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Routes to Complexity Induced by Constraints in Cournot Oligopoly Games with Linear Reaction Functions

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Abstract

Within a classical discrete-time Cournot oligopoly model with linear demand and quadratic cost functions, minimum and maximum production constraints are imposed in order to explore their effects on the dynamic of the system. Due to the presence of such constraints, the dynamic model assumes the form of a continuous piecewise linear map of the plane. The study of Nash equilibria of the oligopoly game, together with an analytical and numerical investigation of the different kinds of attractors of the dynamical system, shows how the presence of production constraints generates so called border collision bifurcations, a kind of global bifurcations recently introduced in the literature on non-smooth dynamical systems, which gives rise to a quite rich spectrum of dynamic scenarios, characterized by drastic changes in the qualitative dynamic properties of the system.

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1 Introduction

Starting from the pioneering mathematical description of duopoly competition proposed by Cournot (1838), oligopoly models have always held a fascination for mathematically inclined economists (as stated in Shubik (1981)) as well as for economically inclined mathematicians. Indeed, after the duopoly model with linear demand and cost functions proposed by Cournot, where a Cournot-Nash equilibrium is achieved in the long run as the game is repeated by two players endowed with naive expectations, oligopoly models have been extended into different directions. A stream of literature studies the stability of oligopolistic markets as the number of competing firms increases (see Teocharis (1960), Hahn (1962), Okuguchi (1964), Okuguchi and Szidarovszky (1999)). In particular, Teocharis (1960) proves that a discrete time Cournot model with linear demand and cost functions is only stable in the case of duopoly. Moreover, McManus and Quandt (1961), Hahn (1962), Okuguchi (1964) show that this statement depends on the kind of adjustment considered and the kind of expectations formation. However, Fisher (1961) stresses that in general "the tendency to instability does rise with the number of sellers for most of the processes considered". Edgeworth (1925), by using prices as decision variables (following the suggestion in Bertrand (1883)) and assuming quadratic costs, stresses that prices may never reach an equilibrium position and continue to oscillate cyclically forever (of course, his conclusions also apply to Cournot models, as quantities and prices are related by an invertible, linear in this particular case, demand function).

Other extensions consider duopoly models with nonlinear demand and/or cost functions, by which several kinds of reaction functions can be obtained, such as non monotonic ones, which may lead to periodic or quasi-periodic or chaotic behaviors (Rand (1978), Dana and Montrucchio (1986), Puu (1991), Kopel (1996), Bischi, Chiarella, Kopel, and Szidarovszky (2010)). In particular, by using a formal approach based on symbolic dynamics, Rand (1978) showed that a Cournot tâtonnement with unimodal reaction functions can be chaotic, i.e. erratic bounded oscillations arise with sensitive dependence on initial conditions. Postom and Stewart (1978), pp. 424-425, claim that "adequate mathematics for planning in the presence of such phenomena is a still far distant goal". Economic motivations for unimodal reaction functions have been given in Van Huyck, Cook, and Battalio (1984) and Van Witteloostuijn and Van Lier (1990) in terms of goods that are strategic substitutes and complements in the sense of Bulow, Geanokoplos, and Klemperer (1985), whereas Dana and Montrucchio (1986) proved that any kind of reaction function can be obtained from a sound economically microfounded problem with suitable demand and cost functions. Puu (1991) shows how an hill-shaped reaction function can be obtained by using linear costs and an hyperbolic demand function,

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i.e. a demand with constant elasticity, and that complex behavior emerges provided that agents are sufficiently heterogeneous; in Kopel (1996) and Bischi and Lamantia (2002) unimodal reaction curves are obtained starting from a linear demand function and a nonlinear cost function with positive cost externalities. In all these papers complex (periodic or chaotic) dynamics arise through the well-known period doubling route to chaos, typical of smooth discrete dynamical systems. Moreover, global dynamical properties have been studied in Bischi, Mammana, and Gardini (2000), Bischi and Kopel (2001), Bischi and Lamantia (2002), Agliari, Gardini, and Puu (2006) where the method of critical curves for continuously differentiable maps is used to bound chaotic attractors and to characterize global bifurcations that cause qualitative modifications of the basins of attraction. Indeed, all these oligopoly models are based on the implicit assumption that firms can adjust outputs to their desired levels, without constraints on minimum and maximum production. Such simplifying assumption implies that the dynamic models obtained are smooth, hence the standard results on stability and bifurcations of differentiable dynamical systems can be applied. Only a few works on the subject relax these assumptions (see for instance Bischi et al. (2010), Puu and Norin (2003), Tramontana, Gardini, and Puu (2011)). As a matter of fact, with such constraints firms' production strategies over time often assume the form of *piecewise smooth* maps, i.e. discrete dynamical systems whose state space can be partitioned into regions where the functional form of the map changes (see Mosekilde and Zhusubaliyev (2003) and Di Bernardo, Budd, Champneys, and Kowalczyk (2008)). This implies that, beside the standard bifurcations (either local or global), well-studied for smooth systems, other interesting dynamic phenomena are possible, such as those related to the existence of *borders* (or *switching manifolds*) in the phase space where the functional form defining the map changes, and thus to discontinuous Jacobian. The collision of an invariant set of the piecewise smooth map with such a border may lead to a bifurcation often followed by drastic changes in the dynamic scenarios. The dynamic phenomena related to these contacts are nowadays called *Border Collision Bifurcations*, a term introduced in Nusse and Yorke (1992) and then adopted by many authors. The simplest case occurs when a fixed point (or a periodic point) crosses a border of non differentiability in a piecewise smooth map. In Banerjee, Karthik, Yuan, and J.A. (2000a) and Banerjee, Ranjan, and Grebogi (2000b), it is shown that such a contact may produce any kind of effect (transition to another cycle of any period or to chaos), depending on the eigenvalues of the two Jacobian matrices involved on the two opposite sides of a border.

In this paper we consider a classical Cournot oligopoly model, proposed in the recent book Bischi et al. (2010), where linear demand and quadratic cost functions (linear cost can be obtained as a particular case) give rise to continuous piecewise linear reaction functions, characterized by the presence of points of non differentiability due to the presence of upper and lower output constraints. Our goal is to show how complex dynamic behaviors may arise through non standard routes, characterized by border-collision bifurcations. We show that the lines of non-differentiability, due to the presence of constraints, can have several effects on the global dynamics observed. In fact, contacts of invariant sets with switching manifolds may lead to the sudden appearance or destruction of periodic or chaotic attractors; moreover such lines of non-differentiability may play the role of critical curves, so that they can be used to characterize some contact bifurcations that change the structure of the basins and, together with their images, they can be used to bound chaotic attractors, as suggested in Mira, Gardini, Barugola, and Cathala (1996), see also Bischi et al. (2010).

In particular, we intend investigate the role of the total number of firms, agents' heterogeneity and the inertia of the adaptive adjustments on the overall dynamic behavior of the model. In order to consider the combination of agents' numerosity and heterogeneity while keeping the model tractable, we assumed that the population of N firms is subdivided into two subsets of identical firms so that each of them can be represented by a representative firm. In this way, some degree of heterogeneity can be introduced and the total number of firms N can be used as a bifurcation parameter. Moreover, we show that when firms are aware of symmetry within the groups, the model exhibits very simple dynamic properties, whereas lack of information brings about the aforementioned Border Collision Bifurcations.

Of course, the fact that an unstable linear model becomes periodic or chaotic when constraints on minimum and maximum production are also considered (floor and ceiling), is not surprising. However, we believe that it is nowadays interesting to relate such phenomena to the rich literature on piecewise smooth dynamical systems arising in relevant applications in electrical engineering, (Di Bernardo, Feigen, Hogan, and Homer (1999), Banerjee and Grebogi (1999), Banerjee et al. (2000a), Banerjee et al. (2000b), Avrutin and Schanz (2006), Avrutin, Schanz, and Banerjee (2006), Tramontana and Gardini (2011)) or physics (see e.g. Zhusubaliyev, Mosekilde, Maity, Mohanan, and Banerjee (2006), Zhusubaliyev, Soukhoterin, and Mosekilde (2007)), and even to the works of some mathematical precursors of Nusse and Yorke that already studied the particular bifurcations associated with piecewise smooth maps, such as Leonov (1959), Leonov (1962), Mira (1978), Mira (1987), Maistrenko, Maistrenko, and Chua (1993), Maistrenko, Maistrenko, Vikul, and Chua (1995), Maistrenko, Maistrenko, and Vikul (1998).

The paper is organized as follows. In section 2 the setup of the Cournot dynamic model with linear reaction functions and adaptive adjustment is introduced; in section 3 the existence of Nash equilibria is discussed with particular emphasis on the role of production constraints; in section 4 some issues on dynamic behavior of the model, and the particular bifurcations observed, are addressed through both analytical and numerical methods. Section 5 concludes and gives suggestions on further researches about the proposed model.

2 The constrained Cournot model with piecewise linear reaction functions

Following Bischi et al. (2010), we introduce a Cournot oligopoly model where each firm k, $k = 1,...,N$, has an upper capacity limit L_k, so that it can choose a quantity inside the interval $[0, L_k]$. Moreover, for firm k, the production of the rest of the industry is given by

$$Q_k = \sum_{\substack{i=1 \\ i \neq k}}^{N} x_i$$

so that $Q = x_k + Q_k$ is the total output of the industry.

In this paper we assume that the inverse demand is a piecewise linear ramp function of the form:

$$p = f(Q) = \begin{cases} A - Q & \text{if } 0 \leq Q \leq A \\ 0 & \text{if } Q > A \end{cases} \tag{1}$$

where A is the market absorbing capacity, and the cost is given by a quadratic function

$$C_k(x_k) = c_k x_k + e_k x_k^2. \tag{2}$$

If $e_i \in (0, +\infty)$ the cost functions are convex and no particular constraint on other parameters should be imposed, whereas if $e_i \in (-1, 0)$, the cost function is strictly concave and it is necessary to add the condition

$$L_k < -\frac{c_k}{2e_k} \tag{3}$$

in order to avoid decreasing production costs when $x_k > -\frac{c_k}{2e_k}$. Of course, for $e_k = 0$ we obtain the particular case of linear cost function. The individual profit function

$$\pi_k(x_1, ..., x_N) = x_k f(x_k + Q_k) - C_k(x_k) = \tag{4}$$

$$= \begin{cases} x_k (A - Q_k - x_k - c_k - e_k x_k) & \text{if } 0 \leq Q_k + x_k \leq A \\ -c_k x_k - e_k x_k^2 & \text{if } Q_k + x_k > A \end{cases} \tag{5}$$

is concave in the strategy set $x_k \in [0, L_k]$ if $e_i > -1$, condition assumed to hold throughout the paper. The firm k maximization problem defines the reaction functions:

$$R_k(Q_k) = \underset{0 \le x_k \le L_k}{\operatorname{argmax}} \pi_k(x_1, ..., x_N)$$

given by the following piecewise linear mapping (see also Bischi et al. (2010), chapter 2):

$$R_k(Q_k) = \begin{cases} 0 \ \text{ if } Q_k \ge A - c_k \\ L_k \text{ if } Q_k \le A - c_k - 2L_k(1+e_k) \\ \frac{A - c_k - Q_k}{2(1+e_k)} \ \text{ otherwise} \end{cases} \tag{6}$$

Notice that $R_k(Q_k)$ is not identically zero whenever $A > c_k$, $k = 1, ..., N$, which also is assumed to hold in the rest of the paper.

In the following, in order to introduce some degree of heterogeneity while keeping the model tractable, we assume that there are $N \ge 2$ agents, subdivided into two groups of homogeneous players, referred to as group 1 and 2, formed by $1 \le n_1 < N$ and $n_2 = N - n_1$ firms respectively.

Note that, by (1), prices are positive as long as

$$A > n_1 L_1 + n_2 L_2. \tag{7}$$

We denote by $x_i(t)$ the quantity produced by a representative agent in group i, $i = 1, 2$ at time t, and we assume naive expectations, i.e. each agent in the first group assumes the current production of the rest of the industry in order to compute its best reply for time $t+1$:

$$Q_1(t) = (n_1 - 1)x_1(t) + (N - n_1)x_2(t) \tag{8}$$

and analogously for a representative agent in group 2

$$Q_2(t) = n_1 x_1(t) + (N - n_1 - 1)x_2(t). \tag{9}$$

3 Equilibria

The individual production strategies at a Nash equilibrium are obtained as solutions of the system

$$\begin{cases} \widetilde{R}_1(x_2) = x_1 \\ \widetilde{R}_2(x_1) = x_2 \end{cases} \text{ such that } x_i \in [0, L_i], i = 1, 2 \tag{10}$$

where the modified best reply $\widetilde{R}_k$ for a firm in group k, as a function of the quantity produced by a representative agent in group h, is obtained by substituting (8) and (9) into (6) to get:

$$\widetilde{R}_k(x_h) = \begin{cases} 0 \ \text{ if } x_h \in \left(\frac{A-c_k}{n_h}, L_h \right] \\ \frac{A-c_k-n_h x_h}{1+n_k+2e_k} \text{ if } x_h \in \left[\frac{A-c_k-(1+2e_k+n_k)L_k}{n_h}, \frac{A-c_k}{n_h} \right] \\ L_k \text{ if } x_h \in \left[0, \frac{A-c_k-(1+2e_k+n_k)L_k}{n_h} \right) \end{cases} ,k,h = 1,2; k \neq h \quad (11)$$

Notice that the interval $[0, L_h]$, is given by the union of the following three intervals,where the definition of $\widetilde{R}_k(x_h)$ changes:

$$L_{h1} = \left[0, \frac{A-c_k-(1+2e_k+n_k)L_k}{n_h} \right) \quad (12)$$

$$L_{h2} = \left[\frac{A-c_k-(1+2e_k+n_k)L_k}{n_h}, \frac{A-c_k}{n_h} \right], \qquad k,h = 1,2; k \neq h \quad (13)$$

$$L_{h3} = \left(\frac{A-c_k}{n_h}, L_h \right]. \quad (14)$$

These are all proper intervals of the real line provided that

$$(1+2e_k+n_k)L_k < A-c_k < n_h L_h, \qquad k,h = 1,2; k \neq h \quad (15)$$

Obviously, for $A-c_k > n_h L_h$, it is $L_{h3} = \emptyset$ and for $(1+2e_k+n_k)L_k > A-c_k$ it is $L_{h1} = \emptyset$. In any case L_{h2} is always a proper interval of the real line. We assume that $L_{h2} = \left[\frac{A-c_k-(1+2e_k+n_k)L_k}{n_h}, L_h \right]$ if $A-c_k > n_h L_h$ and $L_{h2} = \left[0, \frac{A-c_k}{n_h} \right]$ if $(1+2e_k+n_k)L_k > A-c_k$.

The Nash equilibria of the game are located at the intersection points of modified best replies (11), whose definition changes in each of the following nine rectangles

$$A_{ij} = L_{1i} \times L_{2j}, \ \ i,j = 1,2,3 \quad (16)$$

provided they are properly defined for the set of parameters at hand.

In order to stress the role of upper production constraints in the determination of existence and uniqueness of the Nash equilibria, we first neglect the presence of production constraint L_1 and L_2, to get a benchmark case, according to the following Proposition:

Proposition 1. *Consider the Cournot game whose reaction functions are given in (6), with two classes of homogeneous agents, quadratic cost functions (2), with $e_h \in (-1, +\infty)$, and no capacity constraints, i.e. $L_1 \to \infty$ and $L_2 \to \infty$. Then:*

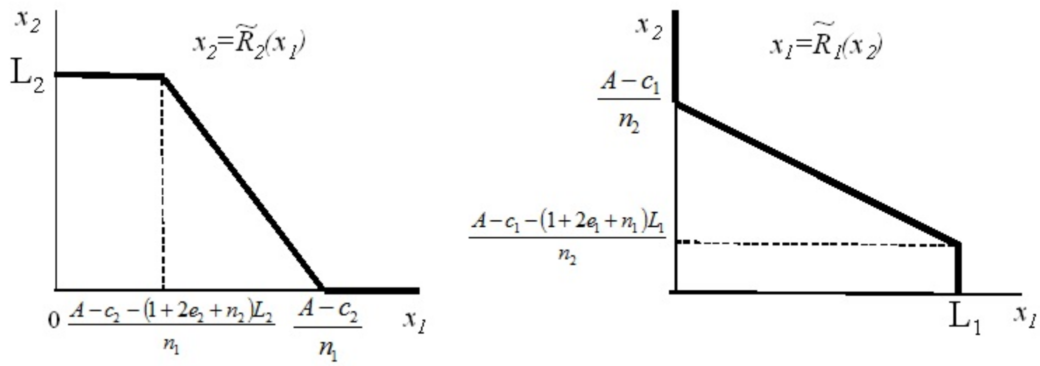

Figure 1: The modified reaction curves whose intersections define the Cournot-Nash equilibria.

1. *if* $e_h > \widehat{e}_h = -\frac{1}{2} - \frac{n_h(c_h - c_k)}{2(A-c_k)}$, $k,h = 1,2; k \neq h$, *then a unique (inner) equilibrium* $E_* = (q_1^*, q_2^*)$ *exists, with components given by*

$$q_k^* = \frac{(A-c_k)(1+2e_h) + n_h(c_h - c_k)}{1+N+2e_h(1+n_k)+2e_k(1+n_h)+4e_ke_h} \text{ , } k,h = 1,2; k \neq h; \quad (17)$$

2. *if* $e_h < \widehat{e}_h$, $h = 1,2; k \neq h$, *then the inner equilibrium (17) exists together with the two boundary equilibria*

$$E_1 = \left(\frac{A-c_1}{1+n_1+2e_1}, 0\right); \; E_2 = \left(0, \frac{A-c_2}{1+n_2+2e_2}\right);$$

3. *otherwise only one boundary equilibrium exists.*

Proof.

Neglecting the presence of capacity constraints, the conditions to have a unique equilibrium, given by the intersection between the decreasing parts of the modified best reply functions (11), is obtained by imposing the condition $\frac{A-c_k}{n_h} > \frac{A-c_h}{1+n_h+2e_h}$, $k,h = 1,2; k \neq h$. Similarly, conditions $\frac{A-c_k}{n_h} < \frac{A-c_h}{1+n_h+2e_h}$, $k,h = 1,2; k \neq h$ lead to part 2 of the proposition (two boundary and one inner equilibrium), whereas conditions $\frac{A-c_k}{n_h} < \frac{A-c_h}{1+n_h+2e_h}$ and $\frac{A-c_h}{n_k} > \frac{A-c_k}{1+n_k+2e_k}$ lead to the part 3 of the proposition.□

Now we consider how the presence of upper capacity constraints affects the equilibria. Of course, in the cases 1 and 2 of proposition 1, the inner equilibrium (17) exists provided that $L_k \geq q_k^*$, $k = 1,2$. However, from proposition 1 several consequences can be obtained when adding capacity constraints.

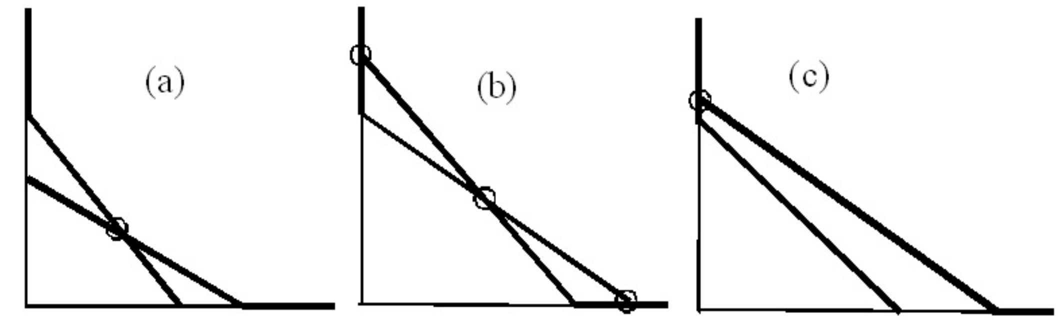

Figure 2: Illustration of Proposition 1, cases 1, 2, 3.

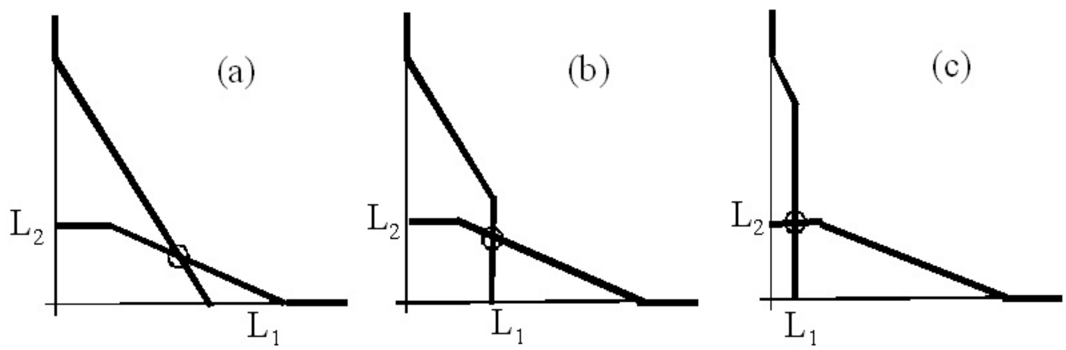

Figure 3: Graphical illustration of Proposition 2.

Proposition 2. *Consider Proposition 1, case 1, with finite values of* L_k, $k = 1,2$. *Also in this case a unique equilibrium always exists, in particular:*

1. *If* $L_k > q_k^*$, $k = 1,2$, *then capacity constraints do not affect the equilibrium* E_* *given by (17);*
2. *If* $L_k \leq q_k^*$, *and* $L_h > q_h^*$ *then the k-th and h-th coordinates of the unique equilibrium are given, respectively, by* L_k *and* $\widetilde{R}_h(L_k) = \frac{A - c_h - n_k L_k}{1 + n_h + 2e_h}$;
3. *If* $L_k \leq q_k^*$, $k = 1,2$, *then the unique equilibrium is* (L_1, L_2).

The proof is straightforward, see also fig. 3.□

Proposition 3. *Consider proposition 1, case 2. The inner Nash equilibrium* E_* *exists provided that* $L_k \geq q_k^*$, $k = 1,2$; *otherwise the inner equilibrium does not exist; in addition, there is at least another equilibrium, whose coordinates are given, respectively, by* $\left(L_1, \widetilde{R}_2(L_1)\right)$ *or* $\left(\widetilde{R}_1(L_2), L_2\right)$, i.e.:

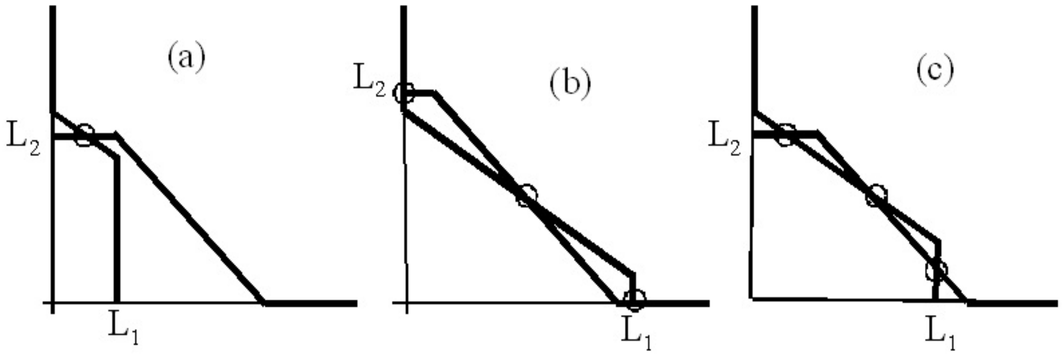

Figure 4: Graphical illustration of Proposition 3.

1. 0 *and* $\frac{A-c_k}{1+n_k+2e_k}$, *if* $\frac{A-c_k}{1+2e_k+n_k} < L_k$;
2. 0 *and* L_k, *if* $\frac{A-c_k}{1+2e_k+n_k} \geq L_k > \frac{A-c_h}{n_k}$;
3. $\frac{A-c_h-n_kL_k}{1+n_h+2e_h}$ *and* L_k, *if* $L_k \leq \frac{A-c_h}{n_k}$.

This proof is also straightforward, see fig. 4.□

Proposition 4. *Consider proposition 1, case 3, where only one border equilibrium exists. For this unique border equilibrium, the same conditions stated in proposition 3 also apply in the presence of constraints.*

It is useful to rearrange the possible steady states of the Cournot game in the following matrix M, where m_{ij} is the solution (if any) of (10) in each rectangle A_{ij}:

$$M = \begin{pmatrix} (L_1, L_2) & \left(\frac{A-c_1-n_2L_2}{1+n_1+2e_1}, L_2\right) & (0, L_2) \\ \left(L_1, \frac{A-c_2-n_1L_1}{1+n_2+2e_2}\right) & (q_1^*, q_2^*) & \left(0, \frac{A-c_2}{1+n_2+2e_2}\right) \\ (L_1, 0) & \left(\frac{A-c_1}{1+n_1+2e_1}, 0\right) & (0,0) \end{pmatrix} \tag{18}$$

Note that equilibrium $(0,0)$ only exists in the unrealistic cases $L_1 = L_2 = 0$ or $A = 0$.

It is worth to notice that the Nash equilibria are not the only invariant sets of the dynamic game considered and, as it is well known in the literature, the long run behavior of this dynamic game can exhibit persistent (i.e. self-sustained) periodic oscillations that never converge to a Nash equilibrium. For example, even in the simplest case of duopoly competition with best reply response, i.e. $N = 2$ and

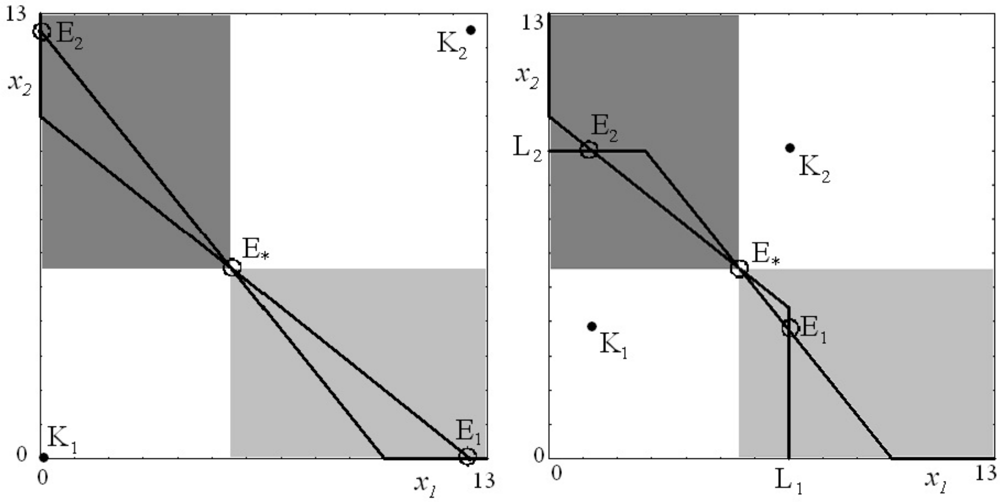

Figure 5: The case of duopoly with best reply and unstable interior equilibrium.

$a_1 = a_2 = 1$, for negative values of e_i coexistence of two stable boundary equilibria and a cycle of period two is obtained provided that $e_h < \widehat{e}_h$, $h = 1,2$, as shown in fig. 5, obtained with parameters $A = 16$, $c_1 = c_2 = 6$; $e_1 = e_2 = -0.6 < \widehat{e}_h = -0.5$. In fig. 5a the upper production constraints are arbitrarily large, i.e. $L_h > (A - c_k)/n_h = 10$, and besides the interior equilibrium, which is unstable (see Bischi et al. (2010), see also the next section), two stable boundary equilibria exist, given by $E_1 = \left(\frac{A-c_1}{1+n_1+2e_1}, 0\right) = (12.5, 2)$; $E_2 = \left(0, \frac{A-c_2}{1+n_2+2e_2}\right) = (0, 12.5)$ as well as a stable cycle of period 2 given by $\{K_1, K_2\} = \{(0,0), (12.5, 12.5)\}$, each with its own basin of attraction, represented in fig. 5 by the light grey, dark grey and white regions respectively. If production limits L_1 and L_2 are lower, such as in fig. 5b, obtained with $L_1 = 7$ and $L_2 = 9$, both the equilibria and the periodic cycles are modified accordingly, as they become $E_1 = \left(L_1, \widetilde{R}_2(L_1)\right)$, $E_2 = \left(\widetilde{R}_1(L_2), L_2\right)$ and $\{K_1, K_2\} = \left\{\left(\widetilde{R}_1(L_2), \widetilde{R}_2(L_1)\right), (L_1, L_2)\right\}$ respectively.

It is plain that for $e_h \geq 0$, $h = 1,2$, i.e. for convex cost functions, the interior Nash equilibrium is unique and always globally stable in the positive orthant (see Bischi et al. (2010) and also the next sections of this paper).

However, for increasing values of N and lower values of adaptive speeds of adjustment a more rich spectrum of dynamic scenarios can be observed, especially when binding upper production constraints are considered. In the following, we analyze the effects induced by the presence of upper production limits L_1 and L_2

on the dynamic behaviors of the Cournot dynamic game with adaptive adjustments. As we shall see, when information is not complete, different kinds of periodic and chaotic behaviors can be observed, and transitions between such dynamic scenarios can be characterized by sudden changes due to border collision bifurcations, as the values of production constraints are gradually modified.

4 Dynamics and bifurcations

As customary, we assume that the time evolution of the repeated production choices in discrete time is based on an adjustment process where each representative firm partially adjusts towards the computed best reply with naive expectations, modelled by the iteration of a two-dimensional *piecewise smooth* map $T : A \to A$, with $A = [0, L_1] \times [0, L_2]$.

However, according to the kind of information available to the firms, we can distinguish the following two cases:

1. Any agent of group i, $i = 1, 2$, is aware that in the industry there are other $n_i - 1$ of its same kind and, consequently, it reacts to the quantity of a representative firm that is of the different type, according to (11); more precisely, since each firm is aware that agents of its own type will play its strategy, firms have perfect rationality on the quantity that will be played by their own group and naive expectations on the quantity produced by firms of the different group. The map assumes the form (19).
2. Any agent of group i, $i = 1, 2$, is unaware that in the industry there are other $n_i - 1$ of its kind, and so it reacts to the quantity produced by the rest of the industry Q_i, on which each agent has naive expectation and best replies according to (6). In this case, the map assumes the form (20).

Both maps are piecewise linear and their equilibria, already detailed above, are the same. However, their dynamic properties are very different, as it is investigated below. In particular, the first model shows very simple dynamic properties and will be analyzed in the next subsection, whereas the rest of this section is devoted to the second model.

4.1 The dynamic oligopoly with complete information

In the first case, each firm reacts to the expected quantity produced by a representative agent of the opposite kind, as the information on homogeneity within groups is known by all players. Under this assumption, the map assumes the form:

$$\widetilde{T}: \begin{cases} x_1(t+1) = \widetilde{T}_1(x_1(t), x_2(t)) = a_1 \widetilde{R}_1(x_2(t)) + (1-a_1)x_1(t) \\ x_2(t+1) = \widetilde{T}_2(x_1(t), x_2(t)) = a_2 \widetilde{R}_2(x_1(t)) + (1-a_2)x_2(t) \end{cases} \tag{19}$$

where $\widetilde{R}_k(x_h)$, $k,h = 1,2$, $k \neq h$, are given by (11), $a_k \in (0,1]$, $k = 1,2$, denote the speeds of adjustment toward the best reply, or equivalently $(1-a_k)$, $k = 1,2$, denote inertia, or anchoring attitude, with respect to the decision of changing the current production at time t into the computed one, according to the maximization problem, i.e. the best reply function. Notice that for $a_k = 1$, $i = 1,2$, each player chooses its best reply strategy with naive expectations. Moreover, if $N = 2$ (and consequently $n_1 = n_2 = 1$), a typical Cournot duopoly game is obtained, with the particular structure $(x_1(t+1), x_2(t+1)) = (R_1(x_2(t)), R_2(x_1(t))$ studied by many authors for its peculiar mathematical properties, see e.g. Dana and Montrucchio (1986), Bischi et al. (2010), Cánovas (2000), Cánovas and Lopez Medina (2010).

Notice that map $\widetilde{T}$, given by (19), has a different specification in each one of the nine rectangles given in (16). The main dynamic properties of the map (19) are summarized in the following Proposition:

Proposition 5 *Consider the map (19) with best replies (11).*

1. *For parameters values as in Proposition 1, part 1, and Proposition 2, the unique inner equilibrium is always locally asymptotically stable;*
2. *For parameters values as in Proposition 1, part 2, and Proposition 3, the unique inner equilibrium is always locally asymptotically unstable and the two boundary equilibria are locally asymptotically stable;*
3. *For parameters values as in Proposition 1, part 3, and Proposition 4, the unique border equilibrium is locally asymptotically stable.*

Proof.

By writing the Jacobian matrices of (19) in the different regions A_{ij}, we obtain that, apart from $J_{|\overset{\circ}{A}_{22}}$, eigenvalues are always given by $1-a_i$, and so we have that a boundary equilibrium, whenever it exists, is stable. Applying to $J_{|\overset{\circ}{A}_{22}}$ the usual stability conditions (see (26) below), we get that the condition for having boundary equilibria implies the local instability of the inner equilibrium.□

Let us consider the crossing of the inner equilibrium from region A_{22} to another region, due to changes in parameters; for instance a reduction of the production constraints L_i, moving the fixed point to regions A_{12} or A_{21}, see (16). This crossing has no effect on the stability of the equilibrium, as stability is granted before and after the crossing by the conditions on the Jacobian matrix. In this cases, these border collisions do not bring any bifurcation. However, with less information on players' heterogeneity, a similar border crossing can lead to a dramatic change of the stability properties of the attractors, as it will be shown below.

4.2 The dynamic oligopoly with incomplete information

In the second case, the out-of-equilibrium dynamics of the Cournot oligopoly with naive expectations and adaptive adjustment of N firms subdivided into two heterogeneous groups can be modelled by the iteration of the two-dimensional map T given by

$$T : \begin{cases} x_1(t+1) = T_1(x_1(t), x_2(t)) = a_1 R_1((n_1-1)x_1(t) + n_2 x_2(t)) + (1-a_1)x_1(t) \\ x_2(t+1) = T_2(x_1(t), x_2(t)) = a_2 R_2(n_1 x_1(t) + (n_2-1)x_2(t)) + (1-a_2)x_2(t) \end{cases} \tag{20}$$

with *piecewise linear* reaction functions (6), which can be written, for the purposes of this section, as:

$$R_1(x_1,x_2) = \begin{cases} 0 \ \text{ if } x_2 \geq \frac{A-c_1-(n_1-1)x_1}{n_2} \\ L_1 \text{ if } x_2 \leq \frac{A-c_1-2L_1(1+e_1)-(n_1-1)x_1}{n_2} \\ \frac{A-c_1-[(n_1-1)x_1+n_2x_2]}{2(1+e_1)} \text{ if } \frac{A-c_1-2L_1(1+e_1)-(n_1-1)x_1}{n_2} < x_2 < \frac{A-c_1-(n_1-1)x_1}{n_2} \end{cases} \tag{21}$$

and

$$R_2(x_1,x_2) = \begin{cases} 0 \ \text{ if } x_2 \geq \frac{A-c_2-n_1x_1}{n_2-1} \\ L_2 \text{ if } x_2 \leq \frac{A-c_2-2L_2(1+e_2)-n_1x_1}{n_2-1} \\ \frac{A-c_2-[n_1x_1+(n_2-1)x_2]}{2(1+e_2)} \text{ if } \frac{A-c_2-2L_2(1+e_2)-n_1x_1}{n_2-1} < x_2 < \frac{A-c_2-n_1x_1}{n_2-1}. \end{cases} \tag{22}$$

Due to the presence of the lower and upper limits, the map (20) has a different specification in each region of the phase space $[0,L_1] \times [0,L_2]$, whose borders (when inside the phase space) are given by the following lines:

$$\begin{aligned} l_1 : x_2 = l_1(x_1) &= \frac{A-c_1-(n_1-1)x_1}{n_2} \\ l_2 : x_2 = l_2(x_1) &= \frac{A-c_1-2L_1(1+e_1)-(n_1-1)x_1}{n_2} \\ l_3 : x_2 = l_3(x_1) &= \frac{A-c_2-n_1x_1}{n_2-1} \\ l_4 : x_2 = l_4(x_1) &= \frac{A-c_2-2L_2(1+e_2)-n_1x_1}{n_2-1} \end{aligned} \tag{23}$$

Notice l_1 and l_2 are parallel, with l_1 entirely located above l_2; l_3 and l_4 are parallel as well, with l_3 above l_4. Moreover, when $n_1 = 1$ then l_1 and l_2 are horizontal lines in the plane (x_1, x_2).

Denote the sets

$$S_i = \{(x_1, x_2) \in [0, L_1] \times [0, L_2] : x_2 < l_i(x_1)\ \forall x_1 \in [0, L_1]\}$$

Obviously, each region inside which the map (20) is differentiable (indeed linear) can be written in terms of S_i sets. For instance, the inner Nash equilibrium E_* given in (17), if it exists, is inside the set

$$I = S_1 \cap \overline{S}_2 \cap S_3 \cap \overline{S}_4 \quad . \tag{24}$$

where $\overline{A}$ denotes the complement of set A. From the economic point of view, in the set S_2 (S_4), firms of the first (second) group produce their upper capacity limit L_1 (L_2), whereas in the set $\overline{S}_1$ ($\overline{S}_3$), firms of the first (second) group do not produce at all, because of (1). The lines l_i constitute the boundaries along which the map is not differentiable, and across such lines (also called borders) the map T assumes different expressions. This is the basic mechanism that causes border collision bifurcations when a portion of an invariant set (e.g. a fixed point or a point of a periodic cycle) has a contact with one of such lines and then crosses it. Its stability properties may undergo a sudden change, as it may be suddenly destroyed or transformed into a completely different kind of invariant set. In particular this may cause the sudden creation, destruction or a qualitative change of an attractor.

4.3 Numerical examples

In this subsection, we show some numerical examples obtained with map (20) to better motivate the following bifurcation analysis. Let us consider the bifurcation diagram shown in fig.6, obtained with parameters $a_1 = a_2 = 0.5$; $A = 16$, $c_1 = c_2 = 6$; $N = 16$, $n_1 = 6$, $e_1 = e_2 = 0$, $L_2 = 1$ and $L_1 \in [0, 1.2]$ as a bifurcation parameter. Notice that in this example the cost functions are both linear being $e_k = 0$, $k = 1, 2$. At $L_1 \simeq 0.56$ the unique (and globally stable) Nash equilibrium E loses stability and a big chaotic attractor suddenly appears.

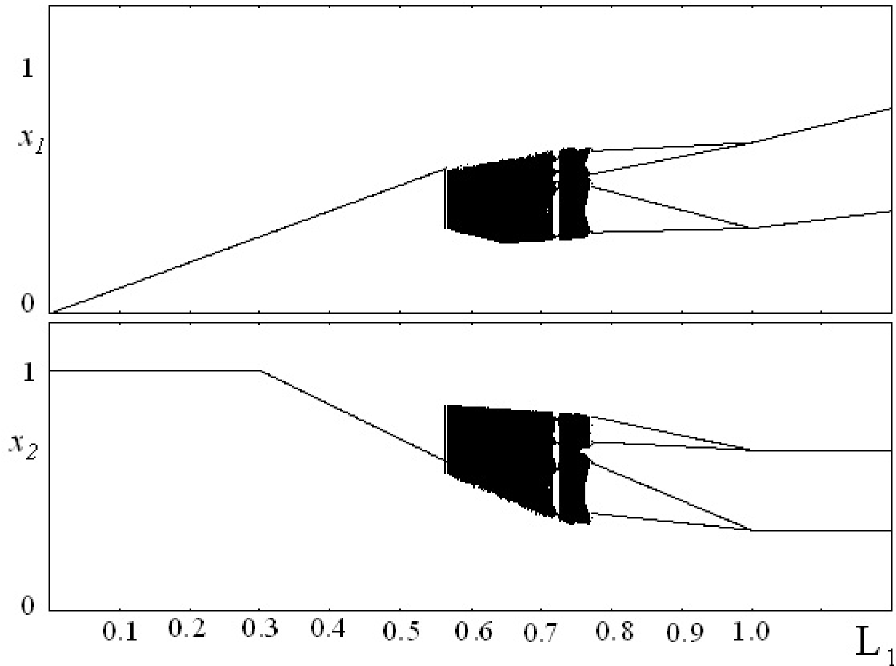

Figure 6: A ”period one to chaos” border collision bifurcation. Parameters are given as follows: $a_1 = a_2 = 0.5, A = 16, c_1 = c_2 = 6, N = 16, n_1 = 6, e_1 = e_2 = 0, L_2 = 1$ and $L_1 \in [0, 1.2]$ as bifurcation parameter.

This is a typical border collision bifurcation that can only occur in a non-smooth dynamical system. In fact, when a stable equilibrium of a smooth dynamical system loses stability, it is generally replaced by an attractor close to it, that may be a stable cycle of period two in the case of flip (i.e. period doubling) bifurcation or a couple of stable fixed points issuing from it in the case or a pitchfork bifurcation or another fixed point merging and gradually departing from it in the case of transcritical (i.e. stability exchange) bifurcation or a small stable closed invariant curve departing from it in the case of a Neimark-Sacker subcritical bifurcation. Instead, in this case we observe a sudden transition from a globally stable fixed point to a fully developed chaotic area of finite amplitude. This is even more evident, and more easily understood, if we observe what happens in the phase plane (x_1, x_2) as the bifurcation parameter L_1 is gradually varied across the bifurcation value. In fig. 7a, obtained for $L_1 = 0.53$, the globally stable equilibrium $E = \left(L_1, \widetilde{R}_2(L_1)\right)$

is represented together with the borders (or switching lines) l_i, $i = 1, ..., 4$, given by (23). If L_1 is gradually increased, then the equilibrium point moves to the right and also the line l_2 moves downwards and they have a contact (border collision) when $E \in l_2$, i.e. $\widetilde{R}_2(L_1) = l_2(L_1)$ that gives $\frac{A-c_2-n_1L_1}{1+n_2+2e_1} = \frac{A-c_1-L_1(1+2e_1+n_1)}{n_2}$ from which the bifurcation value $L_1 = L_{1BC} = 10/17 \simeq 0.588$ can be analytically computed. As soon as L_1 is further increased, a chaotic attractor suddenly appears with the structure shown in fig. 7b, obtained with $L_1 = 0.6$. Fig. 7b also shows that if we consider the portion of switching line l_4 approximately included inside the chaotic area (such as the dashed thick portion of l_4 represented in the figure), its images by the map T, denoted as $T(l_4)$ and $T^2(l_4)$ in fig. 7b, constitute portions of the boundaries of the chaotic area inside which the long run dynamics of the Cournot game are trapped. The same holds for the images of increasing rank of the other switching line, l_2, included inside the chaotic area, that are not represented in the figure. The union of the early images $T^k(l_4)$ and $T^k(l_2)$, allows one to get the whole boundary of the chaotic area, thus giving an estimate of the amplitude of the chaotic oscillations that characterize the long run dynamics of the system. This may be expressed by saying that the switching lines (or borders) at which the expression of the map T changes behave like critical curves, i.e. the curves of vanishing Jacobian that represent the folding manifolds of differentiable noninvertible maps (see e.g. Mira et al. (1996), Agliari, Bischi, and Gardini (2002)). This is another important feature of the non differentiable (piecewise linear) dynamical system due to the presence of upper production constraints.

4.4 Stability of the inner equilibrium and genesis of the border collision bifurcations

The stability properties of the interior Nash equilibrium E_* of the unconstrained duopoly model with best reply, i.e. $N = 2$ and $a_1 = a_2 = 1$ are well known (see e.g. Vives (2001)): it is globally stable whenever it is the unique equilibrium, and loses its stability when the boundary Nash equilibria E_1 and E_2 are created (see proposition 1). In the latter case also a stable cycle of period two coexists with the two stable boundary equilibria, as proved in Bischi et al. (2010), see also fig. 5a. The stabilizing effects of inertia, i.e. decreasing values of speeds of adjustment a_i in a Cournot duopoly with adaptive adjustment is analyzed in Bischi et al. (2010) where also the case of N firms with $n_1 = 1$ is considered. A complete analysis of

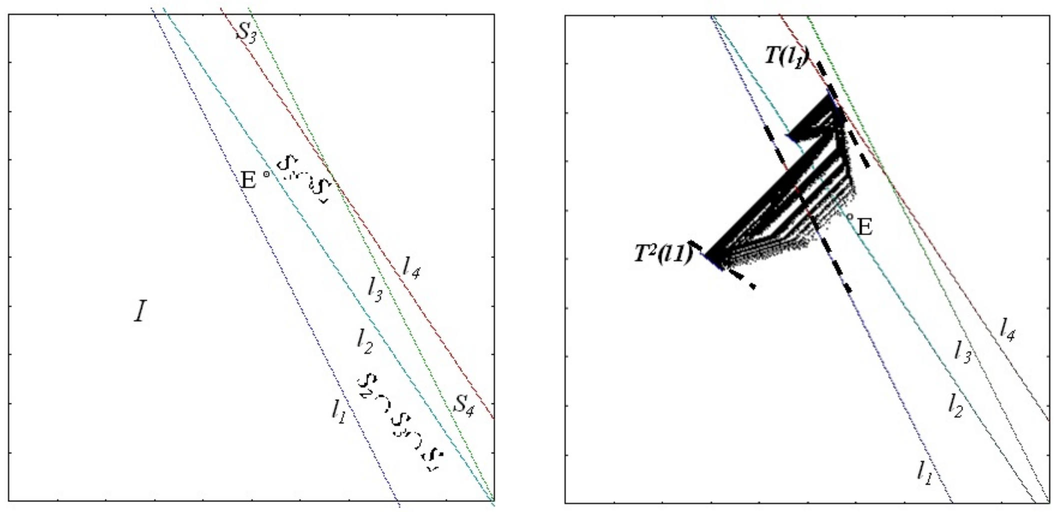

Figure 7: Representation of a border collision bifurcation in the phase space, all parameters as in fig.6, but with a fixed L_1. (a) $L_1 = 0.53$: the stable equilibrium E has not yet crossed the border l_2; (b) $L_1 = 0.6$: just after the crossing of l_2, the fixed point E is unstable and a large chaotic attractor appears, whose bounds can be obtained by iterating suitable portions of the borders l_i.

the stability of the equilibrium points of the Cournot model with constraints is not easy in general, as it requires the localization of the fixed points in the different regions where the map T has different local definitions, and the computation of the corresponding Jacobian matrix

$$J = \begin{pmatrix} (1-a_1)+\frac{\partial R_1}{\partial x_1} & a_1\frac{\partial R_1}{\partial x_2} \\ a_2\frac{\partial R_2}{\partial x_1} & (1-a_2)+\frac{\partial R_2}{\partial x_2} \end{pmatrix} \tag{25}$$

in the region where the fixed point is located, where R_i are the reaction functions given in (21) and (22). Notice that the Jacobian matrix is just the matrix of the coefficients, being the map linear in each region of the phase plane. In each region the usual stability conditions apply, given by

$$\begin{cases} 1+Tr+Det > 0 \\ 1-Tr+Det > 0 \\ Det < 1 \end{cases} \tag{26}$$

where Tr and Det represent the trace and the determinant of the Jacobian matrix respectively. Of course if, due to a variation of one or more parameters, a fixed point crosses a border that bounds the region where it is defined, after the crossing it may no longer be a fixed point of the map T or it may still be a fixed point but with a different stability property due to the fact that a different Jacobian matrix governs the dynamics in the new region. In these cases, we say that a border collision bifurcation occurs. Instead, it may happen that this crossing has no effects on the stability of the equilibrium, if its existence and stability is granted both before and after the crossing by the conditions on the two different Jacobian matrices on the two sides of the border. In this case, we may say that the border collisions causes no qualitative changes, i.e. it is not a bifurcation. However, in general, it is quite difficult to forecast which kind of dynamic scenario will prevail after the crossing, and a trade off between analytical and numerical methods is often necessary to study such situations.

Analogous arguments apply to the case of a cycle of period n, that in general may have periodic points belonging to different regions, e.g. k periodic points in a given region and the remaining $n-k$ periodic points in a different region. Then the stability of the cycle depends on the eigenvalues of the product of the Jacobian matrix at each periodic points, which is, in general, a different matrix inside each region where the periodic points belong. This implies that, whenever a single periodic point moves between two different regions by crossing a border, the periodic cycle may disappear or it may suddenly change its stability property.

Proposition 6. *Consider the map (20) with best replies given in (21)-(22).*

1. *Any fixed point or cycle in regions $S_2 \cap S_4$ or $\overline{S}_1 \cap \overline{S}_3$ is locally asymptotically stable.*
2. *Any fixed point or cycle in regions $S_2 \cap \overline{S}_4$ or $\overline{S}_1 \cap S_3$ is locally asymptotically stable whenever $a_2 \in \left(0, \frac{4(1+e_2)}{1+2e_2+n_2}\right)$.*
3. *Any fixed point or cycle in regions $\overline{S}_2 \cap S_4$ or $S_1 \cap \overline{S}_3$ is locally asymptotically stable whenever $a_1 \in \left(0, \frac{4(1+e_1)}{1+2e_1+n_1}\right)$.*

Proof.

We observe that, in (25), $\frac{\partial R_1}{\partial x_1} = \frac{\partial R_1}{\partial x_2} = 0$ in $\overline{S}_1$ and S_2, and $\frac{\partial R_2}{\partial x_1} = \frac{\partial R_2}{\partial x_2} = 0$ in $\overline{S}_3$ and S_4. In $S_1 \cap \overline{S}_2$, it is $\frac{\partial R_1}{\partial x_1} = -\frac{n_1-1}{2(1+e_1)}$ and $\frac{\partial R_1}{\partial x_2} = -\frac{n_2}{2(1+e_1)}$. In $S_3 \cap \overline{S}_4$, it is $\frac{\partial R_2}{\partial x_1} = -\frac{n_1}{2(1+e_2)}$ and $\frac{\partial R_2}{\partial x_2} = -\frac{n_2-1}{2(1+e_2)}$. Therefore, outside the set I, at least one off-diagonal term of the Jacobian matrix is 0 and the eigenvalues are given by the entries in the main diagonal; furthermore at least one of these eigenvalues is always between zero and one, as it is equal to $1-a_i$. In particular, in $S_2 \cap S_4$ and in $\overline{S}_1 \cap \overline{S}_3$

both eigenvalues are always $1-a_i$ and so any fixed point or cycle in these regions is stable.

With respect to part 2, in regions $S_2 \cap \overline{S}_4$ or $\overline{S}_1 \cap S_3$ (provided they are nonempty) the Jacobian matrix reads

$$J_{|S_2\cap\overline{S}_4} = J_{|S_3\cap\overline{S}_1} = \begin{pmatrix} 1-a_1 & 0 \\ -\frac{a_2 n_1}{2(1+e_2)} & 1-\frac{a_2(1+2e_2+n_2)}{2(1+e_2)} \end{pmatrix}$$

so that stability of a fixed point (or of a periodic cycle) in these regions surely occurs whenever $\frac{a_2(1+2e_2+n_2)}{2(1+e_2)} < 2$; otherwise, it is possible to set a speed of adjustment so that an equilibrium (or a cycle) is unstable, i.e. if $a_2 \in \left(\frac{4(1+e_2)}{1+2e_2+n_2}, 1\right]$, which is a proper interval provided that $n_2 > 3+2e_2$. Analogous arguments apply to regions $S_4 \cap \overline{S}_2$ and $S_1 \cap \overline{S}_3$ by swapping indices 1 and 2, thus proving 3.□

Proposition 6, part 1, deals with the case where firms from both groups sell a constant production, as in $\overline{S}_1 \cap \overline{S}_3$ both groups do not produce ($R_1(x_1,x_2) = R_2(x_1,x_2) = 0$) and in $S_2 \cap S_4$ upper capacity constraints are binding for all players, i.e. $R_1(x_1,x_2) = L_1$ and $R_2(x_1,x_2) = L_2$. Since productions inside these regions are constants, the stability of attractors in these regions is achieved for all parameters values. Instead, proposition 6, case 2 (3), deals with the case where lower or upper constraints are binding only for firms in the first (second) group; here convergence to a fixed point occurs provided that it is sufficiently low the speed of adjustment for firms whose production is not constant.

Now we turn to the problem of stability of the inner Nash equilibrium (17). For sake of simplicity, we study the case with equal speed of adjustments ($a_1 = a_2$) and quadratic cost coefficient ($e_1 = e_2$), so that the difference between the two groups can be their numerosity n_i, their cost coefficient c_i or the capacity constraints L_i, $i = 1,2$.

Proposition 7. *Consider map (20) with best replies given in (21)-(22) and with* $a_1 = a_2$ *and* $e_1 = e_2 > -\frac{1}{2}$.

1. *The inner Nash equilibrium (17) is stable for a speed of adjustment* $a_i \in \left(0, \frac{4(1+e_i)}{1+2e_i+n_1+n_2}\right)$ *and unstable for* $a_i \in \left(\frac{4(1+e_i)}{1+2e_i+n_1+n_2}, 1\right]$;
2. *For a speed of adjustment* $a_2 \in \left(\frac{4(1+e_2)}{1+2e_2+n_1+n_2}, \frac{4(1+e_2)}{1+2e_2+n_2}\right)$, *a border collision bifurcation occurs whenever the inner Nash equilibrium (17) collides with the borders* l_1 *or* l_2 *given in (23).*
3. *For a speed of adjustment* $a_1 \in \left(\frac{4(1+e_1)}{1+2e_1+n_1+n_2}, \frac{4(1+e_1)}{1+2e_1+n_1}\right)$, *a border collision bifurcation occurs whenever the inner Nash equilibrium (17) collides with the borders* l_3 *or* l_4, *given in (23).*

Proof.
Inside the interior region I we have

$$J_{|I} = \begin{pmatrix} 1 - \frac{a_1(1+2e_1+n_1)}{2(1+e_1)} & -\frac{a_1 n_2}{2(1+e_1)} \\ -\frac{a_2 n_1}{2(1+e_2)} & 1 - \frac{a_2(1+2e_2+n_2)}{2(1+e_2)} \end{pmatrix}$$

In this case, it is possible to show that condition (iii) in (26) is always satisfied, and the same holds for condition (ii) whenever $e_i > -\frac{1}{2}$. Assuming $e_i > -\frac{1}{2}$, condition (i) in (26) is fulfilled, and so the inner equilibrium is stable, provided that $a_i \in \left(0, \frac{4(1+e_i)}{1+2e_i+n_1+n_2}\right)$, thus proving 1; from this and from proposition 6, parts 2 and 3, it is immediate to show parts 2 and 3 of this proposition.□

Propositions 6 and 7 are the analytical basis to explain the bifurcations which are responsible of the sudden changes observed in the numerical experiments: if a variation of one or more parameters causes a displacement of the equilibrium point (or of a periodic point of a cycle) into different regions by crossing a border (e.g. l_2) the stability properties of the equilibrium (or of the periodic cycle) suddenly change by a so-called border collision bifurcation.

For instance, an increment in the production constraint L_1 moves down the border l_2 and so a stable cycle in S_2 can cross the border l_2, where the equilibrium (17) is indeed unstable. As we have just shown, the stability properties of fixed points (or cyclic points) are different on opposite sides of these borders, and so border collision bifurcations lead to these jumps in the eigenvalues of the Jacobian matrix.

Instead of going in deeper analytical details, on the basis of these general theoretical arguments we prefer to illustrate some consequences of border collisions through numerical simulations.

Let us start with the following set of parameters: $a_1 = a_2 = 0.35$; $c_1 = c_2 = 6$; $A = 16$; $L_2 = 1$; $N = 16$; $n_1 = 10$; $e_1 = e_2 = -0.4$. Note that with these parameters (3) and (7) are satisfied, so that cost functions are strictly concave and always increasing and prices are nonnegative. Under these parameters the inner equilibrium (17) is $(q_1^*, q_2^*) = (0.617284, 0.617284)$: the two coordinates are equal as agents are homogeneous in the parameters that are relevant to calculate the inner equilibrium.

Now we consider a change in the capacity constraint of firms in group 1, $L_1 \in [0,1]$.

For $L_1 = 0$, region $S_1 \cap \overline{S}_2 = \emptyset$, and for L_1 sufficiently low, firms in group 1 continue to play their capacity constraint L_1; notice that the border l_2 is shifted below as L_1 is increased. Consequently, firms in group 2, given the low production in the whole industry due to the tight capacity constraint of their competitors in group 1, play their capacity constraint $L_2 = 1$: the equilibrium (L_1, L_2), which is in $S_2 \cap S_4$, is played by all the firms and it is always stable, as shown before.

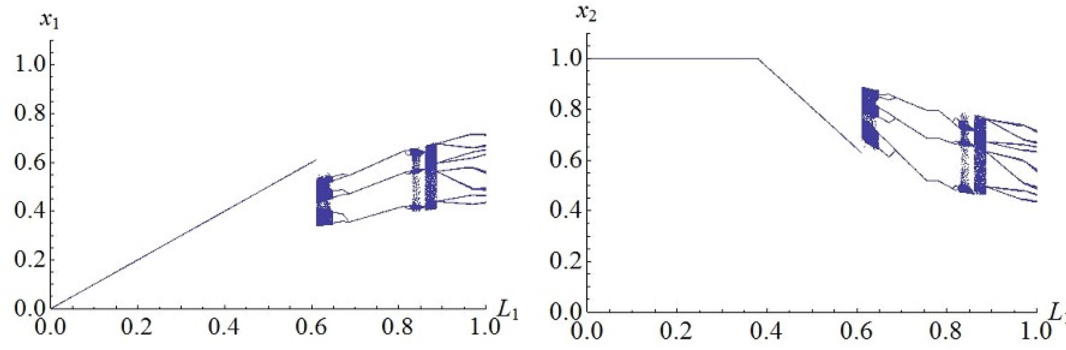

Figure 8: Another ”period one to chaos” border collision bifurcation. Parameters are given as follows: $a_1 = a_2 = 0.35$; $c_1 = c_2 = 6$; $A = 16$; $L_2 = 1$; $N = 16$; $n_1 = 10$; $e_1 = e_2 = -0.4$ and $L_1 \in [0,1]$ as bifurcation parameter.

As the constraint L_1 is relaxed and firms in group 1 can increase their productions, the equilibrium (L_1, L_2) crosses the border l_4 at $L_1 = 0.38$: firms in group 2 start to best reply to L_1, according to (6), and the asymptotic behavior of the model is the convergence to a fixed point of the form $\left(L_1, \frac{A-c_2-n_1L_1}{1+n_2+2e_2}\right)$, see the decreasing branch in the bifurcation diagram for x_2 in fig.8b. As we established analytically before, this new equilibrium remains stable by crossing the border l_4, i.e. in the crossing between region $S_2 \cap S_4$ and $S_2 \cap \overline{S}_4$. Indeed, for such a speed of adjustment a_2, an equilibrium in $S_2 \cap \overline{S}_4$ is stable, but it would be unstable in $\overline{S}_2 \cap \overline{S}_4$, as $a_2 = 0.35 \in (0.148148, 0.387097) = \left(\frac{4(1+e_2)}{1+2e_2+n_1+n_2}, \frac{4(1+e_2)}{1+2e_2+n_2}\right)$. Therefore it is the crossing of border l_2 that causes a border collision bifurcation.

In fact, when $L_1 = 0.617284$, a border collision bifurcation occurs between (17) and l_2: the inner equilibrium (17) enters region I, i.e. becomes feasible, as firms in group 1 can actually produce the ”unconstrained” equilibrium quantity. However, as already mentioned, for such a speed of adjustment this equilibrium is indeed unstable and so convergence to it is not achieved; instead we observe the sudden convergence to a chaotic attractor, which become larger as L_1 is further increased, see fig. 8a,b. We stress that in general it is not easy to predict the kind of dynamic after the border crossing, as it depends on the global properties of the map, which depend on all borders in the phase space.

Another interesting situation is shown in the bifurcation diagram of fig. 9a, obtained with parameter values $A = 16$; $N = 16$, $n_1 = 10$, $a_1 = a_2 = 0.5$; $c_1 = c_2 = 6$; $e_1 = e_2 = 0.5$, $L_2 = 2.5$, and bifurcation parameter $L_1 \in [0, 1.5]$. It is easy to compute the value of L_1 at which the contact between the equilibrium $E = \left(L_1, \widetilde{R}_2(L_1)\right)$

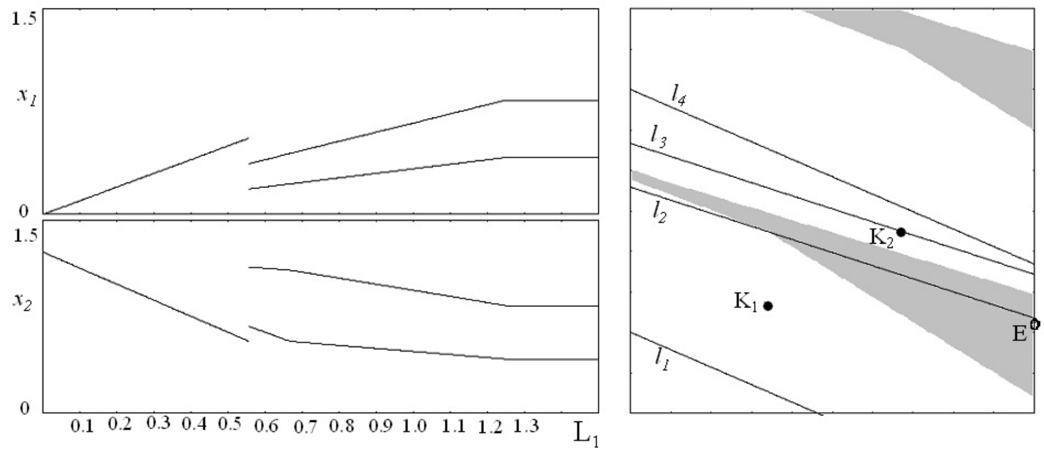

Figure 9: (a) A "period one to period two" border collision bifurcation; Parameters are given as follows: $A = 16$; $N = 16$, $n_1 = 10$, $a_1 = a_2 = 0.5$; $c_1 = c_2 = 6$; $e_1 = e_2 = 0.5$, $L_2 = 2.5$, $L_1 \in [0, 1.5]$; (b) coexistence of attractors with $L_1 = 0.54$: the grey and white regions are, respectively, the basins of attraction of the Nash equilibrium E and the stable 2 cycle K_1 and K_2.

and the border line l_2 occurs, whose crossing, as stated above, causes a the loss of stability of the equilibrium E. In fact, from the condition of border collision $E \in l_2$ we get $L_{1BC} = 5/9 = 0.\overline{5}$. Indeed, from the bifurcation diagram we can clearly see that, at this bifurcation value, a sudden transition between a stable equilibrium value and a stable cycle of period 2 and of finite amplitude suddenly occurs. However, in this case the discontinuous jump between two attractors, located in quite different regions, is related to a situation of coexistence of the two attractors before the occurrence of the border collision. This can be seen in fig. 9b, obtained with the same set of parameters as in the bifurcation diagram of fig. 9a and $L_1 = 0.54$. In the picture, the Nash equilibrium E is shown together with its basin of attraction represented by the grey region, together with the stable cycle of period 2, denoted by the periodic points K_1 and K_2, whose basin of attraction is given by the white region. As the production constraint L_1 is slightly increased beyond the border collision bifurcation value, the equilibrium E becomes unstable and the cycle of period 2 remains the only global attractor. As L_1 is further increased, some non smooth changes of the position of the stable periodic points can be seen, at $L_1 \simeq 0.63$ and $L_1 \simeq 1.25$, due to the crossing of the upper periodic point with the lines l_1 and l_3 respectively, however these border collisions do not cause changes of stability.

Many other different dynamic situations and bifurcations can be investigated, and the effects of variations of different bifurcation parameters can be ana-

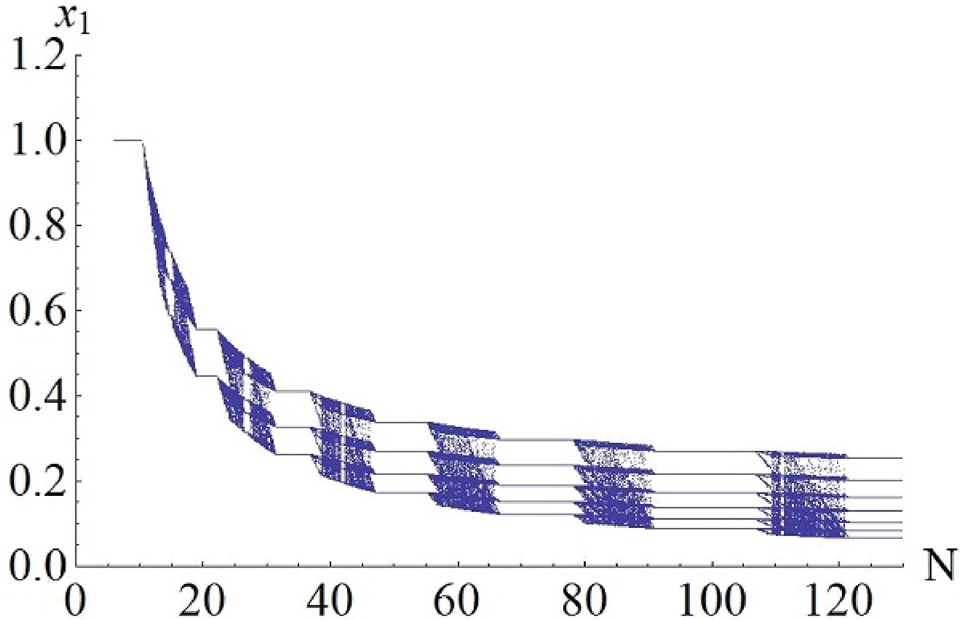

Figure 10: A "Period increment" sequence of quantities produced by agents in the first group. Parameters are given as follows: $A = 16$, $n_1 = 10$, $a_1 = a_2 = 0.2$, $c_1 = c_2 = 6$, $e_1 = e_2 = -0.7$, $L_1 = L_2 = 1$, and the number of firms $N \in [10, 130]$ as bifurcation parameter.

lyzed, leading to different kinds of attractors as well as coexistence among them, with complicated structures of the basins of attraction. As the Cournot game considered here is represented by a linear dynamical system, whenever upper and lower production constraints are neglected, we can say that all these complex dynamic scenarios are directly or indirectly induced by the presence of such constraints. For example, even the complex structure of the bifurcation diagram shown in fig. 10, obtained with parameters $A = 16$, $n_1 = 10$, $a_1 = a_2 = 0.2$, $c_1 = c_2 = 6$, $e_1 = e_2 = -0.7$, $L_1 = L_2 = 1$, and the number of firms $N \in [10, 130]$ taken as a bifurcation parameter, is a consequence of the presence of constraints. In fact, the structure known as "period increment" sequence shown in this picture is typical of non-smooth dynamical system, because the creation of stable cycles with periods that increase in arithmetic progression is due to sequences of border collision bifurcation involving contacts between switching manifolds and periodic points (see e.g. Avrutin and Schanz (2006), Gardini and Tramontana (2010), Gardini, Tramontana, Avrutin, and Schanz (2010)).

5 Conclusions

In this paper, we considered a standard Cournot oligopoly model, with linear demand and quadratic costs, when production constraints are added. The dynamic adjustment, based on quite standard assumptions of naive expectations and an adaptive adjustment with inertia (or anchoring attitude), exhibits simple (and well known) properties if maximum production constraints are neglected, as commonly assumed in the literature. Instead, if such constraints are imposed, interesting dynamic phenomena occur, which are caused by the fact that the dynamical system becomes non smooth, giving rise to global bifurcations known as ”border collisions”, which have recently become a focus topic in the literature on applied dynamical systems.

We have shown how the reduction of production capacity of one or both groups of firms can induce important changes in the kind of attractors that characterize the long-run dynamics of the system as well as in the structure of the basins of attraction when the positive Nash equilibrium is unstable and cyclic dynamics coexist with monopoly equilibria, occurring when one of the groups of firms is pushed out of the market. In the particular case of linear demand and linear cost, we showed that the presence of constraints, together with increasing number of identical firms, can give rise to bounded chaotic behaviors, i.e. an alternative route to complexity with respect to the one marked by the introduction of nonlinearities in the demand or cost functions.

Moreover, this paper has emphasized the effects of binding upper production constraints on the existence and stability of Nash equilibria of the oligopoly game, as well as the non standard bifurcation routes leading to the creation of complex attractors related to the occurrence of border collision bifurcations induced by the gradual variations of production constraints. As the presence of lower and upper constraints (floor and ceiling) is quite important in the dynamic modelling of economic systems, a systematic study of the dynamic scenarios induced by such constraints, by using the methods recently developed in the emerging literature on non smooth dynamical systems, may become an important issue in the economic dynamic literature as well. However, we neglected here other important properties of the model at hand, which may be studied in the future. For example, the role of the speed of adjustment in the adaptive adjustment process, or equivalently the role of inertia of the players in the adoption of the computed best reply, is worth to be investigated, especially in connection with the possibility of considering an increasing number of firms as a bifurcation parameter in the model. Indeed, as often stressed in the literature on oligopoly games, the increase of the number of firms has a destabilizing effects, whereas increasing the inertia, i.e. decreasing the speed of adjustment towards the best reply with naive expectations, has a stabilizing role. So, it is intriguing to analyze these two opposite effects when firms have production constraints and, consequently, the quite new property of non-smoothness of the corresponding dynamical system; of course, in these cases, it is necessary to adopt non-standard methods for studying the stability properties of non differentiable dynamical systems. We recall that very low speeds of adjustment always lead to stability, as the dynamical system becomes a strongly contractive one, whereas high speeds of adjustments (i.e. close to one), corresponding to best reply dynamics with naive expectations, can only have simple attracting sets, such as fixed points and cycles of period two, in a piecewise linear dynamical systems. So, the most interesting and rich dynamic situations can be obtained for intermediate values of the speeds of adjustment. Another interesting investigation of the model proposed in this paper is related to the coefficients of the quadratic cost functions, whose variations in magnitude and sign (related to convexity properties of the cost functions) open up to a wide spectrum of dynamic scenarios when these variations are associated with binding production constraints.

Finally, the role of constraints as ”folding manifolds”, similar to critical curves obtained from the curves of vanishing Jacobian in continuously differentiable maps, is worth to be further analyzed, as they can be employed to get an estimate of trapping regions, i.e. regions of the phase space where the asymptotic dynamics of the economic system are ultimately bounded. These methods can be usefully tested by studying oligopoly models like the ”vintage” one of this paper, allowing to get an intuitive understanding behind the analytical and numerical results based on these new mathematical techniques.

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