, y) = 0$, where in region I

$$DT = \begin{bmatrix} 1-k_1 & k_1\left[\frac{2(n-1)}{\sqrt{5(n-1)y}} - (n-1)\right] \\ k_2\left[\frac{\sqrt{2}}{\sqrt{3[x+(n-2)y]}} - 1\right] & 1-k_2+(n-2)\,k_2\left[\frac{\sqrt{2}}{\sqrt{3[x+(n-2)y]}} - 1\right] \end{bmatrix};$$

in region II

$$DT = \begin{bmatrix} 1-k_1 & k_1\left[\frac{2(n-1)}{\sqrt{5(n-1)y}} - (n-1)\right] \\ 0 & 1-k_2 \end{bmatrix};$$

in region III

$$DT = \begin{bmatrix} 1-k_1 & 0 \\ 0 & 1-k_2 \end{bmatrix};$$

in region IV

$$DT = \begin{bmatrix} 1-k_1 & 0 \\ k_2\left[\frac{\sqrt{2}}{\sqrt{3[x+(n-2)y]}} - 1\right] & 1-k_2+(n-2)\,k_2\left[\frac{\sqrt{2}}{\sqrt{3[x+(n-2)y]}} - 1\right] \end{bmatrix}.$$

As in regions II and III it is $|DT| = (1-k_1)(1-k_2) > 0$, only in regions I and IV we may have points at which the Jacobian determinant vanishes.

As we have already mentioned, in general both kinds of critical curves are important: the lines along which the Jacobian changes the sign and the ones along which the Jacobian changes its expression, i.e., the lines of non differentiability. Sometimes both curves have important effects on the dynamics, but in other cases just one of them is involved. For the sets of parameters we use in the paper, the lines of non differentiability play an important role in bifurcations: as we shall see, collisions with them of some cycles lead to qualitatively different dynamics (i.e., the BCBs are observed). Let us remark that, in general, the lines of vanishing Jacobian influence only the shape of chaotic attractors.

In fact, when a portion of LC_{-1}, independently of its nature, crosses a chaotic set, then its images of increasing rank become portions of the boundaries of the chaotic area, and, as a result, we observe folding of the attractor along such boundaries. Moreover, the critical lines may cover only some parts of such boundaries, because some portions can be bounded by the unstable sets of saddle points or cycles leading to the so-called mixed chaotic areas (see, for example, Fig.5b or Fig.7).

Given that, in the following we mainly focus on the routes to complexity induced by lines of non differentiability, i.e. on the different bifurcation sequences in the model.

3 Fixed points and local stability

In this Section, we discuss the existence and the stability of fixed points of (2.4), already outlined in (Bischi et al., 2010). Regarding existence, it holds the following
Proposition. *Two fixed points of (2.4) exist in region I, given by*

$$O=(0,0) \text{ and } E=(x^*,y^*)=\left(\frac{16\,(n-1)\,(n+4)}{(6n-1)^2},\frac{80\,(n-1)}{(6n-1)^2}\right)$$

with $0<x^*<2$ *and* $y^*<x^*$ *for any* $n\geq 2$*, whereas no fixed points exist in regions II, III, IV.*

Proof. The unique fixed point of T_2 (2.8) is $(0,0)$ that does not belong to its definition region, i.e., to the region II; the unique fixed point of T_3 (2.9) is $(0,0)$ that does not belong to the region III; the unique fixed point of T_4 (2.10) is $\left(0,\frac{8}{3(n-2)}\right)$ that does not belong to region IV being $\frac{8}{3(n-2)}>\widetilde{y}$ for each $x>0$.
So, let us compute the fixed points in region I, which are solutions of the system

$$\sqrt{\frac{16(n-1)y}{5}}-(n-1)y=x,$$
$$\sqrt{\frac{8\,(x+(n-2)y)}{3}}-x-(n-2)y=y,$$

i.e.,

$$16(n-1)y=5\left[x+(n-1)y\right]^2, \qquad (3.1)$$
$$8\left[x+(n-2)y\right]=3\left[x+(n-1)y\right]^2.$$

Dividing we get

$$\frac{2\,(n-1)\,y}{x+(n-2)y}=\frac{5}{3},$$

from which

$$y=\frac{5}{n+4}x.$$

Substituting this expression into the first equation (3.1) we get

$$x=0 \text{ and } x^*=\frac{16\,(n-1)\,(n+4)}{(6n-1)^2}.$$

Moreover $0<x^*<2$ and $y^*<x^*$ for any $n\geq 2$.

□

Two observations are worth noticing. First, the unique non-trivial fixed point E corresponds to the Cournot-Nash equilibrium of the oligopoly game, whose existence for this particular model is granted by the results in (Rosen, 1965)[2]. Second, at the Cournot point, the firm with the

[2] In fact, the strategy space of each agent is a nonempty, compact interval of $\mathbb{R}$ and each player's individual profit is concave in his own strategy sets.

lower marginal cost sells a greater quantity than any other competitor ($x^* > y^*$) and gains the highest payoff, being

$$\pi_1(x^*, y^*) = \frac{(n+4)^2}{(1-6n)^2} > \frac{25}{(1-6n)^2} = \pi_2(x^*, y^*),$$

which is very easy to justify from an economic point of view (see also (Puu, 1998)).
To investigate the local stability of the positive fixed point E, we employ Jury's stability conditions

$$\begin{cases} P(1) = 1 - tr + det > 0, \\ P(-1) = 1 + tr + det > 0, \\ P(0) = det < 1, \end{cases}$$

where tr and det denote, respectively, the trace and the determinant of the Jacobian matrix at fixed point. The Jacobian matrix of the function T_1 evaluated in E is

$$DT_1(E) = \begin{bmatrix} 1-k_1 & k_1 \frac{(9-4n)}{10} \\ k_2 \frac{(11-6n)}{12(n-1)} & 1 - k_2 + (n-2)\, k_2 \frac{(11-6n)}{12(n-1)} \end{bmatrix},$$

whose trace and determinant are given by

$$\begin{aligned} tr &= 2 - k_1 - k_2 + (n-2)\, k_2 \frac{(11-6n)}{12(n-1)} = \\ &= 2 - k_1 - k_2 - (n-2)\, k_2 \frac{(n - \frac{11}{6})}{2(n-1)}, \end{aligned}$$

and

$$\begin{aligned} det &= (1-k_1)\left(1 - k_2 + (n-2)\, k_2 \frac{(11-6n)}{12(n-1)}\right) - k_1 k_2 \frac{(9-4n)}{10} \frac{(11-6n)}{12(n-1)} = \\ &= (1-k_1)(1-k_2) - (1-k_1)\,(n-2)\, k_2 \frac{n - \frac{11}{6}}{2(n-1)} - k_1 k_2 \frac{(4n-9)}{10} \frac{(n-\frac{11}{6})}{2(n-1)}. \end{aligned}$$

Thus, for $n > 2$ the condition $det < 1$ is always satisfied, and regarding the other stability conditions we have:

$$\begin{aligned} P(1) &= k_1 k_2 \left(1 + (n-2)\frac{(n-\frac{11}{6})}{2(n-1)}\right) - \frac{(4n-9)}{10}\frac{(n-\frac{11}{6})}{2(n-1)} = \\ &= k_1 k_2 \left(1 + \frac{(6n-11)^2}{120(n-1)}\right) > 0, \end{aligned}$$

which implies that a bifurcation with eigenvalue equal to $+1$ cannot occur to the fixed point E, while

$$P(-1) = 2(2-k_1-k_2) + k_1 k_2 - (2-k_1)\,(n-2)\, k_2 \frac{(n-\frac{11}{6})}{2(n-1)} - k_1 k_2 \frac{(4n-9)}{10}\frac{(n-\frac{11}{6})}{2(n-1)},$$

and it is clear that for $n > 2$ the condition $P(-1) > 0$ will not be satisfied and the fixed point E undergoes a flip bifurcation.
Regarding the local stability of the origin, we notice that the map is not differentiable in $O = (0,0)$. However the trace of the Jacobian matrix tends to infinity as a point approaches the origin, denoting that the eigenvalues are increasing so that, as it is usual in duopoly games, we can consider this fixed point unstable.

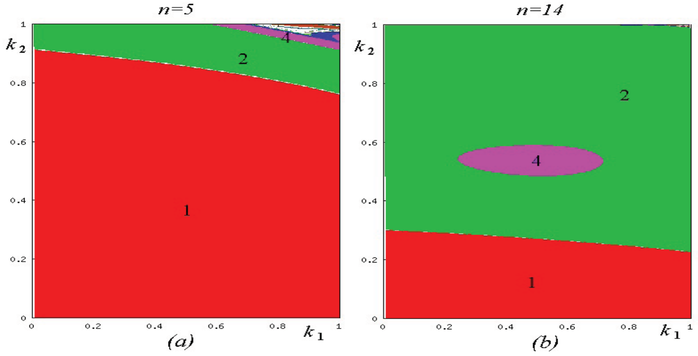

Figure 1: Two-dimensional bifurcation diagram in the (k_1, k_2)-parameter plane for (a) $n = 5$ and (b) $n = 14$. Numbers denote periods of the related stable cycles.

4 Routes to complexity in the Cournot game

Let us consider the dynamic behaviors of the model as a function of the parameters n, k_1, k_2. In general, as long as the number of competitors (n) is low, the model is quite stable, in the sense that we have the unique positive equilibrium, which is locally stable for a wide range of values of the other parameters (see on this points also (Puu and Panchuk, 2009)). As already noted, for $n < 7$ all the 4 regions (and thus the 4 maps) are involved in the dynamics, but we mainly have convergence to a fixed point or to a stable cycle derived from flip bifurcations of it. As an example, let us consider the bifurcation diagram in the parameters (k_1, k_2) at a fixed value of n, say $n = 5$, as reported in Fig.1a. It can be seen that the attractor changes from a stable fixed point to a stable cycle of period 2 (it occurs via flip bifurcations), and then to cycles of period 4 and 8 (again through flip bifurcations). Only in the upper corner of the bifurcation diagram of Fig.1a we can see the existence of a region with more complicated dynamics, that is, for high values of both k_1 and k_2 some complex dynamics may occur, and the borders are involved in some border-collision bifurcations (as explained below).

For higher values of n, the phase plane involves only three regions (and thus only three maps), as region IV is in the negative half-plane. In Fig.1b ($n = 14$) we can see that the dynamics are mainly stable (converging to a stable fixed point or to a stable cycle of period 2 or 4), and the existence of such a stable cycle for high values of k_2 is due to the piecewise smooth nature of the model. However, as n is further increased, also complex phenomena can be observed: in the white regions in Fig.2 the dynamics is often chaotic and the attracting set is a chaotic attractor.

In Fig.2b ($n = 18$) we can see the existence of an area associated with a stable cycle of period 3, which reflects the role played by the borders of the three maps; in fact, such a cycle is associated with the piecewise smooth nature of the map, and its appearance and destruction are caused by border collision bifurcations. The role played by such a 3-cycle becomes more relevant when the number n is further increased, as the portion of parameter values in which it exists, becomes wider and wider. An example of it is shown in Fig.3a ($n =$

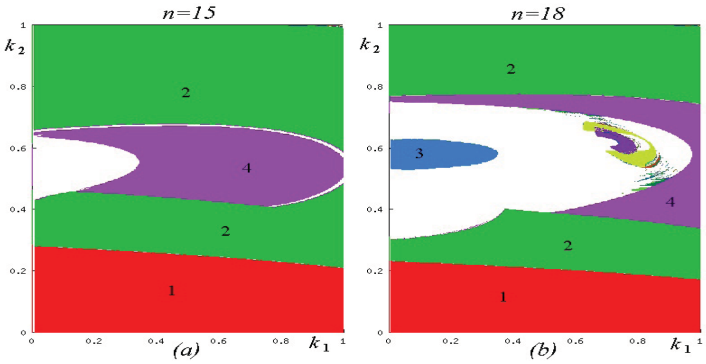

Figure 2: Two-dimentional bifurcation diagram in the (k_1, k_2)-parameter plane for (a) $n = 15$ and (b) $n = 18$.

23): the white region including chaotic behavior is reduced to two thin strips, inside which complicated periodic behaviors occur (see in particular the enlargement of Fig.3b). However, the predominant dynamics is related to a stable cycle of period less than 5. It is worth to note that the bifurcations associated with the dynamics of our model are mainly nonstandard with respect to those occurring in smooth maps. Interesting examples can be obtained, for instance, increasing k_2 and keeping constant k_1. From the point of view of the oligopoly model, it means that, in deciding their own outputs, the $n-1$ homogeneous firms increase the weight of best reply quantities, whereas the single heterogeneous firm maintains the same weights. In this case, whenever an attracting set has a contact with the lines on the boundary of definition of the maps (i.e. the critical lines) a border collision bifurcation occurs, whose effects is not easy to be predicted.

Regarding our model we can show a few of such nonstandard bifurcations, both from the point of view of the BCB theory, and also with respect to the investigation of the structure of the attracting set (which also may depend on the constraints).

To this scope, as an example, let us fix $k_1 = 0.4$ in Fig.3a and increase k_2. At $k_2 \approx 0.1653$ the fixed point E undergoes a flip bifurcation, which creates a stable 2-cycle. Soon after the bifurcation the two periodic points belong to region I and are close to the saddle E. As the parameter k_2 is further increased the two periodic points move far away from E, and one periodic point intersects the boundary of region I, i.e. the critical line LC_{-1}. This first merging could produce any kind of effect, however here no bifurcation occurs: one of the periodic points moves in region II (while the other persists in region I) and the two-cycle is still attracting. This is an example of border collision without qualitative dynamic changes. At $k_2 \approx 0.2462$ the 2-cycle undergoes a flip bifurcation and a stable 4-cycle appears. As before, soon after the bifurcation the four periodic points are close to the 2-cycle saddle, and far from the critical lines. However, as the parameter k_2 is increased more, one of the periodic points moves towards the critical line and at $k_2 \approx 0.2466$ one periodic point intersects the boundary of region I, i.e. the critical line LC_{-1} (in Fig.4a, the points of the stable 4-cycle and of the 2-cycle saddle are represented in the phase space as filled and unfilled circles respectively). This merging is a true BCB whose

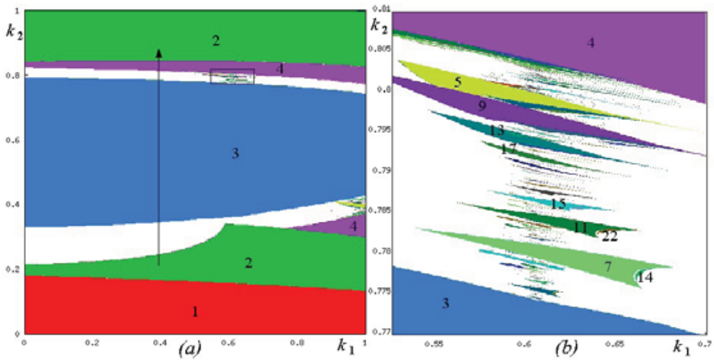

Figure 3: Two-dimentional bifurcation diagram in the (k_1, k_2)-parameter plane for $n = 23$ (a), and its enlarged part (b).

effect is the transition to a 4-pieces chaotic attractor, say G_4 (see Fig.4b)
As it can be seen in Fig.4, at first the attracting set involves only regions I and II: region III is above the second critical line, called LC'_{-1}, and for now it plays no role.
From Fig.4b, we can stress another property of piecewise smooth maps: when the invariant area including the (chaotic) attracting set intersects the critical line LC_{-1} (as it can be seen for example in Fig.4b), then the boundary of the area includes points belonging to critical curves of higher rank (obtained by taking the images under the map of the involved portion of LC_{-1}).
The saddle 2-cycle in between, as well as the saddle fixed point E, will be relevant for the structural changes of the chaotic attractor. In fact, as k_2 is increased more, the four pieces become wider and approach the stable set of the saddle 2-cycle. A contact of the chaotic area with the stable set causes the reunion of the pieces by pair, leaving a two-pieces chaotic attractor G_2; at the same time, this corresponds to the first homoclinic bifurcation of the saddle 2-cycle. In fact, at the contact bifurcation the stable and unstable sets of the saddle have infinitely many contact points (and this occurs at $k_2 \approx 0.2765$), while soon after the contact (see Fig.5a) the stable and unstable sets of the saddle have infinitely many transverse intersections, in homoclinic points. Similarly, increasing further the same parameter, the two pieces of the chaotic attractor enlarge, approaching the stable set of the saddle E, and a contact of the chaotic area with the stable set of E causes the reunion of the two pieces, thus leaving a one-piece chaotic attractor G_1: at the same time, this corresponds to the first homoclinic bifurcation of the saddle point E. In fact, at the contact bifurcation, the stable and unstable sets of the saddle E have infinitely many contact points (and this occurs at $k_2 \approx 0.2965$, see Fig.5b), while soon after the contact the stable and unstable sets of E have infinitely many transverse intersections in homoclinic points. Moreover, after this bifurcation, the boundary of the chaotic area G_1 is given by the images of suitable portions of the involved critical curves LC_{-1} and LC'_{-1} and portions of the stable set of the saddle E.
In addition, Fig.5b shows that, when the latter bifurcation occurs, also another border has been crossed, without any qualitative dynamic change. That is, the two pieces of the chaotic attractor G_2 increase and a contact with the critical line LC'_{-1} occurs, before the reunion into G_1. As

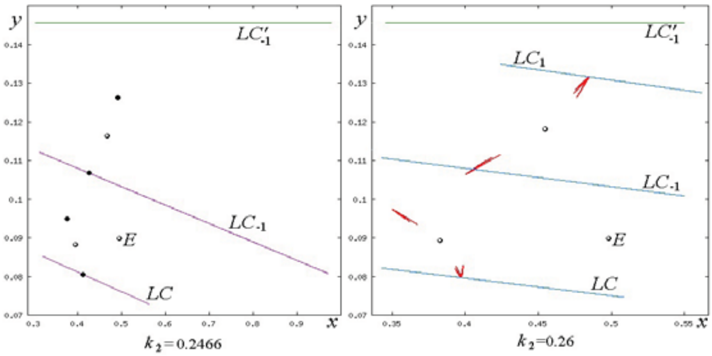

Figure 4: Phase portrait of the map T at $k_1 = 0.4$ and (a) $k_2 = 0.2466$; (b) $k_2 = 0.26$.

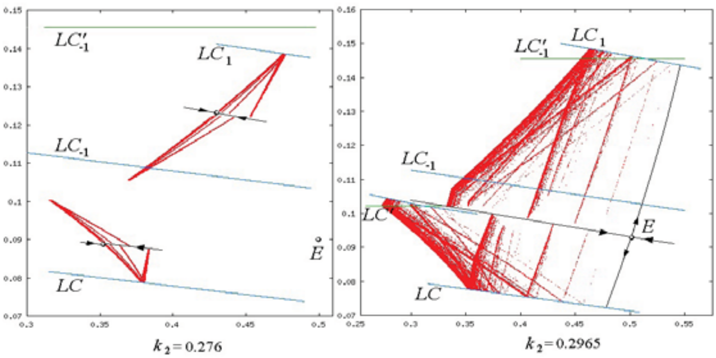

Figure 5: Phase portrait of the map T at $k_1 = 0.4$ and (a) $k_2 = 0.276$; (b) $k_2 = 0.2965$.

said, this border-crossing has no dynamic effect: the attractor is a two pieces chaotic attractor G_2 both before and after the crossing. What is to be emphasized is that after this crossing all the three regions are involved in the asymptotic behavior, i.e. all the three maps are applied for a trajectory on the chaotic attractor. Thus, the only difference is in the boundary of the chaotic area, which before the contact involves only the images of a suitable portion of LC_{-1}, while after the crossing it involves the images of suitable portions both of LC_{-1} and of the critical line LC'_{-1} (more precisely, the images of the pieces involved in the chaotic area).

Instead, a true BCB is the one causing the destruction (or disappearance) of the chaotic attractor G_1. In fact, as k_2 is further increased, we suddenly observe the appearance of a stable 3-cycle (this occurs at $k_2 \approx 0.3448$), as depicted in Fig.6a, where one point of the three-cycle seems on the critical line LC_{-1}. Then the three periodic points move from the critical set, and a stable 3-cycle exists for a wide interval of values of the parameter k_2. Its disappearance occurs at $k_2 \approx 0.783$ through a second BCB (see Fig.6b), whose effect is the transition again to a one-piece chaotic attractor G_1, an example of which is shown in Fig.7.

After this bifurcation we observe a decreasing in the complexity of the attracting set, as reverse bifurcations occur, both for the saddle fixed point E and for the saddle 2-cycle. That

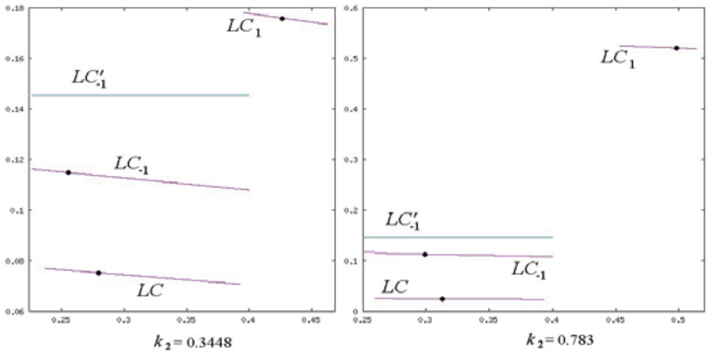

Figure 6: Phase portrait of the map T at $k_1 = 0.4$ and (a) $k_2 = 0.3448$; (b) $k_2 = 0.783$.

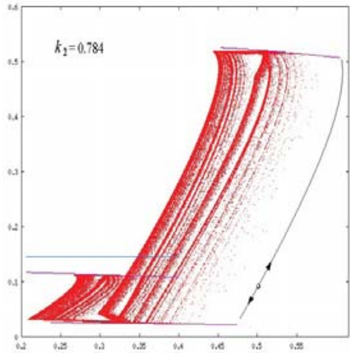

Figure 7: Phase portrait of the map T at $k_1 = 0.4$ and $k_2 = 0.784$.

is, increasing k_2 we get the transition of the one piece chaotic attractor G_1 into a two-pieces chaotic attractor G_2 by disappearance of all the homoclinic points of E: when the stable and unstable set of E become disjoint again, we have the transition of G_1 into G_2 (see Fig.8a).
In addition, also the homoclinic points of the two-cycle disappear, so that, when the stable and unstable sets of the two-cycle become in touch and then disjoint, we have the transition of the two-pieces chaotic attractor G_2 into a four-pieces chaotic attractor G_4 (see Fig.8b).
Ultimately, a border collision bifurcation gives rise to a stable cycle of period 4, which disappears by a reverse flip bifurcation leading to a stable 2-cycle again, but with periodic points in regions I and III.

5 Conclusions

In this paper, we showed as a simple economic model of oligopoly can generate rich dynamic scenarios, as constraints are added to the original formulations. In fact, these constraints lead to piecewise smooth maps, for which we can have all standard bifurcations of smooth maps and the so-called border collision bifurcations, proper of these models.

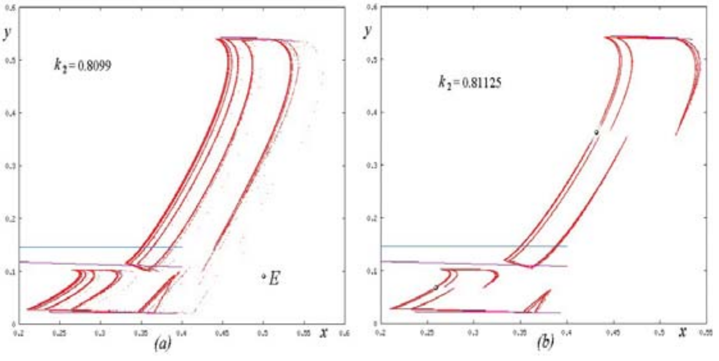

Figure 8: Phase portrait of the map T at $k_1 = 0.4$ and (a) $k_2 = 0.8099$; (b) $k_2 = 0.81125$.

In particular, we run through an example in (Bischi et al., 2010), exploring the dynamic of the model as the number of players and their speeds of reaction are changed. When the number of firms is low, the dynamic mainly converges to fixed points or low-period cycles. However, when the number of competitors is increased, then changes in agents' speed of reaction lead to homoclinic bifurcations and to BCB, as those associated with the sudden appearance or disappearance of chaotic attractors. The main example has been carried out by fixing the speed of reaction of an agent while increasing the other ones. In particular, we observed an increment in the complexity of the model, due to BCB, when the speeds of adjustment are increased over a given amount; convergence to simpler attractor, as a consequence of reverse BCB, are reported for higher values of the same parameter. In these cases, we explored how these bifurcations are created by closely following the attracting sets of the map and their contact with the critical lines, which represent an extremely useful instrument of analysis. As a final remark, this specific example could also serve for didactic purposes.

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