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A dynamic model of oligopoly with R&D externalities along networks. Part I.

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Abstract

This paper formulates and analyzes a two-stage oligopoly game where firms can invest in cost-reducing R&D activity with the possibility of sharing R&D results with partner firms as well as gaining knowledge for free through spillovers. Firms are arranged within networks (or districts) inside which they can cooperate by bilateral agreements for sharing knowledge and compete in the market. An adaptive dynamic mechanism is proposed to describe how firms in a two-networks system repeatedly decide their R&D efforts over time. This adaptive adjustment may converge to a Nash equilibrium in the long run, or exhibit more complex dynamic behaviors. Analytical results about stability of equilibrium points are given, as well as numerical simulations to show global dynamical properties, including coexistence of attractors and complicated structures of their basins. In a second paper (Part II) some analytical results will be given for some relevant benchmark cases, together with numerical experiments that stress the role of the level of connectivity (i.e. the collaboration attitude) inside networks, as well as the effects of involuntary knowledge spillovers inside each network and among different competing networks.

Keywords: R&D cooperation; Networks; Repeated games; Knowledge spillovers

1. Introduction

When firms compete in a global market, their efforts are mainly devoted to gain knowledge in order to adopt new technologies and improve production standards. In many cases, such efforts can be identified with expenditures in Research and Development (R&D) activities with cost-reducing effects. However, R&D activities are often more efficient, and their results more effective, if firms collaborate and share information on innovation and research results.

Along this line, the seminal paper by D'Aspremont and Jacquemin [11] proposes a two-stage game, where in the first stage two identical firms optimize their investment in cost-reducing R&D, with possible R&D spillover from the rival; then, in the second stage, firms compete in a homogeneous Cournot duopoly game.

Following this line of research, Kamien et al. [21] propose four different models, again in the form of two-stage games, where firms decide their effective R&D cost-reducing investments with or without formation of research joint ventures, and then they are engaged in either Cournot or Bertrand competition. All these models admit that research

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efforts are subject to various degrees of knowledge spillovers, and they conclude that the formation of research joint ventures, associated with competition in the product output, is the most desirable policy because it leads to higher profits and lower product prices; along the same line of research see also Qiu [28], Suzumura [30] and Amir et al. [2]. A dynamic version of the static game examined in Ref. [11] has been recently proposed by Cellini and Lambertini [9], in the form of a differential game, where it is shown that R&D cartelization dominates competition. Related issues in a difference game set-up are analyzed by Petit and Tolwinski [26,27].

As a matter of fact, many empirical studies show that partnerships among firms have significantly increased in recent years (see, e.g. Gauvin [16] and Goyal and Moraga-Gonzales [19]), and often this partnership is in terms of bilateral agreement for sharing information on R&D results and technological collaboration. This is the main motivation for setting up oligopoly models with R&D networks structures. Indeed, traditional models of oligopoly are centered on markets and neglect the presence of such R&D networks.

The research on networks in economics has become popular in the last decade. The importance of R&D networks is well explained in Goyal and Joshi [18], where different structures of bilateral collaborative links in firms' networks are described, as well as in Cowan and Jonard [10], where the relationship between the network architecture and knowledge transmission is explored, mainly by numerical analysis. For an extensive survey of the network literature we refer to Vega-Redondo [31] and Goyal [17].

The most related work to ours is the well-known contribution on R&D networks by Goyal and Moraga-Gonzales [19], which addresses the issue of collaborative ties formation starting from symmetric cases. In oligopolies with R&D networks, agents have several sources of strategic interdependence. First of all, oligopolists are strategically related on the demand side, as they all operate in the same market or in dependent markets. Moreover, also network externalities arise, as each firm's payoff is influenced by the R&D efforts by neighbors ('local' effects), non-neighbors ('global' effects) and by the whole set of connections in the industry. Of course, even without any agreement, knowledge may spill from one firm to its competitors, due to the difficulty of protecting intellectual properties (see Refs. [29,11,5]). This clearly introduces another externality between agents and a free-riding dilemma into firms' relationships, so that a trade-off between partial (and often asymmetric) involuntary knowledge spillovers, and complete (symmetric) information share, associated with bilateral agreements, may arise.

In this paper, we consider a repeated two-stage oligopoly where N ex-ante identical firms are subdivided into one or more groups ('sub-networks'). In these sub-networks couples of firms ('neighbors') have bilateral ties to share R&D results. Hence, effective R&D of each firm includes not only its own R&D but also neighbors' results. Moreover, a given firm can receive two types of knowledge spillovers: internal (from non-neighbors inside its sub-network) or external (from non-neighbors outside its sub-network).

Each discrete time step is ideally subdivided into:

- A precompetitive stage, where each firm selects cost-reducing R&D efforts in the direction of increasing individual profits, along the steepest ascent direction (so-called 'gradient process');
- A (Cournot-)competitive stage, where each firm sets its 'optimal' quantity, taking into account the level of efforts of other firms and the networks' structures, i.e. its effective cost.

These disjointed sub-networks can be interpreted as different Countries or industrial districts or groups of firms linked by ownership ties, characterized by different rules for partnership or patent protection or different abilities to take advantage from knowledge spillovers. This kind of structure is actually described in empirical works on the configuration of national R&D networks.[1]

All in all, the time evolution of the oligopoly can be described in pretty much the same way of standard dynamic Cournot models, with adaptive players revising their efforts over time (see Bischi et al. [4] for a complete treatment of oligopoly theory). In order to reduce the complexity of firms' decisional processes, the second stage decisions (on quantities) are univocally determined by firms' decisions on efforts by backward induction. Indeed, we mainly develop

[1] For instance, Gauvin [16] stressed that the formation of research alliances is an increasing phenomenon with coalitions that tend to be domestic rather than international, with some nations that show higher propensities to form coalitions, such as Japan and Germany, with respect to, e.g., USA and Canada. However, the connection between network relationships and innovation intensity in empirical cases is somehow controversial, as pointed out by Love and Roper [22].

the model positing regular ('symmetric') sub-networks, widely employed in the literature (see Galeotti et al. [15]); in this case, all agents belonging to the same sub-network have the same degree (i.e. the same number of neighbors) and are ex-post in a similar situation. Within this context, we first perform an analysis on network strategic effects of efforts. Then the evolution of effort's choices in each sub-network is studied by means of a discrete dynamical system. For this stylized model of networks competition, it is possible to obtain analytical results on existence and stability of equilibria as well as on social efficiency. In addition to the local stability analysis, we also perform some simulation when the system fails to converge to an equilibrium. From these simulations, interesting global properties typical to noninvertible maps of the plane (e.g. path dependence) can be detected, according to the theory in Mira et al. [25].

In this paper we try to address the following research questions: (i) How do the exogenous structural properties of competing networks influence aggregate outcomes? (ii) Under which circumstances (level of collaboration, degree of knowledge spillovers) is the equilibrium stable, and how can it lose stability? The companion paper Bischi and Lamantia [6] (Part II henceforth) will be mainly devoted to the following question: (iii) What are the influences of the degree of collaboration and knowledge spillovers on profits, social welfare and, more generally, on overall efficiency?

This paper is organized as follows. Section 2 is devoted to the static formulation of the model and comparative statics. In Section 3 a dynamic version of the model is proposed with general conditions on local stability of steady states and numerical simulations to show some global dynamical properties and disequilibrium dynamics. Section 4 concludes.

2. Static analysis

2.1. Model formulation and equilibrium solution

We consider a homogeneous-product oligopoly where $N \geq 2$ quantity setting firms operate in a market characterized by a linear inverse demand function $p = a - bQ$, $a, b > 0$, Q being the total output in the market.[2]

These N firms are ex-ante partitioned into h groups, (called *sub-network* in the following). We say that two firms of the same sub-network are neighbors (or that they are linked) if they have a direct tie, i.e. they form a bilateral agreement for a complete sharing of R&D results. Two firms without a direct tie are called non-neighbors. Note that each sub-network is an undirected graph, as ties are bidirectional.

Each of these h sub-networks, say s_j, $j = 1, \ldots, h$, is formed by n_j firms, where of course $N = \sum_{j=1}^{h} n_j$. For sake of simplicity, we assume that each sub-network s_j is symmetric [3] of degree k_j, with $0 \leq k_j \leq n_j - 1$, i.e. every firm in s_j has the same number of collaborative ties k_j, a parameter that represents the level of connectivity (or collaborative attitude) of sub-network s_j.

Let us assume that, given a network structure, firms have to decide their R&D efforts and quantities to produce. In this section we derive the profit functions to be maximized. Then, based on the results of this section, in the next section we shall describe a dynamic model where firms' strategies are dynamically chosen at each (discrete) time period.

Each firm decides its R&D effort, whose cost-reducing effects are totally shared with neighbors; moreover, R&D results within a sub-network can spill over for free to non-neighbors inside the same sub-network s_j (internal spillovers) as well as to 'rival' sub-networks s_k with $k \neq j$ (external spillovers). Assuming a linear cost function $c_i q_i$ for a firm i that produces q_i, a representative firm in sub-network s_j has a marginal cost c_i of the form

$$c_i(s_j) = c - e_i - k_j e_{l_i} - \beta_j e_{l_{-i}}\left[\left(n_j - 1\right) - k_j\right] - \beta_{-j} \sum_{m \notin s_j} e_m \tag{1}$$

where c is the marginal cost without R&D efforts (equal for all firms), e_i represents R&D effort of firm i, $k_j e_{l_i}$ represents the total effort exerted by firms with whom i is linked in s_j, $\beta_j \in [0, 1)$ are related to knowledge spillovers with non-neighbors in network s_j, and $\beta_{-j} \in [0, 1)$ regulate external spillovers, i.e. originating from non-neighbors out of s_j towards firm i. Hence, the last two terms in (1) are, respectively, the spilled effort by l_{-i} and m, i.e. representative

[2] We disregard the presence of inventories and, in general, intertemporal demand interactions. These extensions could be added in the model following Bischi et al. [4].

[3] We recall that in a 'symmetric' (or 'regular') network all nodes (firms) have the same first(-order) degree (i.e. the same number of bilateral R&D collaboration links).

non-neighbors inside s_j and outside s_j respectively. In order to ensure strictly positive prices, we impose from now on condition $a > c$.

In any case, all N firms are rivals in the market place (see Refs. [11,19]), and they calculate their optimal outputs by solving individual profit maximization problems. Then, given optimal quantities as functions of efforts by backward induction, they assess R&D efforts to increase their individual profits; due to R&D networks of collaboration and spillovers, each firm calculates these cost-reducing efforts taking into account the whole cost structure of the industry.

Following Ref. [19], we assume that each oligopolist i in sub-network s_j maximizes a profit function of the form

$$\pi_i(s_j) = \left\{ a - b[q_i(s_j) + \sum_{p \neq i} q_p] - c_i(s_j) \right\} q_i(s_j) - \gamma e_i^2(s_j)$$

where $q_i(s_j)$ and $e_i(s_j)$ are, respectively, the quantity produced and the R&D effort by agent i in sub-network s_j, and γe_i^2, $\gamma > 0$, is the cost of effort (see Ref. [11]).

The optimal quantity of firm i in sub-network s_j is

$$q_i(s_j) = \frac{a - Nc_i(s_j) + \sum_{p \neq i} c_p}{b(1+N)} \tag{2}$$

with corresponding optimal profit

$$\pi_i(s_j) = \left[\frac{a - Nc_i(e_i) + \sum_{p \neq i} c_p}{\sqrt{b}(1+N)} \right]^2 - \gamma e_i^2 \tag{3}$$

Given this setting, each firm tries to maximize its individual profit with respect to its own R&D efforts. Substituting the cost functions of representative agents in each sub-network, we can reformulate the optimal profit for firm i in sub-network s_j (3) as the following quadratic functions of efforts only (see Appendix A for details):

$$\pi_i(s_j) = \left[\frac{a - Nc_i(s_j) + k_j c_{l_i}(s_j) + (n_j - 1 - k_j) c_{l_{-i}}(s_j) + \sum_{w=1, w \neq i}^{h} n_w c_v(s_w)}{\sqrt{b}(1+N)} \right]^2 - \gamma e_i^2 \tag{4}$$

By standard arguments, it is possible to establish the existence of a Nash equilibrium. In fact, if each firm i chooses an effort level $e_i \in [0, c]$, the strategy space is a compact and convex set. Moreover profit functions are continuous with respect to strategies of all players and, for large γ, concave in own strategies. Then by theorem 3.1 in Ref. [17], there exists a Nash equilibrium in pure strategies (see also Ref. [19]).

Under concavity of payoffs in own strategies, the FOCs $\frac{\partial \pi_i}{\partial e_i} = 0$, $i = 1, \ldots, h$ are necessary and sufficient for an interior optimum E^*. As sub-networks are symmetric, all firms belonging to the same sub-network exert the same effort, (i.e. $e_{l_i} = e_{l_{-i}} = e_i$). So in general we obtain a system of h reaction functions (one for each sub-network) in h unknowns. This system of equations is *linear*, as profits are quadratic functions of efforts; consequently the existence and uniqueness of a Nash equilibrium can be given in terms of non-singularity of a $h \times h$ matrix, as we show in the following section. When all firms start the game exactly at the Nash equilibrium, then there is no unilateral incentive to deviate and the model reduces to a one shot game. Otherwise a dynamic mechanism for updating R&D effort over time must be defined. Before doing so, we briefly analyze the main relationships between marginal profits and effort in the proposed multi-network competition.

2.2. Effects of increasing collaboration level and spillovers on profits

From the expression (4) for the optimal profit, it is clear that individual efforts exerted by any firm in any network influence, with different degrees, profits of all other firms. Here we show that efforts do not necessarily increase or decrease when collaboration level or knowledge spillovers are increased, as they depend on several factors. Let us begin by considering the direct effect on the marginal profit of firm i in network s_j, $\pi_i(s_j)$, from changing its own R&D effort, i.e.

$$\frac{\partial^2 \pi_i(s_j)}{\partial e_i^2} = \frac{2\left(N - k_j(1-\beta_j) - \beta_j(n_j - 1) - \sum_{w=1, w \neq j}^{h} \beta_{-w} n_w\right)^2}{b(1+N)^2} - 2\gamma \tag{5}$$

Hence, as effort e_i is increased, both marginal revenues and marginal costs (whose slopes are the first and second components in Ref. (5)) increase, even if an increase in degree k_j raises the number of neighbors with which the research efforts are shared, and this lowers marginal returns to firm i.

This can be explained by noting that when a firm has more collaborators, an increase in its effort not only lowers its own costs, but it lowers the costs for neighbors as well, and, consequently, they become tougher competitors in the marketplace. Exactly the same effect, for similar reasons, is observed as internal spillovers β_j in network s_j and/or external spillovers β_{-w} in networks s_w, $w = 1, \ldots, h$, $w \neq j$, increase, since marginal revenues are decreasing in β_j and β_{-w}. In both cases, again, R&D effort by firm i is discouraged as this spills over to competitors.[4]

From this point of view, a firm could try to act as a free rider and exploit the effort by partners without actually exerting its own. To understand better this point, it is useful to consider the so called strategic effects, i.e. whether R&D efforts between firms are strategic complements or strategic substitutes, see Refs. [8] and [15]. Moreover, these strategic effects can be 'local' if they come from neighbors or 'global' if they originate from non-neighbors (see Ref. [17]). In the present model, different global effects arise depending on the network they originate, as shown below.

First of all, we consider 'local' strategic effect; in this case, efforts by linked firms are always strategic complements; in fact, denoting by e_{l_i} the effort of a generic neighbors of i (as in the previous section), the expression for $\frac{\partial^2 \pi_i}{\partial e_i \partial e_{l_i}}$ is always strictly positive, being identical to the first component in Eq. (5). This has an immediate economic justification, as incremental efforts by neighbors always reduce unitary costs to i for free.

Now we consider how an effort variation by non-neighbors impacts i's payoff, i.e. the global network effects. In general, mixed partial derivatives of profits with respect to efforts are quadratic functions in spillover parameters. However, when all spillover effects are neglected, these derivatives with respect to unlinked firms reduce to

$$\left.\frac{\partial^2 \pi_i}{\partial e_i \partial e_{l_{-i}}}\right|_{\beta=0} = -\frac{2(n_j - k_j - 1)(N - k_j)}{b(1+N)^2} < 0 \tag{6}$$

and

$$\left.\frac{\partial^2 \pi_i}{\partial e_i \partial e_m}\right|_{\beta=0} = -\frac{2n_p(N - k_j)}{b(1+N)^2} < 0 \tag{7}$$

where $e_{l_{-i}}$ and e_m are, respectively, R&D efforts by a non-neighbor in s_j (i.e. the same network of i) and in a different network s_p (with n_p firms and degree k_p). In the notation, it has been emphasized that no spillovers are present. These relations say that, without spillovers, efforts of non-neighbor firms are strategic substitutes, which is a very intuitive property in the context of the model.[5] Moreover, by Eqs. (6) and (7), the higher the degree k_j, the lower the magnitude of non-neighbors's strategic substitutability is.

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[4] This result is an extension of proposition 4 in Ref. [19], where $\beta_j = 0$ and $n_w = 0$, $w = 1, \ldots, h$, $w \neq j$, are assumed.

[5] The expression in Ref. (6) is always strictly negative, because it vanishes only at $k_j = n_j - 1$, i.e. when all firms in s_j are neighbors.

However the situation overturns as knowledge spillovers are considered. A complete characterization of strategic effects would be very lengthy and therefore we omit it here. We just observe that when all knowledge spills over firms in network s_j, ($\beta_j = 1$; with all other βs equal to zero), it is

$$\frac{\partial^2 \pi_i}{\partial e_i \partial e_{l_{-i}}}_{|\beta_j=1} = \frac{2(1+N-n_j)^2}{b(1+N)^2} > 0 \tag{8}$$

and so, by continuity, there exists an intermediate internal spillover level $\overline{\beta_j} \in (0, 1)$ such that efforts in network s_j cross from strategic substitutes to strategic complements. Again, this is very easy to justify: when internal spillovers are not present, an increment of the R&D activity by a non-neighbor of i in s_j reduces the unitary cost of that firm without providing an advantage to i; all the same, if internal spillovers are present, a cost reduction to firm i is granted. Hence, incremental effort can become strategic complement for sufficiently high internal spillovers. In the limiting case of full internal spillovers, it is as if the network structure disappear, as all firms inside the network fully share their knowledge through spillovers, and so, the first component of Eq. (5) with full connections ($k_j = n_j - 1$) coincides with Eq. (8). These results are analogous to the ones reported in Ref. [11].

A similar argument can be applied to external spillovers. In fact, we have that

$$\frac{\partial^2 \pi_i}{\partial e_i \partial e_m}_{|\beta_{-j}=1} = \frac{2(N-k_j)}{b(1+N)^2} > 0 \tag{9}$$

so that, again, efforts in a network different to that i belongs to become strategic complements if external spillovers are sufficiently high (all other βs are equal to zero), i.e. substantial cost reductions are granted to firms inside network s_j as a consequence of spillovers from network s_p to network s_j. Notice that, depending on the number of firms in each sub-network, their degree and spillovers, we can have pure local effects (only neighbors' actions matter), pure global effects (actions of all individuals have the same effects) and different combinations of them (see Ref. [17]).

In addition, individual effort by firm i in network s_j influences profits to firms outside i's network. As a matter of fact, an increment of efforts by firm i in network s_j affects the profit function of firm g in any competing network s_p; from $\pi_g(s_p)$, we obtain that marginal revenue grows linearly in e_i with slope

$$\frac{\partial^2 \pi_g(s_p)}{\partial e_i^2} = \frac{2n_j^2(-1-k_j(1-\beta_j)-\beta_j(n_j-1)+\beta_{-p}(1+n_j))^2}{b(1+N)^2} \tag{10}$$

As this expression represents a nonnegative and convex parabola in k_j, that vanishes at $k_j^{\min} = \frac{-1-\beta_j(n_j-1)+\beta_{-p}(1+n_j)}{1-\beta_j}$, we can conclude that without external spillovers β_{-p}, or when they are sufficiently small, an increment in effort e_i by a firm in s_j gives more advantages to competitors in network s_p as the number of links k_j in s_j increases. This fact is exactly the mirror image of the previous case: in fact, as the number of links k_j in s_j increases, marginal revenue in s_j declines and this is an advantage for their competitors in network s_p. However, as β_{-p} is increased, this effect can be inverted or we can observe that the benefit to $\pi_g(s_p)$ is minimized for intermediate levels of collaborative activity k_j. This can be explained by noting that when $\beta_{-p} > 0$, firms in s_p has substantial cost reductions when their rivals in s_j invest in R&D; however as k_j increases and with high external spillovers β_{-p}, firms in s_j tend to invest less in R&D, in order not to advantage firms in s_p, so that less knowledge spills over from network s_j to s_p.

Now we turn to total profit. Ref. [19] shows that when knowledge spillovers are absent and only one network operates ($\beta_j = \beta_{-j} = 0$ and $n_i = 0$, $i = 2, \ldots, h$), then total profit $\pi(s_j)$ is always maximized for an intermediate level of collaborative activity k_j. As along an invariant axis (e.g. $e_i = 0$, $i = 2, \ldots, h$) our model reduces to Ref. [19], we can also say that profits at steady states are maximized for intermediate number of collaborative arrangement k_j. However it is not easy to answer this question in the general model with multi-network competition, since payoffs depend on the overall compensation between several opposing forces, as mentioned. For these reasons, in the following section we address the problem of disequilibrium dynamics, where firms repeatedly modify their R&D decisions toward the direction of increasing profits, and we analyze, the existence and stability of equilibria in the case of two interacting R&D sub-networks.

3. A dynamic adaptive model for R&D efforts

3.1. Myopic disequilibrium dynamic and equilibria

Due to the complex network structure of R&D collaborations and spillover externalities, it is unlikely that agents are able to play the Nash equilibrium strategy in one shot. Consequently, we assume that firms behave myopically, i.e. each player cares about immediate payoffs and believes that actions of other players in the current period are the same as the actions in the immediately preceding period (see Ref. [17]). In this setting, agents adaptively adjust their efforts over time towards the 'optimal' strategy, following the direction of the local estimate of expected marginal profits, according to the so called "gradient dynamics" (or "gradient process", see Refs. [3,14,12,7,13])

$$e_j(t+1) = e_j(t) + \alpha_j(e_j)\frac{\partial \pi_j}{\partial e_j};\ \ j = 1, ..., h \tag{11}$$

where $e_j(t)$ represents the R&D effort at time period t of a representative firm belonging to the sub-network s_j; α_j are positive functions that represent speeds of adjustment. So, efforts at time t, which are observable by all agents, lead to the choice of next period R&D activities, through a repeated adaptive process (11). It can easily be seen that the Nash equilibria are also equilibrium points for the dynamic process (11). If such an equilibrium is stable, then we can say that the adaptive agents are able to learn, in the long run, how they can choose R&D efforts in an optimal way. However, as we shall see, these equilibria are not always stable under the gradient dynamics (11).

In the following we focus on the case of only two sub-networks s_1 and s_2 with n_1 and n_2 firms and connection degrees k_1 and k_2 respectively. Moreover, we assume linear speeds of adjustment $\alpha_j(e_j) = \alpha_j e_j$, i.e. the relative effort change $[e_j(t+1) - e_j(t)]/e_j(t)$ is posited to be proportional to the expected marginal profit.

Under these assumptions the dynamical system that describes the time evolution of the efforts chosen by the two representative firms is given by

$$e_i(t+1) = e_i(t) + \frac{\alpha_i e_i(t)}{b(1+n_i+n_j)^2}[A_i + B_i e_j(t) + C_i e_i(t)],\ i, j = 1, 2; i \neq j \tag{12}$$

where the following aggregate parameters have been introduced:

$$\begin{aligned} A_i &= 2(a-c)[(n_i - k_i)(1-\beta_i) + \beta_i + n_j(1-\beta_{-j})] \\ B_i &= 2n_j[(1-\beta_i)(n_i-k_i) + \beta_i + n_j(1-\beta_{-j})][-\beta_j(n_j-k_j-1) + \beta_{-i}(n_j+1) - k_j - 1] \\ C_i &= 2\left\{(-k_i + \beta_i(1+k_i-n_i) + N - \beta_{-j}n_j)(1+k_i+n_j+k_i n_j - \beta_{-j}n_i n_j - \beta_i(1+k_i-n_i)(1+n_j)) - b\gamma(1+N)^2\right\} \end{aligned} \tag{13}$$

with $N = n_1 + n_2$. It is useful to notice that $A_i > 0$ for all economic meaningful parameters, and the sign of B_i is the same as the sign in the second term in square brackets. In particular, if $\beta_{-i} \to 0$, i.e. network i has a low capacity to gain knowledge from network j for free, then $B_i < 0$; on the other hand, if $\beta_{-i} \to 1$ and $\beta_j \to 0$ then $B_i > 0$; in this case network i is able to gain knowledge for free from network j whereas the opposite does not hold for network j. With respect to C_i it is immediate to observe that condition $C_i < 0$ is equivalent to $\frac{\partial^2 \pi_i}{\partial e_i^2} < 0$ so that the profit function $\pi_i(e_i)$ is strictly concave and the FOC for a maximum is also sufficient. As observed before, this condition holds for a sufficiently high level of effort cost γ.

The dynamical model (12) always admits three boundary equilibria:

$$O = (0, 0),\ E_1 = (-A_1/C_1, 0),\ E_2 = (0, -A_2/C_2), \tag{14}$$

located on the invariant coordinate axes, with nonzero coordinate strictly positive if and only if the corresponding profit function $\pi_i(e_i)$ is strictly concave, and a unique interior equilibrium

$$E^* = \left(\frac{A_2B_1 - A_1C_2}{C_1C_2 - B_1B_2}, \frac{A_1B_2 - A_2C_1}{C_1C_2 - B_1B_2}\right) \tag{15}$$

which is obtained from the system

$$\begin{cases} C_1e_1 + B_1e_2 = -A_1 \\ B_2e_1 + C_2e_2 = -A_2. \end{cases} \tag{16}$$

provided that $C_1C_2 - B_1B_2 \neq 0$.

We observe that, in general, at a boundary equilibrium E_i only n_i agents invest in R&D even if N agents sell their product in the market. Equilibrium E^* is obtained by unilateral profit maximization by representative firms in the two sub-networks and correspond to the Nash equilibrium solution described in the previous Section. We characterize the equilibrium E^* in some benchmark cases in Part II.

3.2. Stability properties of equilibria with two sub-networks

Now we tackle the problem of local stability of the equilibria of model (12), whose Jacobian matrix is given by

$$J(e_1, e_2) = \begin{bmatrix} 1 + \frac{\alpha_1}{b(1+N)^2}(A_1 + B_1 e_2 + 2C_1 e_1) & \frac{\alpha_1 B_1 e_1}{b(1+N)^2} \\ \frac{\alpha_2 B_2 e_2}{b(1+N)^2} & 1 + \frac{\alpha_2}{b(1+N)^2}(A_2 + B_2 e_1 + 2C_2 e_2) \end{bmatrix} \tag{17}$$

From standard local stability analysis we get:

Proposition 1. *Equilibrium $O = (0, 0)$ is a repelling node.*[6]

In fact, $J(0, 0)$ is a diagonal matrix, and the stability conditions for the equilibrium O, given by $-2 < \frac{\alpha_i A_i}{b(1+N)^2} < 0$, $i = 1, 2$, never hold.

This means that for a sufficiently low initial R&D level (no matter how low) there will be at least one network exerting a positive (and increasing with time) level of effort. We can restate this result by saying that it is always convenient, for at least one network, to invest in R&D.

We now consider the equilibrium E_i located on the invariant axis e_i, along which the dynamics are described by the unidimensional map

$$e_i(t+1) = e_i(t) + \frac{\alpha_i e_i(t)}{b(1+N)^2}[A_i + C_i e_i(t)] \tag{18}$$

Of course, it is also important to study the stability along the direction transverse to e_i, as it explains under which conditions one network (the second if $i = 1$) progressively reduce its efforts to zero. The main conclusions are summarized in the following

Proposition 2. *Equilibrium E_i, $i = 1, 2$, is attracting along the e_i axis as long as $A_i < 2b(1+N)^2/\alpha_i$. At $A_i = 2b(1+N)^2/\alpha_i$ equilibrium E_i undergoes a flip bifurcation and, as $A_i > 2b(1+N)^2/\alpha_i$, cascades of period doubling bifurcations are created leading to a chaotic regime.*

In the direction transverse to e_i, E_i is stable if condition

$$-2 < \frac{\alpha_j(A_j C_i - B_j A_i)}{C_i b(1+N)^2} < 0$$

holds. At $\alpha_j(A_j C_i - B_j A_i) = -2bC_i(1+N)^2$ equilibrium E_i undergoes a flip bifurcation along the transverse eigendirection, whereas at $A_j C_i = B_j A_i$ a transcritical bifurcation occurs at which equilibria E_i and E^* merge.

Proof. See Appendix B.

In terms of original parameters we can restate the previous proposition by saying that if the cost of effort γ is sufficiently high (thus $C_i < 0$ and profits are concave in efforts) and only network i invest in R&D, then this network will tend to adopt a constant level of efforts provided that its reaction coefficient α_i or the aggregate parameters A_i are sufficiently small: we can relate "small" A_i values to a small differences between the maximum selling price a and the maximum marginal cost coefficient c and/or to a great cost reduction within network i (due to many collaborative arrangements within network i or high internal spillovers). Otherwise equilibrium E_i looses stability through a period

[6] By "repelling node" we mean an equilibrium with both eigenvalues real and with modulus greater than one.

doubling cascade of bifurcations, as in the standard logistic model, and R&D efforts inside network i are characterized by increasing levels of unpredictability.

When E_i is stable also in the direction transverse to e_i, then we observe a decrease in R&D efforts by network j till its effort goes to zero, and only network i will invest at the equilibrium level E_i.Now we tackle the more difficult problem of the stability of the inner equilibrium E^*, where both sub-networks exert a positive R&D effort. We first analyze its local stability in terms of aggregate parameters, then, with the help of numerical explorations, we investigate the role of some original economic parameters, such as k_i, β_i, β_{-i}, as well as the kinds of dynamic behaviors that can be observed when E^* is not stable.

The following proposition states necessary conditions for the local stability of E^*.

Proposition 3. *If $C_1C_2 \geq B_1B_2$, with $C_i < 0, i = 1, 2$ and equilibrium $E^* = \left(e_1^*, e_2^*\right)$ is stable then*

$$\frac{\alpha_1 C_1 e_1^* + \alpha_2 C_2 e_2^*}{b(1+N)^2} \geq -4 \tag{19}$$

holds.

Proof. See Appendix C.

It is worth noticing that the case $C_i > 0$ is not interesting from an economic point of view, as it corresponds to the case of strictly convex profit function and the equilibrium E^* is not of Nash type, as profits are there in a global minimum.

From an economic point of view, Proposition 3 says that when the marginal demand b is low or speeds of adjustments α_1 and/or α_2 are sufficiently large or C_1 and/or C_2 are sufficiently negative, then the fixed point E^* will be unstable, i.e. inequality (19) will not be satisfied. For example, the aggregate parameters C_i are increasing functions of k_i, the degree of collaboration, so they became more negative for decreasing values of the respective R&D collaboration parameters. In other words, more collaboration will help to obtain the stability of E^*.

Remaining in a general case but with homogeneous agents (so that all symbols can be written without subscripts as they coincide), let us consider the internal equilibrium $E^* = \left(\frac{-A}{C+B}, \frac{-A}{C+B}\right)$, with $C+B<0$, so that both coordinates of E^* are strictly positive. Conditions (i) in (C.2) with strict inequality is equivalent to $C-B<0$. Condition (ii) and (iii) in (C.2) hold as strict inequalities in the following cases: if $B \leq 0$, then it must be $b > \frac{\alpha A}{2(1+N)^2}$; if $B>0$, then $b > \frac{\alpha A(C-B)}{2(C+B)(1+N)^2}$. Thus in the homogeneous case, when $C-B<0$, and b is sufficiently high (and/or α sufficiently low), the inner equilibrium is always stable.

3.3. Numerical simulations of the model with two sub-networks

The results of the three propositions given above, obtained through the standard linearization procedure, only concern local stability of the equilibrium points, and give no information about the global dynamics of the system. For this reason, in this subsection we carry on some numerical experiment for the dynamical system (12) to investigate the possible basins of attraction, the kinds of disequilibrium dynamics that prevail when an equilibrium point, in particular E^*, loses stability and the possible coexistence of several attractors, each with its own basin of attraction. As we shall briefly see, some typical feature of nonlinear dynamical system can be numerically evidenced, in particular some global structures of the attractors and the basins which are typical of noninvertible maps of the plane (see, e.g. [25]).

First of we consider a fixed set of parameters:

$$\begin{aligned} \alpha_1 &= 0.27, \alpha_2 = 0.26; n_1 = n_2 = 10; k_2 = 5; \gamma = 6; \\ \beta_1 &= \beta_2 = 0.1, \beta_{-1} = \beta_{-2} = 0.02; a = 200; c = 80; b = 1 \end{aligned}$$

and the collaboration parameter $k_1 \in [0, 9]$ as a bifurcation parameter[7]. As shown in the bifurcation diagram of Fig. 1, the set of parameters considered (in particular the low value of b and the sufficiently high values of the speed of reaction α_i) are such that all the four equilibria exist and are unstable, and chaotic dynamics characterizes the asymptotic behavior

[7] Of course, k_1 must assume integer values, however we shall consider fractional values as well in the numerical computation of the bifurcation diagram.

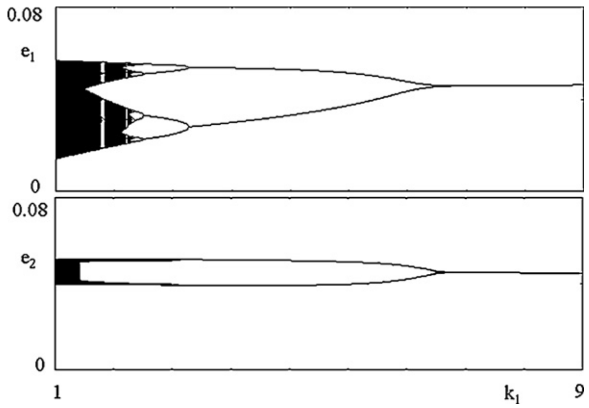

Fig. 1. Bifurcation diagram with bifurcation parameter the $k_1 \in [0, 9] \cap \mathbb{Z}$ and fixed parameters $\alpha_1 = 0.27$, $\alpha_2 = 0.26$; $n_1 = n_2 = 10$; $k_2 = 5$; $\beta_1 = \beta_2 = 0.1$, $\beta_{-1} = \beta_{-2} = 0.02$; $a = 200$; $c = 80$; $\gamma = 6$; $b = 1$ with initial condition taken close to the positive equilibrium E^*.

of the adaptive system for low values of k_1, i.e. low degrees of $R\,\&\,D$ collaboration in the first network. As k_1 increases, a sequence of period-halving (or backward flip) occurs, leading to stability of E^* for sufficiently high levels of $R\,\&\,D$ sharing in the first network. This confirms the stability analysis given in the previous subsection. From an economic point of view, this numerical example outlines that a parameter proper to a network (its degree) can strongly influence the long-run R&D efforts of the other network, so that the analysis for a single R&D network can be somehow misleading if such interrelations are neglected.

It is also interesting to analyze the kind of chaotic attractors and the basin of attraction that characterize, respectively, the long run dynamics and the role of the initial conditions, i.e. the path dependence, of the adaptive system. The example depicted in Fig. 2 is obtained with the same set of parameters as in Fig. 1 except higher speeds of reaction $\alpha_1 = 0.35$, $\alpha_2 = 0.3$ and with $k_1 = 5$. The attractor is a 2-cyclic chaotic attractor, and the white region represents its basin of attraction, while the grey shaded region represents the set of initial conditions that generate diverging trajectories. Therefore, following the direction of higher profits, firms in each networks alternate low and high R&D investments over time without converging to an equilibrium.

However, by slightly changing the set of the parameters of the last example, it is obtained an interesting case of coexistence between two distinct attractors, whose basins share the region of initial conditions that generate bounded

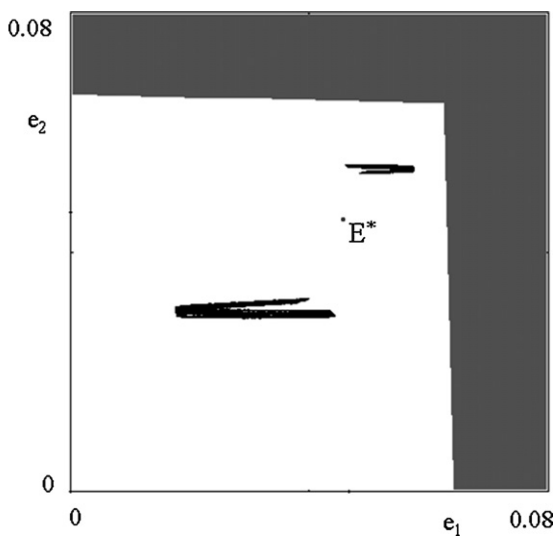

Fig. 2. A 2-cyclic chaotic attractor with its basin of attraction (white region) represented in the phase plane $(e_1(t), e_2(t))$ for the set of parameters as in Fig. 1 but $\alpha_1 = 0.35$, $\alpha_2 = 0.3$ and $k_1 = 5$. The initial conditions taken in the grey shaded region generate diverging trajectories.

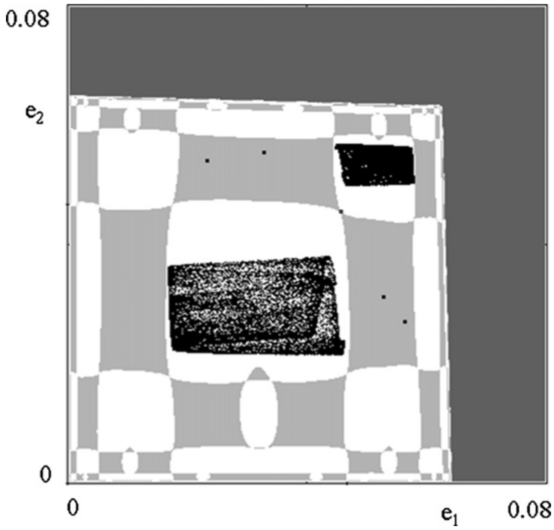

Fig. 3. Two coexisting attractors: a 2-cyclic chaotic attractor and a 4-cycle, each with its own basin of attraction represented by the white and light grey regions respectively. The dark-grey region represents the basin of diverging trajectories, as in the previous picture. Parameters are as in Fig. 2 but $\alpha_2 = 0.325$ and $\beta_1 = 0.2$.

trajectories. In Fig. 3, obtained with parameters $\alpha_1 = 0.35$, $\alpha_2 = 0.325$, $n_1 = n_2 = 10$, $k_1 = k_2 = 5$, $\beta_1 = 0.2$, $\beta_2 = 0.1$, $\beta_{-1} = \beta_{-2} = 0.02$, $a = 200$, $c = 80$, $\gamma = 6$, $b = 1$, a 2-cyclic chaotic attractor (white basin) coexist with a periodic cycle of period 4 (intermediate grey region), whereas the dark grey region denotes, as usual, the points that generate diverging trajectories. The complicated topological structure of the basins is typical of noninvertible maps (see, e.g. Refs. [25,1,4]) and implies a strong path dependence, i.e. even a negligible displacement of the point in the space of efforts $(e_1(t), e_2(t))$ may cause the crossing of a basin boundary, so that the asymptotic dynamics of the trajectory may be quite different; this phenomenon is also denoted as “final state sensitivity” following Ref. [20].

Even the structure of the chaotic attractor is typical of a noninvertible map, like folded veils due to the folding action of the critical curves (see Ref. [1]). This is even more evident in Fig. 4, obtained with a slight increase of $\alpha_2 = 0.36$, which shows the merging of the two coexisting attractors in a bigger chaotic attractor so that R&D effort dynamics becomes highly unpredictable.

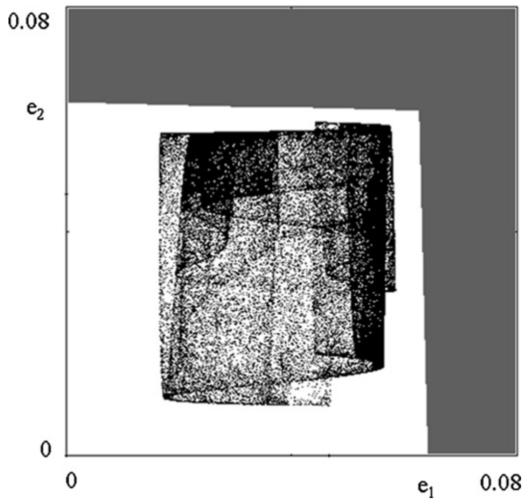

Fig. 4. A unique chaotic attractor with its basin obtained after the merging of the two coexisting ones; parameters as in Fig. 3 but $\alpha_2 = 0.36$.

4. Conclusions

In this paper we have introduced a repeated two-stage game, for describing the competition between firms constituting Research Joint Ventures, in the form of R&D networks. As often postulated in the literature, R&D efforts have a cost-reducing effect and they are decided in a pre-competitive stage; in most cases, they can be carried out by firms linked in networks. We assumed that firms act non-cooperatively at the second stage, where quantities to sell are decided in order to maximize their individual profit, as in a Cournot oligopoly game.

Our model was mainly motivated by Ref. [19], however it departs from Ref. [19] in several points. Firstly, we focus on network effects but we do not address here the issue of network formation and we assumed that several (sub-)networks of collaboration coexist and are exogenously given. Secondly, we consider an adaptive dynamic adjustment process in the first stage, given by a step by step adaptive process to decide R&D efforts. In other words, as suggested in Ref. [21], we assume that the decision about R&D efforts involves an heuristic trial and error process, just following local signals of profit gradients. In fact, given the structure of the game, the agents' rationality requirements for a one-shot positioning to a Nash equilibrium are very high; instead, with myopic dynamic adjustments, the only requirement is that a representative agent of each sub-network is able to assess how small variations in its R&D efforts influence its expected profit.

After analyzing the effect of individual R&D efforts on profit of the various rival firms in the general multi-network framework, we proposed a dynamical system for studying R&D effort over time in the case of two competing networks of firms. Although this model can be employed just to perform equilibrium analysis, as in Ref. [11], i.e. supposing that agents are able to select, in one shot, the proper Nash equilibrium level, we adopted the weaker assumption of a repeated adaptive process for updating R&D effort choices over time. The long run outcome of such adaptive process may coincide with an underlying Nash equilibrium of the game. However, when several equilibria are present, as it occurs in the model proposed when inner and boundary equilibria coexist, an *equilibrium selection* problem arises, so it is crucial to analyze the path dependent dynamic transition toward an equilibrium, as the initial condition of the system plays a role in the long run behavior of the model. This point recalls Nash's concern, expressed in his thesis, about a possible evolutionary interpretation of the concept of Nash equilibrium, see Ref. [23].

In this way, we stated several results on the convergence to a Nash equilibrium in terms of the parameters of the system. The last part of the paper briefly deals with the global dynamics of the system when an equilibrium loses stability and more complex attractors arise, also showing the possible path dependence of R&D efforts in these cases. The set-up of the model as well as the analytical results obtained in this paper are relevant for the analysis carried out for the two-networks case the in the companion paper [6].

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Appendix A. Model derivation

A.1. Cost functions

By the assumption of homogeneity of firms within sub-network s_j and analogously to Eq. (1), we can write the marginal cost for a firm $l_i \neq i$ linked to i in the sub-network s_j as

$$c_{l_i}(s_j) = c - k_j e_{l_i} - e_i - \beta_j e_{l_{-i}}[(n_j - 1) - k_j] - \beta_{-j} \sum_{m \notin s_j} e_m \tag{A.1}$$

Analogously, the marginal cost for l_{-i}, in s_j and non-neighbors of i, is

$$c_{l_{-i}}(s_j) = c - (k_j + 1)e_{l_{-i}} - \beta_j \{e_{l_i}[(n_j - 2) - k_j] + e_i\} - \beta_{-j} \sum_{m \notin s_j} e_m \tag{A.2}$$

A firm v_i in sub-network $s_w \neq s_j$ (formed by n_w firms) has k_w links with firms in its sub-network and a marginal cost of the form

$$c_v(s_w) = c - (k_w + 1)e_{v_i} - \beta_w e_{v_{-i}}[(n_w - 1) - k_w] - \beta_{-w}(e_i + \sum_{\substack{m \notin s_w \\ m \neq i}} e_m) \tag{A.3}$$

where $e_{v_{-i}}$ is effort by a non-neighbors of v_i in s_w.[8]

All in all, by Eqs. (1), (A.1), (A.2), (A.3) and the assumption of symmetry, for firm i in network s_j, the cost of production for the rest of the industry in Eq. (3) is given by

$$\sum_{p \neq i} c_p = k_j c_{l_i}(s_j) + \left(n_j - 1 - k_j\right) c_{l_{-i}}(s_j) + \sum_{\substack{w = 1 \\ w \neq j}}^{h} n_w c_v(s_w)$$

A.2. Profit functions with two sub-networks

Let us consider the case of only two competing sub-networks, say s_1 and s_2, with n_1 and n_2 firms and connection degrees k_1 and k_2 respectively. We can rewrite Eq. (1) as

$$c_i(s_1) = c - e_i - k_1 e_l - \beta_1 e_l[(n_1 - 1) - k_1] - \beta_{-1} n_2 e_m$$

and (A.1), the cost function for a firm linked to i, as

$$c_{l_i}(s_1) = c - k_1 e_l - e_i - \beta_1 e_l[(n_1 - 1) - k_1] - \beta_{-1} n_2 e_m = c_i(s_1)$$

whereas the cost function (A.2) for a firm untied to i but still in network s_1 reads as

$$c_{l_{-i}}(s_1) = c - (k_1 + 1)e_l - \beta_1\{e_l[(n_1 - 2) - k_1] + e_i\} - \beta_{-1} n_2 e_m$$

and the marginal cost (A.3) for a representative firm in s_2 is

$$c_m(s_2) = c - (k_2 + 1)e_m - \beta_2 e_m[n_2 - 1 - k_2] - \beta_{-2}[(n_1 - 1)e_l + e_i]$$

In all these expressions we emphasized the dependence on the e_i, i.e. the effort exerted by firm i in s_1.

All in all, for firm i in network s_1, the average cost of production for the rest of the industry to substitute in (3) is given by

$$\sum_{p \neq i} c_p = k_1 c_{l_i}(s_1) + (n_1 - 1 - k_1) c_{l_{-i}}(s_1) + n_2 c_m(s_2)$$

Substituting all these cost functions into Eq. (3), optimal profit for firm i in sub-network s_1 (for sub-network s_2 the derivation is analogous) can be written as

$$\pi_i(s_1) = \left[\frac{a - N c_i(s_1) + k_1 c_{l_i}(s_1) + [n_1 - 1 - k_1] c_{l_{-i}}(s_1) + n_2 c_m(s_2)}{\sqrt{b}(1 + N)}\right]^2 - \gamma e_i^2$$

Appendix B. Proof of proposition 2

All stability results follow from standard local analysis and are left to the reader. With respect to the period doubling route to chaos, we observe that the map (18) is topological conjugate to the well known Myrberg quadratic map $q(x) = x^2 - c$ (see, e.g. [24], or [25], chapter 2) through the linear homeomorphism $\tau(x) = \frac{\alpha C_i}{b(1+N)^2} x + \frac{\alpha A_i}{2b(1+N)^2} + \frac{1}{2}$,

[8] Notice that in (A.3) the possible dependence of $c_v(s_w)$ on e_i through external spillovers is evidenced.

with $c = \frac{1}{4}\left(1+\frac{\alpha A_i}{b(1+N)^2}\right)^2 - \frac{1}{2}\left(1+\frac{\alpha A_i}{b(1+N)^2}\right)$. The first flip bifurcation for $q(x)$ at $c = \frac{3}{4}$ translate for Eq. (18) to the condition aforementioned.

Appendix C. Proof of proposition 3

By condition (16), Jacobian matrix (17) computed at E^*, given by Eq. (15), reads

$$J(E^*) = \begin{bmatrix} 1+\dfrac{\alpha_1 C_1 e_1^*}{b(1+N)^2} & \dfrac{\alpha_1 B_1 e_1^*}{b(1+N)^2} \\ \dfrac{\alpha_2 B_2 e_2^*}{b(1+N)^2} & 1+\dfrac{\alpha_2 C_2 e_2^*}{b(1+N)^2} \end{bmatrix}$$

Necessary conditions for stability of E^* can be expressed by

$$\begin{cases} 1 - Tr + Det \geq 0 \\ 1 + Tr + Det \geq 0 \\ \quad Det \leq 1 \end{cases} \tag{C.1}$$

where Tr and Det represent the trace and the determinant of $J(E^*)$ respectively.[9]

These conditions become

$$\begin{cases} \dfrac{\alpha_1\alpha_2 e_1^* e_2^*}{b^2(1+N)^4}(C_1C_2 - B_1B_2) \geq 0 & (i) \\ 4 + \dfrac{2\alpha_1 C_1 e_1^*}{b(1+N)^2} + \dfrac{2\alpha_2 C_2 e_2^*}{b(1+N)^2} + \dfrac{\alpha_1\alpha_2 e_1^* e_2^*}{b^2(1+N)^4}(C_1C_2 - B_1B_2) \geq 0 & (ii) \\ -\dfrac{\alpha_1 C_1 e_1^*}{b(1+N)^2} - \dfrac{\alpha_2 C_2 e_2^*}{b(1+N)^2} - \dfrac{\alpha_1\alpha_2 e_1^* e_2^*}{b^2(1+N)^4}(C_1C_2 - B_1B_2) \geq 0 & (iii) \end{cases} \tag{C.2}$$

From the first condition we get that $C_1C_2 \geq B_1B_2$ is necessary for stability; moreover, if also the last two conditions hold true, by adding them we get $4 + \frac{\alpha_1 C_1 e_1^* + \alpha_2 C_2 e_2^*}{b(1+N)^2} \geq 0$, i.e.

$$4 + \frac{2\alpha_1 A_2 B_1 C_1 + 2C_2(A_1\alpha_2 B_2 - \alpha_1 A_1 C_1 - \alpha_2 A_2 C_1)}{b(C_1C_2 - B_1B_2)(1+N)^2} \geq 0$$

It is interesting to notice that when $C_i \geq 0$, $i = 1, 2$ the first condition implies the second one that in turn is incompatible with the third condition. Hence when $C_i \geq 0$, $i = 1, 2$, equilibrium E^* is never stable.

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[9] Notice that these three conditions, all taken as strict inequalities, give sufficient conditions for stability of E^*.

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