Sliding and oscillations in fisheries with on–off harvesting and different switching times

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Abstract

In this paper, we propose a fishery model with a discontinuous on–off harvesting policy, based on a very simple and well known rule: stop fishing when the resource is too scarce, i.e. whenever fish biomass is lower than a given threshold. The dynamics of the one-dimensional continuous time model, represented by a discontinuous piecewise-smooth ordinary differential equation, converges to the Schaefer equilibrium or to the threshold through a sliding process. We also consider the model with discrete time impulsive on–off switching that shows oscillations around the threshold value. Finally, a discrete-time version of the model is considered, where on–off harvesting switchings are decided with the same discrete time scale of non overlapping reproduction seasons of the harvested fish species. In this case the border collision bifurcations leading to the creations and destruction of periodic oscillations of the fish biomass are studied.

1. Introduction

The exploitation of unregulated open access resources, such as fisheries, is characterized by a typical prisoner dilemma, often denoted as the ‘tragedy of the commons’ after [1]. In fact, free entry and individual profit maximization eventually lead the stock to very low levels, such as the open access equilibrium (see [2]). Thus, the main consequences are overfishing and fleets with overcapacity, leading to biological and economic inefficiencies, i.e. low levels of both fish and profits. Therefore, central institutions (fisheries management agencies) usually enforce various forms of regulation, ranging from setting harvesting restrictions (limiting the level of fishing effort, setting a total annual catch quota (TAC), establishing vessels buy-back programs, etc.), to imposing taxes on catches or limiting the kinds of species to be caught or the regions where exploitation is allowed (see e.g. [2–4]).

Over the last 50 years advanced mathematical bioeconomic models have been proposed to regulate the sustainable management of fisheries, assuming that central authorities solve optimal control problems to take into account biological, economic and social constraints. In practice, the real management of fisheries is achieved by trying to stabilize the biomass around a target level $B$, typically the Maximum Sustainable Yield (MSY). In 2002, during the Earth Summit, the EU member states committed themselves to “maintain or restore stocks to levels that produce the MSY with the aim of achieving these goals for depleted stocks on an urgent basis and where possible not later than 2015”.1 In other cases, the target level has been identified as the more conservative Maximum Economic Yield (MEY), as reported in [5]. The estimate of a target level is usually

1 See the Johannesburg Plan of Implementation, article 31a at http://www.johannesburgsummit.org/html/documents/summit_docs/2309_planfinal.htm, last accessed on December the 28th, 2012.

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based on a mathematical model for the single population under consideration (for instance employing the Schaefer model [6]) and depends on the regulator’s goals and preferences.

This idea of stabilizing the stock around a target level is supported by the theory. In fact, as established by Clark and Munro in [7], if a regulator tries to maximize the present value of economic benefits derived from fishing, the resulting optimal control is a bang-bang policy to steer the system at the desired target: harvesting effort should be at a maximum level if the resource is above the target level, whereas no effort must be exerted if the resource is below the target (see Appendix A for details). However, the target level is greatly influenced by economic parameters (especially by the regulator’s time preference). To make things more complicated, these resources are often shared among several countries (for instance in the case of transboundary stocks) and so the definition of a proper target is taken by several decision makers. Once the kind of target is decided, it is extremely difficult to correctly assess it for single fish populations, as stock dynamics depend on the whole ecosystem and on environmental factors as well (see on this point [8]). Moreover, a bang-bang rule is hardly accepted by fishermen, especially if they have overcapacity and because of the strategic interaction among exploiters (the ‘tragedy of the commons’). In any case there is the problem of preventing bycatches, which endanger the whole ecosystem. Not surprisingly, most fisheries managed to achieve the MSY level are overexploited.

Simple adaptive and self-regulating rules of thumbs are quite often employed in fishery management. One such rule is the so-called ‘Management by reference points’, commonly applied in many North American fisheries (see [9] for an overview of the method and its principal drawbacks). For instance, the 40–10 harvest control rule (or the analogous 25–5 rule) of the Pacific Fisheries Management Council imposes constant harvesting when biomass is above 40% of the virgin stock size; when biomass is between 10% and 40% fishing effort must be progressively reduced and any harvesting activity must be stopped when biomass is below 10%.

In this paper we address a simplified version of this rule, suggested by Clark in [10] to mimic real-world fishery management. This rule states that harvesting effort \( E \) can be proportional to fish biomass if the latter is above a threshold \( B > 0 \); however, total catch must be drastically reduced (for example to zero) whenever biomass falls below the threshold \( B \) (see also [11,12]). Considering the severe overexploitation in fisheries, this model can be considered as a good approximation of a management by reference points (given that the highest reference point is hardly crossed). Clark suggests that this threshold typically is less than or equal to the regulator’s target biomass level (the MSY, the MEY or another target, see Appendix A for details). This threshold policy clearly introduces a discontinuity in the harvesting function, so that the dynamic model that represents this kind of fishery becomes piecewise continuous, as the threshold \( B \) separates the state space into two adjacent regions, the one where harvesting is allowed from that where it is forbidden (or strongly reduced).

The aim of this paper is to explore the main consequences of this specific on–off harvesting control for the resource dynamics and to show the route of complexity in the model because of this apparently simple rule. In particular, we make various assumptions on the time scale of natural growth and harvesting as well as on the time when the regulator assesses if the threshold \( B \) has been crossed or not. In this way, we can study various specifications of the model, to carry out an analysis which is more consistent with the real problem at hand. In the case of a continuous time scale, this kind of dynamical system may give rise to the so-called sliding motion-stabilization by means of a very rapid switching between the application and the interruption of the harvesting activity, leading to a fast convergence towards the threshold \( B \). However, even if the long run behavior of such a system may appear to be a good one, as it is fully controllable by the regulator, it is quite unrealistic because it implies that an infinity of actions takes place in a finite amount of time, a property denoted as “Zeno” (see e.g. [13]); hence it is violated the fundamental requirement that alternating decisions of activating and suppressing harvesting cannot be infinitely fast. Indeed, harvesting decisions cannot be continuously revised, and are taken at discrete times, with a minimum time interval, say \( \Delta t \), between two successive switchings. This leads to the second model under consideration, which is an hybrid dynamical system, as growth of fish species is modeled in continuous time whereas policy decisions occur at discrete time pulses, and causes oscillations around the threshold value \( B \), with amplitude that depends on effort jump \( \Delta E \) and switching time \( \Delta t \). The study of hybrid and piecewise continuous dynamical systems is not easy, and is mainly performed numerically.

The third formulation of the model is in discrete time. As a matter of fact, population dynamics models are often formulated in such a way (see for instance [14,15]). Therefore, if we assume discrete time growth of fish population, i.e. seasonal and non overlapping birth rates, together with harvesting activities and control with the same discrete time scale, the dynamics of the system can be modeled through the iteration of a discontinuous one-dimensional map. For this model, we obtain some analytical results and characterize in a more complete way the periodicity that arises from the above mentioned threshold policy. In this case the creation of periodic cycles around \( B \) can be fully characterized by the study of border collision bifurcations (see e.g. [16–18]) and periodicity regions in the space of control parameters \( B \) and \( \Delta E \) can be obtained.

So, the model proposed in this paper to analyze a very simple (even if frequently used in practice) on–off fishery regulatory policy gives us the opportunity to explore some interesting mathematical properties of discontinuous dynamical systems under different time scales, from continuous to discrete through the hybrid model as an intermediate case. This mathematical exercise allows us to explore analytically and numerically some dynamical properties of the model and to
use some analytic and geometric methods for the study of discontinuous dynamical systems that are at the center of a recent and fast developing literature, see e.g. [19,20].

The plan of the paper is as follows. Section 2 deals with the case of continuous time growth and on–off harvesting. In Section 3, the assumption that the regulator can assess at any instant of time if the no-harvesting barrier is crossed or not is replaced by the more realistic assumption that this can happen only at fixed time periods. In Section 4 the same model in discrete time is formulated and analyzed. Section 5 concludes.

2. Continuous time growth and on–off harvesting

Following Clark [10], let us consider the simplest and most employed model of commercial fishery, commonly called the Schaefer Model (after Schaefer [6] who applied it to various fisheries), describing a fish population which is assumed to grow according to a logistic law and is harvested with constant effort with a catch-per-unit-effort hypothesis. In this model the uptake is proportional to the stock level and the stock dynamics is given by

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - qEX. \quad (1)$$

where $X(t)$ represents the fish stock at time $t$, $r$ is the intrinsic growth rate and $K$ the carrying capacity of the fish population in the environment considered, $E$ is the fishing effort (a measure of number of vessels, fishing hours per day, etc.) and $q$, called catchability coefficient, is a proportionality constant whose value depends on the efficiency (i.e. technological sophistication used). All these parameters are positive, and the aggregate parameter $qE$ is often referred to as fishing mortality.

It is straightforward to see that in the absence of harvesting, i.e. $E = 0$, the carrying capacity $K$ is the only positive (and stable) equilibrium of the fish population. Without loss of generality, this natural equilibrium value $K$ can be taken as the unit to measure fish stock, so we introduce the new adimensional variable $x = X/K$ and the dynamic model (1) becomes

$$\dot{x} = f(x) = G(x) - qEx = rx(1 - x) - qEx \quad (2)$$

where $G(x)$ is the natural rate of growth and the dot, as customary in the notation of dynamical systems, denotes the time derivative. This model has a unique nonzero equilibrium

$$x^* = 1 - \frac{qE}{r} \quad (3)$$

which is positive (and stable) provided that the fishing mortality does not exceed the intrinsic growth rate, i.e. $qE < r$. The extinction equilibrium $x^0 = 0$ is unstable whenever $x^*$ is positive, as $f'(0) = r - qE > 0$ for $qE < r$, and becomes stable, thus leading to extinction, when the complementary inequality holds (a typical transcritical, or stability exchange, local bifurcation). However, danger of extinction (or severe depletion) even occurs at values of $qE$ below $r$ due to intrinsic uncertainty in the control of $qE$ or the evaluation of $r$, as it may be influenced by exogenous factors such as pollution, predators, climate, etc. So, more prudent control policies are needed. One of the simplest (see e.g. [10]) imposes that fish cannot be harvested whenever the stock becomes too scarce, i.e. the stock goes below a given threshold $0 < B < 1$. In other words, the on–off harvesting policy prescribes to stop fishing whenever $x(t) \leq B$ and to allow it whenever $x(t) > B$. The inclusion of this feedback control in the Schaefer model (2) leads to the following nonlinear piecewise-continuous differential equation

$$x = \begin{cases} fL(x) = rx(1 - x) & \text{if } x(t) \leq B \\ fR(x) = rx(1 - x) - qEx & \text{if } x(t) > B \end{cases} \quad (4)$$

Of course, the value of the threshold $B$ is important in the policy adopted, as a value too small may be dangerous for the fish population survival, and a too high value may be economically and socially unsuitable. As a reference value, one may consider the fish stock $x_{MSY}$, corresponding with the Maximum Sustainable Yield (MSY), i.e. the equilibrium fish stock at which the catch rate is maximum. This value can be easily obtained by replacing the equilibrium (3) in the catch rate (or harvesting) $C = qEx$, thus getting

$$C = \begin{cases} qE(1 - \frac{qE}{r}) & \text{if } Eq < r \\ 0 & \text{if } Eq \geq r \end{cases} \quad (5)$$

Total catch attains its maximum for $E_{MSY} = \frac{c}{2}$, from which $C_{MSY} = \frac{c}{4}$ and $x_{MSY} = 1 - \frac{qE_{MSY}}{r} = \frac{1}{2}$. So, according to [10], we may guess that the threshold value $B$ may be placed around $\frac{1}{4}$. However, a more prudent attitude may suggest a value $\frac{1}{3} < B < 1$, for instance the Maximum Economic Yield (MEY) where the stock is $x_{MEY} = \frac{1}{3} + \frac{c}{6r}$. Of course, a less prudent attitude implies that $0 < B < \frac{1}{2}$.

4 This is the case for the above mentioned 40–10 or 25–5 rules of the Pacific Fisheries Management Council.
The ordinary differential Eq. (4) presents a point of discontinuity, so existence and uniqueness of solutions are not guaranteed even if an analytic solution can be easily obtained (the well known logistic function) when the motion is confined to the higher left branch \( f_L \) or the lower right branch \( f_R \). However, the global asymptotic behavior can be qualitatively explained from the study of the sign of the time derivative \( x \) for \( x > 0 \). If the discontinuity point \( B \) is such that \( 0 < B < 1 - \frac{q_E}{r} \), then \( x = 1 - \frac{B}{q_E} \) is still a globally stable equilibrium. In fact, in this case \( x^* \) belongs to the lower right branch, which changes its sign in it. So, the qualitative asymptotic dynamics is not affected by the presence of the on–off policy. It is plain that the robustness of the equilibrium is enhanced because when the fish stock decreases too much for any reason, the presence of the no-harvesting branch \( f_L \) prevents the fish stock from going extinct, as the extinction equilibrium \( x^* = 0 \) is always unstable being \( f_L(0) = r > 0 \). However, as stressed by Clark in [10], such a fishery control policy may be not so effective, especially if the threshold \( B \) is small, due to the difficulties in the estimation of current fish stock in order to decide if harvesting must be stopped (Clark writes “fish in the sea are hard to count”) as well as the inevitable delays between observation and policy application. Indeed, catch per unit effort \( C/E = qx \) is used as a proxy (i.e. an approximate indicator) of the current fish stock. So a prudent attitude suggests to increase the threshold value. So, let us assume that \( 1 - \frac{q_E}{r} < B < 1 \). In this case \( x^* = 1 - \frac{B}{q_E} \) is no longer an equilibrium point, as it belongs to the region where the upper left branch governs the dynamics, so that \( f_L(x^*) > 0 \), as indeed \( f_L(x) > 0 \) for each \( x < B \). Instead, on the right of the discontinuity point \( B \) we have \( f_R(x) < 0 \) for each \( x > B \). So, whenever the fish stock \( x(t) < B \) it increases, i.e. it moves to the right, whereas when \( x(t) > B \) it decreases, i.e. it moves to the left. The overall result is a global convergence towards the threshold point \( B \). However, this convergence occurs in a quite peculiar and unrealistic way. In fact, due to the discontinuity point associated with finite values of opposite signs of the time derivative \( x \) (a situation often denoted as Filippov’s system, after Filippov’s book [21]), namely positive on a left neighborhood of the discontinuity point and negative on a right one, a typical sliding motion is obtained (see [21,22]). The latter is characterized by on–off harvesting effort switches at high frequency, which induce a so fast oscillatory behavior that the density of the exploited population remains practically constant at the threshold value (see Fig. 1).

In other words, an infinity of on–off decisions takes place in a finite amount of time, a property denoted as “Zeno” (see on the point [13]). This violates a fundamental requirement that alternating activation and suppression decisions cannot be infinitely fast. Instead, in real systems harvesting decisions cannot be continuously revised, and are taken at discrete times, with a minimum time interval.

3. Continuous time growth with discrete time on–off switches

In this section we consider the more realistic case in which the on–off harvesting decisions cannot be continuously revised, i.e. on–off harvesting switches cannot occur at any time instant, but at discrete time intervals; in other words, a minimum time interval, say \( \Delta s \), must separate two successive switches. The dynamic model (4) becomes

\[
x = \begin{cases} 
    f_L(x) = rx(1-x) & \text{if } x(\lfloor \frac{t}{\tau} \rfloor) \leq B \\
    f_R(x) = rx(1-x) - qEx & \text{if } x(\lfloor \frac{t}{\tau} \rfloor) > B
\end{cases}
\]

(5)

where \( \lfloor \tau \rfloor \) is the largest integer not greater than \( \tau \) (i.e. the integer part, or floor, of \( \tau \)). So, the dynamic variable \( x \) evolves in continuous time, whereas the switches between the two possible fishing strategies (no harvesting and constant effort harvesting), or equivalently between the two branches \( f_L \) and \( f_R \) separated by the discontinuity threshold \( B \), occur in discrete time, at time periods of length \( \Delta s \). Such a hybrid dynamic model is quite difficult to be studied analytically (see e.g. [23–25]), so we perform some numerical simulations to show the qualitative effects of the harvesting control occurring at discrete time. Fig. 2 shows the dynamics of the hybrid model (5) with \( \Delta s = 1 \) and the same set of parameters \( r \) and \( qE \) as in Fig. 1. The trajectory in the upper panel is obtained with threshold parameter \( B = 0.4 > x^* = 1/3 \). In this case persistent oscillation around \( B \) are clearly visible, and the same can be seen in the middle panel where \( B = 0.6 \). In both cases, the...
oscillations around the threshold $B$ and the inversions of the motion occurring at any switching time are clearly visible. Instead, the trajectory shown in the bottom panel, obtained with $B = 0.3 < x^*$, exhibits convergence to the equilibrium point in the long run, according to the dynamics imposed by $f_R$, even if the initial condition $x(0) = 0.2$ belongs to the region where harvesting is forbidden.

Fig. 3 depicts a trajectory of the hybrid model (5) with the smaller time interval between two successive decisions $\Delta s = 0.3$, and the same set of parameters as before. As expected, the oscillations now become more frequent and with smaller amplitude. Of course, in the limiting case $\Delta s \to 0$ the model in continuous time (4) is obtained, i.e. as $\Delta s$ decreases the oscillations assume higher and higher frequency, as well as smaller and smaller amplitude, thus converging to the Zeno process described in the previous section.

**Fig. 2.** For the hybrid model (5) with $\Delta s = 1$ and the same set of parameters $r$ and $qE$ as those used in Fig. 1, three trajectories are shown with different values of the threshold parameter $B$: $B = 0.4, B = 0.6$ (both greater than $x^*$) and $B = 0.3 < x^*$.

**Fig. 3.** A trajectory of the hybrid model (5) with smaller $\Delta s = 0.3$ and the same set of parameters of the previous examples: $r = 1.5, qE = 1, B = 0.4$. 


4. Discrete time growth and on–off harvesting

In this section we consider a discrete time version of the Schaefer model (2). In this case the fish population reproduces seasonally with non overlapping generations, see e.g. [26,27,14,15]. Under this assumption, the evolution of the resource is described by the map:

\[ x(t + 1) = f(x(t)) = x(t) + rx(t)[1 - x(t)] - qEx(t) \]  

(6)

Notice that the steady states (or fixed points) of (6) are the same as in the corresponding continuous time model (2), namely \( x' = 0 \) and \( x' = 1 - \frac{q}{r} \). However, the non-negativity of the trajectories starting from positive initial conditions is no longer guaranteed. In fact, in order to have non negative (and bounded) trajectories we have to impose \( x(0) \in [0, \frac{1}{r - qE}] \) and \( r - qE < 3 \). Moreover, the positive equilibrium \( x' \) is stable for \( r - qE < 2 \). In fact, differently from the continuous time Schaefer model, in this case damped oscillations of the fish stock around \( x' \) can be obtained if the multiplier of the map (6) at the equilibrium is negative and less than one in modulus, i.e. \( -1 < f'(x') < 0 \), that is, \( 1 < r - qE < 2 \). The equilibrium becomes unstable for \( r - qE > 2 \), through a flip bifurcation occurring at \( r - qE = 2 \). As it is well known, if \( r - qE \) is further increased, a period-doubling route to chaos takes place until the boundary crisis occurring at \( r - qE = 3 \); after this point, the generic trajectory is unbounded.

In the following, we shall assume that the unharvested population equilibrium is stable, i.e. \( r < 2 \) (as remarked in [14] chaos is not easily observed in nature with unexploited populations).

If we also assume that on–off harvesting decisions are taken at discrete time periods with the same time scale, i.e. \( \Delta s = 1 \), the time evolution of the controlled system can be obtained by the iteration of the following piecewise continuous map

\[ x(t + 1) = \begin{cases} f_L(x(t)) = x(t) + rx(t)[1 - x(t)] & \text{if } x(t) \leq B \\ f_R(x(t)) = \max(0, x(t) + rx(t)[1 - x(t)] - qEx(t)) & \text{if } x(t) > B \end{cases} \]  

(7)

where a constraint is imposed in the right branch in order to guarantee the non-negativity of trajectories starting with \( x(0) \in [0, \frac{1}{r - qE}] \), like in the unharvested logistic model.

In the following we study the dynamic behavior and the main properties of the model (7). In particular we are interested in the long run evolution of the fish stock by fixing \( r \) and varying the “policy parameters” \( B \) and \( qE \). Indeed, several different asymptotic dynamic scenarios can be observed. A typical example is shown in Fig. 4, where \( r = 0.8 \), \( qE = 1.1 \), the threshold \( B = 0.4 \) and the initial condition is \( x(0) = 0.1 \). As it can be seen in the staircase diagram on the left, after a transient oscillatory motion, the asymptotic dynamics settle on a periodic cycle of period three (see also the same trajectory represented versus time in the right panel of Fig. 4).

It is easy to realize that by a proper tuning of the model’s parameters, i.e. by shifting the discontinuity point \( B \) as well as the shape of the left and right parabolic branches, regulated by the parameters \( r \) and \( qE \), different stable periodic cycles and even chaotic trajectories can be obtained. However, the bifurcations at which these cycles are created or destroyed are not the usual and well known local bifurcations occurring in smooth maps, as qualitative changes of piecewise continuous maps are often caused by global (or contact) bifurcations known as border collision bifurcations (after [16]), henceforth abbreviated as BCBs. For piecewise continuous maps, new developments concerning such bifurcations have been recently obtained, even if they cannot be considered standard results yet (see e.g. [19,28,29]).

In order to explain the basic mechanism that gives rise to periodic motions like the one shown in Fig. 4, let us focus on a case in which, without harvesting, the fish stock is at the stable positive fixed point \( x^* = 1 \). For expository purpose, we consider a fixed value of the biologic parameter \( r = 0.8 \) and we analyze the qualitative changes occurring as the “control parameters” \( qE \) and \( B \) are let to vary. Of course, only values of \( B < 1 \) are interesting because otherwise the fixed point attracts all the orbits.

Fig. 4. Left panel: The staircase diagram of the trajectory of the discrete-time model (7) with parameters \( r = 0.8 \), \( qE = 1.1 \), \( B = 0.4 \) and initial condition \( x(0) = 0.1 \). Right panel: The same trajectory is represented versus time.
In Fig. 5, a global overview of the asymptotic dynamic properties of the model (7) is summarized by a two-dimensional bifurcation diagram in the parameters’ space \((B, qE)\), where the different colors represent different periods of the stable cycles reached by the trajectory of (7) with initial condition \(x(0) = 0\).

Let us start by considering the blue area at the bottom of the bifurcation diagram. Here the blue color indicates points in the parameter space where the generic trajectory of (7) converges to a fixed point of the right branch of the map, i.e. the branch with harvesting activity. Recall that the fixed point of \(f_L(x(t))\) (besides the null fixed point) is \(x^* = 1 - qE/r\). When \(x^* > 0\), the fixed point is also locally stable in the range of parameters considered, as we assumed that \(r < 2\). Thus, if the threshold value \(B < x^*\), then \(x^*\) attracts the trajectories that eventually converge to \(x^*\), going from the left branch to the right one. So the blue region must be bounded by the curve in which the values of the parameters are such that \(B = x^*\), that is:

\[
\Phi_k : qE = r(1 - B) \quad (8)
\]

For the fishery model this means that the higher the threshold parameter \(B\) is chosen, the lower the fishing mortality \(qE\) must be in order to ensure convergence to the steady state \(x^*\).

The grey region in the upper-right portion of Fig. 5 represents convergence to the fixed point \(x^0 = 0\). At a first sight it appears quite strange that trajectories may converge towards this fixed point, because positive values of \(r\) are considered, and consequently \(x^0\) is unstable. Nevertheless, the extinction equilibrium \(x^0\) can still be reached by some trajectories, even starting from a positive measure set of initial conditions, so that it can be an attractor in the Milnor’s sense (see [30]). In fact, even if \(x^0\) is unstable, it is possible that orbits coming from the left branch of the map go suddenly (in one iteration) to the extinction value because negative values of fish are ruled out (in (7) look at the “max” operator in \(f_L(x(t))\)). In other words, even if local stability does not hold in a topological sense, the extinction equilibrium may attract a set of initial conditions of positive measure. This is indeed the case whenever the jump at the discontinuity point between the left and the right branches is high enough to have the scenario represented in Fig. 6, which is obtained with \(B = 0.45\) and \(qE = 1.292\).

Cases like the one depicted in Fig. 6 share the common feature that the highest points of the left branch are mapped into \(X^0\) (i.e. to the extinction) in one iteration. So the border of this region in the two-parameters bifurcation diagram is obtained by setting the image of the highest point of the left branch equal to 0. The highest point of the left branch coincides with the threshold value \(B\) when it is lower than the maximum of \(f_L\), which is \(X^L = \frac{1}{2r}\) or it is the maximum itself whenever \(B > X^L\). So the border is given by:

\[
\Phi_0: \begin{cases}
  f_k \circ f_l(B) = 0 & \text{if } B < \frac{1}{2r} \\
  f_k \circ f_l(X^L) = 0 & \text{if } B > \frac{1}{2r}
\end{cases}
\]

\[
\Phi_k : \begin{cases}
  qE = 1 + r[1 - (B + rB(1 - B))] = qE_L^{(i)} & \text{if } B < \frac{1}{2r} \\
  qE = 1 + r\left[1 - \frac{(1 + r^2)}{4r}\right] = qE_L^{(ii)} & \text{if } B > \frac{1}{2r}
\end{cases}
\]

(9)

In Fig. 7, both \(\Phi_k\) and \(\Phi_0\) are drawn.

Let us consider now the region of the parameters’ plane \((B, qE)\) between \(\Phi_k\) and \(\Phi_0\). In this case trajectories do not convergence to any fixed point. We can prove the following proposition:
Proposition. If the parameters’ values are such that
\[ r \left( \frac{1}{C_0} B \right) < q_E < q_E^{(i)} \] for \( B < 1 + \frac{r^2}{r} \), then trajectories are bounded and oscillate between the two branches.

Proof. Let us first consider initial conditions belonging to the right branch of the map. Given that \( q_E > r(1-B) \), then \( f_R(B) < B \), i.e. the right branch is below the diagonal in the plane \((x(t), x(t+1))\) so each point is mapped in one iteration to the left branch of the map. So we only need to consider what happens to orbits starting from the left side. These orbits would converge towards \( x_M^L = 1 \), but this is a “ghost” (or virtual) fixed point as \( B < 1 \), so sooner or later the value of \( x(t) \) will be higher than \( B \). This means that these orbits will cross the threshold value \( B \), entering in the right region in which harvesting occurs. Then in one iteration they will come back to the left branch where harvesting is not allowed and the process will continue. The iterated point enters the trapping interval \( I = [I_m, I_M] \), bounded by the highest point of the left branch and the lowest value between its image and \( f_R(B) \), and never escapes from it. So, we have that:

\[
I_M = \begin{cases} 
   f_L(B) & \text{if } B < x_M^L \\
   f_L(x_M^L) & \text{if } B > x_M^L 
\end{cases}, \quad I_m = \min \{ f_R(I_M), f_R(B) \}
\]

\[ \square \]

Corollary. Two or more consecutive time periods in which harvesting occurs are not possible in this region of the parameters’ space.
This corollary follows from the fact that points of the right branch are mapped in one iteration to the left one. In order to analyze the dynamics in this region of the parameters’ plane, we identify four possible scenarios related to the slopes of $f_L$ and $f_R$ in a neighborhood of the discontinuity point $B$:

(i) **Increasing/increasing.** This happens when the value of the discontinuity point $B$ is lower than the maxima of both functions: $B < \min(x_M^L, x_M^R)$;

(ii) **Increasing/decreasing.** In this case $B$ is lower than the maximum of the left branch but higher than the maximum of the right one: $x_M^L < B < x_M^R$;

(iii) **Decreasing/increasing.** This case is opposite to the previous one and requires a discontinuity located after the maximum of the left branch and before the maximum of the right one: $x_M^L < B < x_M^R$;

(iv) **Decreasing/decreasing.** This scenario is characterized by a value of $B$ higher than both the maxima: $B > \max(x_M^L, x_M^R)$.

We can easily exclude scenario (iii). In fact we have that:

$$x_M^L = \frac{1 + r - qE}{2r} \leq x_M^L = \frac{1 + r}{2r} \text{ for } qE \geq 0$$

so it is impossible to satisfy the condition required by this case, i.e. to find a value higher than $x_M^L$ but lower than $x_M^R$.

Moreover, we are also not interested in scenario (iv) because we are considering a value of $r < 1$ and this case would imply $x_M^L > 1$. In other words, scenario (iv) requires a value of $B > 1$ so that all the orbits are attracted by the fixed point $x_I$.

The first two scenarios are analyzed in the following subsections by considering their similarities with the corresponding linear cases.

### 4.1. The increasing/increasing case

As we have just discussed, this scenario requires that we are in the portion of the parameters’ plane characterized by bounded trajectories with a value of the discontinuity $B$ such that $B < x_M^L$. So the curve:

$$\Phi_B : B = x_M^L = \frac{1 + r - qE}{2r}$$

is the right border of this region in the $(B, qE)$ parameters’ plane (the grey region in Fig. 8).

This case (increasing/increasing) has been studied by Keener [31], who shows that if the map is invertible in the trapping interval $I$, then only stable and coexisting cycles may occur. More recently, in [32] it is described how to obtain the BCB curves of the so-called period adding structure. Roughly speaking, we can define a period adding structure as a situation in which when we have two cycles of different periods $p$ and $q$ for different parameters’ values, there must also exist a region in the parameters’ plane in which a cycle of period $(p + q)$ exists, and this applies iteratively. Moreover, bistability cannot occur. Gar-

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5 Given that $x_M^L < x_M^L$, the condition $B < x_M^L$ is sufficient to have also $B < x_M^L$.

6 We stress that for a discontinuous map a BCB occurs when a periodic point collides with the discontinuity point.
dini et al. [29] and Avrutin et al. [19] studied the increasing/increasing linear case in which the BCB curves can all be analytically obtained. In our case the periodicities of the cycles corresponding to this region of the parameters’ plane (and one enlargement) are numerically calculated and shown in Fig. 9.

In the enlargement (Fig. 9b), it is possible to look at the adding structure. If we consider, for instance, the region of the 2-cycle and the region of the 3-cycle, we can see the 5-cycle region between them, and the 7-cycle region between the 2 and the 5-cycle regions, and so on. Even if in our case the map is made up by nonlinear branches, if the points of the cycles are close enough to the discontinuity or, even better, if the maximum of \( f_R \) is outside the trapping region (i.e. the slopes are increasing/increasing in the whole trapping interval), then the mechanism described above works. Moreover, when the decreasing branch of \( f_R \) starts playing a role, then something different can happen.

As a consequence of the nonlinearity of the branches of our map, we cannot analytically calculate the BCB curves, but it still possible to give their equations in implicit form. Let us consider the periodicity regions associated with cycles of the so-called first complexity level. According to the notation by Leonov, see [33, 34] and also adopted in Gardini et al. [29], these regions are characterized by cycles of period \( n + 1 \) with only one periodic point in one of the two regions and \( n \) points in the other one. A possible way for identifying cycles consists in using symbolic sequences according to the following rule: the letter \( L \) (respectively \( R \)) denotes a periodic point on the left (right) side of the discontinuity point \( B \). Cycles of first complexity level are characterized by the symbolic sequence \( LR \) or \( RL \) for any \( n \geq 1 \).

If we start considering the cycles with symbolic sequence \( LR \), it is possible to determine the unique periodic point on the left side of \( x = B \) by using the equation \( f_R \circ f_L (x) = x \). The implicit equations of the periodicity regions of the cycles are given by:

\[
f_R^n \circ f_L (B) = B; \quad f_R^{n-1} \circ f_L \circ f_R (B) = B
\]

In order to obtain the implicit equations of borders in the case of periodic cycles with symbolic sequence \( RL \), we only need the repeat the procedure with \( L \) instead of \( R \) and vice versa. These implicit equations are given by:

\[
f_L^n \circ f_R (B) = B; \quad f_L^{n-1} \circ f_R \circ f_L (B) = B
\]

In Fig. 10 we can seen the numerically calculated BCB curves corresponding the region of the increasing/increasing case.

We also know that between any pair of consecutive periodicity regions of the first complexity level, two infinite families of periodicity regions of second complexity level exist. The periods and rotation numbers of these cycles follow the Farey summation rule (see [35]). The process continues for further (an infinity of) complexity levels.

This means that in the upper side of the region we are considering, we can only find cycles with an higher number of points corresponding to time periods in which harvesting is allowed. The opposite is true for the lower side of the regions.

It is also important to note how different the effects of small changes in the policy parameters can be. For high values of the threshold \( B \), a small change in \( B \) or \( qE \) can at most cause a change of one in the periodicity (e.g. from 2 to 3-cycle or the opposite); however, the same change becomes more relevant if one starts from low values of \( B \).

4.2. The increasing/decreasing case

In the remaining portion of the parameters’ plane, the left branch of the map is increasing and the right one is decreasing. Note that in this case it is not necessary to specify “in a neighborhood of the discontinuity point” because the maxima of the two quadratic functions are not involved. As established before, in this case it is \( x_m^n < B < x_M^n \).

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7 That is above the 2-cycle region.
In this region we can apply the results by Gardini et al. [29], who describe the so-called period increment structure. With respect to the already seen period adding structure, here only an infinite sequence of cycles with increasing periods is possible (incremented by a fixed constant equal to 1 in this case). Differently from the period adding structure, now the coexistence of cycles is possible. In particular, bistability can occur. Fig. 11 shows the numerically calculated periodicity regions.

BCB curves can be obtained in implicit form by the same procedure discussed before for the previous case. In Fig. 12 the numerically obtained BCB curves for both the cases represented in Fig. 11 are drawn. In the grey regions of Fig. 12b there is coexistence of attractors. One such example, in which a 4-cycle coexists with a 5-cycle, is represented in the staircase diagram of Fig. 13.

Note that in these grey regions, it is possible that cycles exist but are locally unstable. This is what happens in the white area visible in Fig. 7a, just below the \( \Phi_0 \) curve. Therefore a certain number of periodic points are located in the decreasing branch of the map \( f_R \). This means that flip bifurcations and chaotic motion are possible for some combinations of parameters.

4.3. Mixed scenarios

Until now we have considered the two possible scenarios separately. However, by varying a parameter, it is possible to move from a scenario to another. For instance, let us consider the case obtained by fixing \( r = 0.8 \) and \( B = 0.65 \). The bifurcation diagram in Fig. 14 shows what happens by increasing the parameter \( qE \) from 0.1 to 1.5.

As we can see, for low values of \( qE \) the system converges to the fixed point \( x_C \). By increasing the bifurcation parameter, it enters the increasing/increasing region by crossing \( \Phi_0 \), so that periodic motion takes place. A further increasing in the value of \( qE \) leads the system inside the increasing/decreasing region, which contains a chaotic subregion. Finally, when \( \Phi_B \) is crossed, the fixed point of extinction is reached after a few iterations. We emphasize that the sudden transition from a cycle

![Fig. 10. Boundaries of periodicity regions of the first complexity level in the increasing/increasing scenario.](image1)

![Fig. 11. Left panel: Periodicities of the cycles for the increasing/decreasing scenario in the parameters’ space \((B,qE)\). Right panel: The enlargement shows some periodicity regions with period increment structure.](image2)
of low period (even a fixed point) to a more complicated (for example chaotic) attractor is a typical feature of BCB of piecewise smooth systems, like the one we are dealing with here.

5. Conclusions

In this paper, a piecewise continuous one-dimensional dynamic model has been proposed to mimic one of the simplest and quite naive policy control rules for fisheries. This on–off harvesting policy prescribes that harvesting must be stopped
whenever fish stock goes below a given threshold. Despite its simplicity (or thanks to that), this is one of the most commonly adopted policies for real world fisheries. In this paper we have investigated how different time scales, from continuous to discrete passing through the ‘hybrid’ intermediate case, can affect the dynamic behavior of the model. In particular, in the continuous time model, described by Clark in [10], the convergence to the Schaefer equilibrium or to the imposed threshold is always achieved; however the latter case is characterized by an unrealistic high-frequency of on–off switches that cannot be realized in practice. We then considered the case of a minimum interval between two successive switches, thus obtaining a hybrid model where biological growth in continuous time is associated with regulator’s policy decisions in discrete time. The occurrence of oscillations has been numerically shown in the more realistic cases. Finally, a discrete time model is proposed, based on the assumption that fish population reproduce seasonally with non overlapping generations, and on–off harvesting decisions are taken with the same time-interval. Even if this may appear a quite particular assumption, it allowed us to perform a detailed analytic and geometric study of the border collision bifurcations which constitute the basic mechanism for the creation/destruction of stable periodic cycles that characterize the asymptotic behavior of the fish stock. In particular, the results for the model in discrete time suggest that an on–off policy can be effective provided that the fishing mortality coefficient is sufficiently low, i.e. that overall harvesting is kept low. Otherwise the system undergoes periodic or chaotic motion without achieving the desired target and with several periods of no harvesting. In any case, our exercise shows that the use of reference points for real-world management can be problematic even from a theoretical point of view in addition to the criticisms expressed in [9].

The analysis has been possible by employing results recently obtained in the literature on piecewise continuous discrete dynamical system, which is a flourishing and promising stream of literature involving pure and applied mathematicians, as well as physicists, economists and social scientists dealing with the dynamic modeling of systems with constraints and bang-bang controls, see e.g. [13,17,18,36,37].

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Appendix A. Target biomass levels

In this Appendix, we review the principal target biomass levels employed in fisheries in addition to MSY. For further details we refer to [7,2,15].

**Maximum Economic Yield (MEY)**

The resource is owned by a single agent (or equivalently regulated by an authority), who selects an effort level to maximize profits

\[ \pi(E) = pC - cE = E(pqE - c) \]

assuming that biomass is in equilibrium. The harvested resource is sold in the market at a constant price \( p \) and the cost of effort is assumed linear. Thus, inserting the equilibrium condition (3) in (11), the required level of effort solves

\[ E_{MEY} = \arg \max_{E} \pi(E) = p \left( 1 - \frac{E_{Q}}{E} \right) qE - cE \]

whence \( E_{MEY} = \frac{1}{2} \left( \frac{E_{Q}}{q} + \frac{E}{q} \right) < E_{MSY} \) with resource at the level \( x_{MEY} = \frac{1}{2} + \frac{E_{Q}}{qE} > x_{MSY} \) and total catch \( C_{MEY} = \frac{1}{2} \left( 1 - \frac{E_{Q}}{pE} \right) < C_{MSY} \).

**Open access (OA) equilibrium**

In this case, the regulator does not establish any limit in the quantity of resource to harvest, i.e. the resource is ‘open access’. Consequently, an agent will exert fishing effort as long as profits (11) are positive (thus obtaining the situation called the ‘tragedy of the commons’). More precisely, the bionomic equilibrium is a biomass steady state characterized by the condition \( \pi = 0 \), which occurs in the model specification here considered at \( x_{OA} = \frac{E}{pE} < x_{MEY} \). From (3), total effort is

\[ E_{OA} = r \left( \frac{1}{2} - \frac{E_{Q}}{pE} \right) = 2E_{MEY} \]

with catches \( C_{OA} = r \left( \frac{1}{2} - \frac{E}{pE} \right) \).

**Biomass targets through dynamic optimization**

The regulator fixes an effort level from time to time in order to maximize the present value of profits derived by harvesting \( (\delta \in (0, +\infty)) \) is a fixed discount factor, i.e. the problem is to maximize the functional

\[ \int_{0}^{\infty} e^{-\delta t} E(t)(pqE(t) - c)dt \]

dynamics (2) and control \( E(t) \in [0, E_{MAX}] \). In [7] it is shown that this optimal control problem has a solution of the form

\[ E(t) = \begin{cases} E_{MAX} & \text{if } x(t) > x_{d} \\ E_{d} & \text{if } x(t) = x_{d} \\ 0 & \text{if } x(t) < x_{d} \end{cases} \]
where the biomass target \( x_d \) satisfies the ‘modified golden rule’ condition

\[
G'(x_d) - \frac{cG(x_d)}{x_d(c - pqx_d)} = \delta
\]

with corresponding effort \( E_\delta = \frac{G(x_d)}{\delta \psi} \). \( G(x) \) given in \([2]\). In other words, the regulator aims at stabilizing the biomass at the target \( x_d \) as quickly as possible. Depending on the discount factor \( \delta \), Clark \([2]\) shows that \( x_d \in (x_{OA}, x_{MEY}) = \left( \frac{H}{K}, \frac{1}{2} + \frac{H}{K} \right) \), with

\[
\lim_{\delta \to 0} x_d = x_{MEY} \text{ and } \lim_{\delta \to \infty} x_d = x_{OA}.
\]

The analysis in discrete time is similar, see \([15]\) for details.

References

[34] Leonov N. Discontinuous map of the straight line. Dolk Ahad Nauk SSR 1962;143(5):1038–41.