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Equilibrium selection in a nonlinear duopoly game with adaptive expectations

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Abstract

We analyze a nonlinear discrete time Cournot duopoly game, where players have adaptive expectations. The evolution of expected outputs over time is generated by the iteration of a noninvertible two-dimensional map. The long-run behavior is characterized by multistability, that is, the presence of coexisting stable consistent beliefs, which correspond to Nash equilibria in the quantity space. Hence, a problem of equilibrium selection arises and the long run outcome strongly depends on the choice of the players' initial beliefs. We analyze the basins of attraction and their qualitative changes as the model parameters vary. We illustrate that the basins might be nonconnected sets and reveal the mechanism which is responsible for this often-neglected kind of complexity. The analysis of the global bifurcations which cause qualitative changes in the topological structure of the basins is carried out by the method of critical curves.

JEL classification: C72; D43; D83

Keywords: Equilibrium selection problem; Duopoly games; Adaptive expectations; Basins of attraction; Critical curves

1. Introduction

In many economic models, multiple equilibria emerge and stability arguments are often used to select among them. The idea behind this approach is that an equilibrium point is a convention that might arise among players interacting repeatedly. As unstable equilibria are unlikely to be observed as the result of such an evolutionary process, only stable equilibria

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have to be considered. If this stability argument selects a single equilibrium, we can abstract from the process itself with its undesirable dependence on historical accident. However, oftentimes many equilibria survive this refinement, and a situation of strategic uncertainty prevails. The selected equilibrium is then path-dependent, and the observed equilibrium depends on the initial condition.

To address this issue, we consider a simple Cournot-type duopoly market where competitors produce goods which are perfect substitutes and offer them at discrete time periods $t = 0, 1, 2, \dots$ on a common market. The duopolists act in a situation of strategic interdependence. Hence, at each time period each player must form an expectation of the rival's output in the subsequent period in order to determine the corresponding profit-maximizing quantities for period $t+1$. If we denote by $q_i(t)$ the output of firm i at time period t, then the optimization problems through which the firms determine their quantities $q_i(t+1)$ are represented by $\mathrm{Max}_{q_1} \Pi_1(q_1, q_2^e(t+1))$ and $\mathrm{Max}_{q_2} \Pi_2(q_1^e(t+1), q_2)$, respectively, where the function $\Pi_i(\cdot, \cdot)$ denotes the profit of firm i and $q_i^e(t+1)$ the expectation of the competitors' output. If we assume that these optimization problems have unique solutions, then

$$\begin{aligned} q_1(t+1) &= r_1(q_2^e(t+1)) \\ q_2(t+1) &= r_2(q_1^e(t+1)) \end{aligned} \tag{1}$$

where r_1 and r_2 are often referred to as *Best Replies* (or reaction functions).

A lot of work has been done on characterizing the dynamical properties of such a type of system under various assumptions on the expectation formation. Cournot (1838) considered the case of *naive expectations*, $q_i^e(t+1) = q_i(t)$, and studied the resulting dynamics of the system $(q_1(t+1), q_2(t+1)) = (r_1(q_2(t)), r_2(q_1(t)))$, where $r_1(\cdot)$ and $r_2(\cdot)$ were linear functions. Furthermore, oligopoly models with naive expectations and monotone decreasing reaction curves (resulting from assumptions on the demand and cost functions) have been studied intensively in the literature, where the focus has been on *local* stability properties. [1] Since, these models are linear, the stability of the Nash equilibrium is then global, i.e. independent of the initial quantities chosen by the players.

In contrast to this literature, we will focus on different issues. First, we will introduce nonlinearities into Eq. (1) and we will analyze the *global* dynamical properties. In the presence of nonlinearities, local stability of an equilibrium does not imply its global stability. Al-Nowaihi and Levine (1985) emphasize this fact: "However, the Cournot process is an attempt at a solution under the more plausible assumption of partial information. Near equilibrium, it can be claimed to be a reasonable approximation to a richer kind of consistent-expectation formation, and for this reason the Cournot adjustment process is still of interest. It has been suggested that from this argument local stability, but not global stability should be studied. However, this is incorrect. Local stability establishes the stability of a system in some region around the equilibrium, but this region can be so small that *stability in any practical sense does not exist*. Global stability in some defined finite region,

[1] Theocharis (1960) solved the problem of local stability of the unique Nash equilibrium in a linear oligopoly model with n competing firms. His stability result in linear oligopoly games has been generalized to situations where the firms are assumed to use partial adjustment to the Best Response and form adaptive expectations. See, e.g. (Fisher, 1961; Okuguchi, 1976; Szidarovszky and Okuguchi, 1988; Szidarovszky and Yen, 1991).

thus, *strengthens* the local stability and the *observability* of a Cournot equilibrium" (p. 308, emphasis added).

Second, we are interested in situations, where several (Nash) equilibria coexist and an equilibrium selection problem arises. Although, multiple (Nash) equilibria might also occur when the Best Replies are linear, the issue of global dynamics is still easy to analyze (see, e.g. Rassenti et al., 1995). On the other hand, it is well-known that various economic conditions give rise to nonmonotonic Best Replies. Among them are nonlinearities or externalities in the cost function, demand conditions which can be captured by a constant elasticity demand relation or the fact that competitors regard their products as strategic complements. As a consequence, this nonmonotonicity often leads to multiple Nash equilibria (see Furth, 1986; Puhakka and Wissink, 1995; Bulow et al., 1985a; Bulow et al., 1985b; Cooper and John, 1988). The global dynamical properties of such a nonlinear game has not been analyzed so far.[2] Third and finally, following the criticism on the assumption of naive expectations, in this paper we will assume that firms revise their beliefs according to the adaptive expectations rules

$$\begin{aligned} q_1^e(t+1) &= q_1^e(t) + \alpha_1(q_1(t) - q_1^e(t)) \\ q_2^e(t+1) &= q_2^e(t) + \alpha_2(q_2(t) - q_2^e(t)) \end{aligned} \tag{2}$$

where $\alpha_i \in [0, 1]$ are referred to as the *adjustment coefficients*. Adaptive expectations have been proposed in many contexts as a more sophisticated kind of learning rule with respect to naive expectations.[3]

We will introduce a nonlinear Cournot model, where several stable equilibria exist, and demonstrate that the final outcome of the repeated game with adaptive expectations may depend on both the values of the parameters and the *starting conditions of the game*. In this sense, our work can be seen as a contribution to the equilibrium selection problem in dynamic economic models (see, e.g. Van Huyck et al., 1990; Van Huyck et al., 1994; Van Huyck et al., 1997). We will also show that the basins of attraction of coexisting equilibria might have a complex topological structure. Although, the corresponding dynamics may be simple, it will become clear that the basins of these equilibria might be quite complicated sets. With the help of *critical curves* (see, e.g. Mira et al., 1996 or Abraham et al., 1997), we study some global bifurcations which mark the route to an increasing complexity of the basins' structures. Our work can also be considered as a continuation and extension of the work by Day (1994). In his contribution, Day studies multiple-phase dynamical systems in one dimension, and regime-switching, stable, unstable and escape sets. As will become clear later, a lot of the phenomena mentioned in Day's work can be described and illustrated by properties of noninvertible maps.

The organization of the paper is as follows. In Section 2, we give a specification of the model and in Section 3, we present its equilibria. We then deal with the case where players are homogeneous with respect to their Best Replies and their expectation formation. Section

[2] Recently, the analysis of economic dynamic models has been increasingly focused on global dynamic properties. See, e.g. (Brock and Hommes, 1997; Bischi et al., 2000a; Bischi et al., 2000b; Bischi and Naimzada, 1999; Bischi et al., 1999; de Vilder, 1996).

[3] There is empirical evidence that individuals form adaptive expectations (see, e.g. Marimon and Sunder, 1993; Marimon and Sunder, 1994; Yen and Szidarovszky, 1995).

4 gives results on the local stability of the equilibria. Section 5 shows how to analyze the global stability, i.e. the basins of attraction of the equilibria and their qualitative changes as the parameters of the model change. The case of heterogeneous players is studied in Section 6. Finally, Section 7 briefly discusses the issue of chaotic consistent expectations equilibria. All proofs are given in the Appendix.

2. The model

If we consider the Best Replies in (1) together with Eq. (2), which describe how the beliefs of the firms are updated as new information emerges, it seems that our model is represented by a four-dimensional dynamical system. However, it is possible to reduce the dimension of the system by inserting Eq. (1) into Eq. (2). We then obtain the following two-dimensional dynamical system in the belief space [4]

$$\begin{aligned} q_1^e(t+1) &= (1-\alpha_1)q_1^e(t) + \alpha_1 r_1(q_2^e(t)) \\ q_2^e(t+1) &= (1-\alpha_2)q_2^e(t) + \alpha_2 r_2(q_1^e(t)). \end{aligned} \tag{3}$$

The quantities chosen by the competitors are then given, for each $t \geq 0$, by the transformation

$$\begin{aligned} q_1(t+1) &= r_1(q_2^e(t+1)) \\ q_2(t+1) &= r_2(q_1^e(t+1)), \end{aligned} \tag{4}$$

which is a mapping from the belief space into the action space. The steady states of the dynamical system (2), defined by $q_i^e(t+1) = q_i^e(t)$, $i = 1, 2$, i.e.

$$\begin{aligned} q_1^e(t) &= r_1(q_2^e(t)) \\ q_2^e(t) &= r_2(q_1^e(t)) \end{aligned} \tag{5}$$

are located at the intersections of the two reaction curves and are independent of the adjustment coefficients α_1 and α_2. In other words, a steady state is a situation where beliefs are not further revised and, hence, quantities do not change. More importantly, from Eq. (5) it now follows that at the steady states the expected outputs coincide with the realized ones. Hence, in belief space we are considering a situation where beliefs are in this sense consistent and this corresponds to a Nash equilibrium in the quantity space.

In order to get a complete description of our dynamic Cournot game, we have to specify the reaction functions. As mentioned in the introduction, we are interested in microeconomic foundations which gives rise to nonmonotonic Best Replies. Several specifications can be found in the literature (see, e.g. Furth, 1986; Van Witteloostuijn and Van Lier, 1990). However, they all share the disadvantage that an analytical expression for the Best Responses cannot be given or that it is quite complicated. Since, we want to keep the mathematical

[4] Hommes (1998) has taken a similar step in order to study the dynamics of price (expectation) paths in a Cobweb model with adaptive expectations and nonlinear supply and demand. Originally, the dynamics of the system were described by two equations, one for the expectation formation and one for the price dynamics. This system can be reduced to a single difference equation capturing the dynamics in belief space.

analysis tractable, we will use the following well-known type of functions as an illustrative example instead:

$$\begin{aligned} r_1(q_2) &= \mu_1 q_2(1-q_2) \\ r_2(q_1) &= \mu_2 q_1(1-q_1) \end{aligned} \tag{6}$$

Although, the properties of the dynamical system depend on the exact specification of the Best Replies, assuming a quadratic form has the advantage that it gives a simple framework in which we can focus on the main points of our analysis-global dynamics, multiple equilibria and equilibrium selection. Note that it has been shown elsewhere (Kopel, 1996) that the functions given in Eq. (6) can be derived as Best Responses if the competitors regard their products as strategic complements over a certain range of the set of admissible actions. The parameters μ_i, $i = 1, 2$ then measure the intensity of the positive externality the actions of one player exert on the payoff of the other player.

To simplify the notation, let $x(t) = q_1^e(t)$ and $y(t) = q_2^e(t)$. Inserting the reaction functions specified in Eq. (6) into Eq. (3), the time evolution of the competitors' beliefs is obtained by the iteration of the two-dimensional map $T : (x, y) \to (x', y')$ defined by

$$\begin{aligned} x' &= (1-\alpha_1)x + \alpha_1\mu_1 y(1-y) \\ y' &= (1-\alpha_2)y + \alpha_2\mu_2 x(1-x) \end{aligned} \tag{7}$$

where ($'$) denotes the unit-time advancement operator. That is, if the right hand side variables represent the beliefs held by the competitors at time period t, then the left hand side represents the beliefs at time $(t+1)$. Starting from given *initial beliefs* $(x(0), y(0)) = (x_0, y_0)$, the iteration of Eq. (7) uniquely determines a *trajectory of beliefs* in the belief space

$$\tau(x_0, y_0) = (x(t), y(t)) = T^t(x_0, y_0), \quad t = 0, 1, 2, \ldots \tag{8}$$

from which the corresponding sequence of realized outputs is obtained by Eq. (4). In what follows, we will assume that the initial beliefs are chosen in the strategy space $\mathcal{S} = \{[0, 1] \times [0, 1]\}$. Clearly, negative values of quantities or beliefs have no economic meaning, and the reaction functions considered imply that producing more than one is a strictly dominated strategy.

The map (7) contains four parameters: $\mu_i > 0$, $i = 1, 2$, and the adjustment coefficients $\alpha_i \in [0, 1]$, $i = 1, 2$. We will focus on the case $\mu_i \in [1, 4]$, $i = 1, 2$. It is easy to see that if $\mu_i \in [0, 4]$, $i = 1, 2$, then the region $\mathcal{S}$ is trapping for each value of α_i. Any trajectory of beliefs which starts inside the strategy space $\mathcal{S}$ remains inside for each $t \geq 0$. The same is true for the realized outputs, computed by Eq. (4) with reaction functions (6).

3. Equilibrium beliefs and Nash equilibria

Recall that the fixed points of the map (7) defined by $(x, y) = T(x, y)$ are pairs of consistent beliefs. They are obtained as the solutions of the algebraic system $x = \mu_1 y(1-y)$, $y = \mu_2 x(1-x)$, and it is easy to see that these consistent beliefs coincide with the Nash equilibria of the duopoly game (see Kopel, 1996). So, in what follows, we will often use the terms *fixed point, consistent or equilibrium beliefs* and *(Nash) equilibrium* interchangeably.

A simple analytical solution can be found under the assumption that the players are homogeneous with regard to their Best Replies, i.e.

$$\mu_1 = \mu_2 = \mu. \tag{9}$$

Throughout the paper, we will assume that Eq. (9) holds, and that heterogeneity of the players arises only with respect to their belief formation. Under this assumption, besides the trivial solution $O = (0, 0)$, a positive symmetric equilibrium exists for $\mu > 1$, given by

$$E_S = \left(1 - \frac{1}{\mu}, 1 - \frac{1}{\mu}\right),$$

Two further equilibrium beliefs

$$E_1 = (\bar{x}, \bar{y}) \quad \text{and} \quad E_2 = (\bar{y}, \bar{x}) \tag{10}$$

are created at $\mu = 3$, where $\bar{x} = (\mu + 1 + \sqrt{\psi})/2\mu$, $\bar{y} = (\mu + 1 - \sqrt{\psi})/2\mu$ and $\psi = (\mu + 1)(\mu - 3)$. For $\mu > 3$ they are located in symmetric positions with respect to the diagonal Δ. The corresponding Nash equilibria have the same entries. Notice that in equilibrium E_1, the beliefs are such that firm 1 will dominate the market and the realized quantities confirm these beliefs. In the equilibrium E_2 firm 2 dominates.

As mentioned in the introduction, in the presence of multiple Nash equilibria the problem of equilibrium selection arises (see Van Huyck and Battalio, 1998), and this naturally leads to the question of stability (see Van Huyck et al., 1994; Van Huyck et al., 1997). The dynamic process becomes path-dependent, i.e. which equilibrium is chosen in the long run depends on the initial beliefs of the players. In the remainder of the paper we will study the impact of homogeneous and heterogeneous beliefs on the long run properties of the game. We will try to answer the question how the global dynamic properties of the system — structure of the basins in the strategy space $\mathcal{S} = [0, 1]^2$ — depend on the extent of the heterogeneity of the players' beliefs.

4. Homogeneous adaptive expectations

In this section, we will study the local stability of the equilibria under the assumption that players are homogeneous with respect to their expectation formation. Local stability of some equilibrium beliefs means that if the initial beliefs are not too far from the equilibrium, as the game with boundedly rational players is played repeatedly it will eventually reach a situation of consistent beliefs, where no player can gain by unilateral deviation from the realized equilibrium quantities. On the other hand, if an equilibrium is unstable, then, even if the initial expectations are very close to it, repeated choices of the players will move them away from this equilibrium. The trajectory of beliefs may then converge to other equilibrium beliefs or it will fail to converge at all. In the latter situation, players do not learn to play the Nash equilibrium assignments and expectations are consistently wrong.

4.1. Naive expectations as a benchmark case

As pointed out above, many authors have been interested in the dynamics of the Cournot tâtonnement process which is obtained under the assumption that players have naive expectations. Naive expectations are obtained from Eq. (2) in the limiting case $\alpha_1 = \alpha_2 = 1$. In this case, the time evolution of the expected outputs coincides with that of the realized outputs with a delay of one period. [5] The following result provides a starting point for the analysis which follows.

Proposition 1. *For the map (7) with homogeneous Best Replies and naive expectations, i.e.* $\mu_1 = \mu_2 = \mu$ *and* $\alpha_1 = \alpha_2 = 1$, *the following results hold:*

- *for* $0 \le \mu \le 4$ *each trajectory starting from initial beliefs inside the region* $\mathcal{S} = \{[0, 1]\times[0, 1]\}$ *is ultimately bounded inside the trapping square* $\mathcal{R} = [0, \mu/4]\times[0, \mu/4]$*;*
- *for* $1 < \mu < 3$ *the fixed point* $E_S = (1 - 1/\mu, 1 - 1/\mu)$ *is the only stable equilibrium belief;*
- *for* $3 < \mu < 1 + \sqrt{6}$ *three coexisting attractors exist: the stable fixed points* $E_1 = (\bar{x}, \bar{y})$ *and* $E_2 = (\bar{y}, \bar{x})$ *, given by* Eq. (10), *and the stable 2-cycle* $C_2 = (\bar{x}, \bar{x}), (\bar{y}, \bar{y})$.

From the viewpoint of economic dynamics, this proposition is interesting because it states that the existence of multiple equilibrium beliefs implies the existence of 2-cyclic beliefs as well. Moreover, if two stable equilibrium beliefs are present then necessarily a stable cycle of period 2 coexists with them. Accordingly, depending on the initial beliefs, players either learn their Nash equilibrium assignments (and beliefs are consistent) or they end up in a situation where they rely on cyclic inconsistent beliefs repeatedly.

To illustrate our results, in Fig. 1, we present the basins of attraction of the two equilibria E_1 and E_2 and the 2-cycle $C_2 = \{C_2^{(1)}, C_2^{(2)}\}$. The light and dark grey regions represent the basins of E_1 and E_2, respectively, the white region represents the set of points which generate trajectories converging to the 2-cycle. The reason why the basins obtained in this case have such a particular structure is explained in Bischi et al. (2000b). Fig. 1 reveals that *learning is path-dependent*: if the initial beliefs of the players are in the set of light grey points around E_1 (immediate basin), then beliefs converge to E_1 and play converges to a Nash equilibrium in which firm 1 dominates the market. Similarly, if the players hold initial beliefs which are situated in the immediate basin of E_2 then play converges to E_2 , where firm 2 dominates the market. In contrast to this, if initial beliefs are in the set given by the white region around the 2-cycle, then the learning process fails to converge to consistent beliefs and players fail to learn equilibrium assignments. In this case, in the long run players predict situation $C_2^{(1)}$, but instead situation $C_2^{(2)}$ is materialized. Subsequently, they predict $C_2^{(2)}$, but instead $C_2^{(1)}$ is realized, and so forth. It should be noted that each *immediate basin* — the larger connected portion of a basin which includes the attractor itself — is given by a rectangle. Accordingly,

[5] The iterated map under naive expectations, $\Phi : (x, y) \to (r_1(y), r_2(x))$, has been analyzed by, e.g. (Rand, 1978; Dana and Montrucchio, 1986; Bischi et al., 2000b), where it is shown that the resulting dynamical system has some peculiar properties. These are due to the fact that the second iterate is a decoupled map: $\Phi^2(x, y) = \Phi(r_1(y), r_2(x)) = (F(x), G(y))$, where $F(x) = r_1(r_2(x))$ and $G(y) = r_2(r_1(y))$.

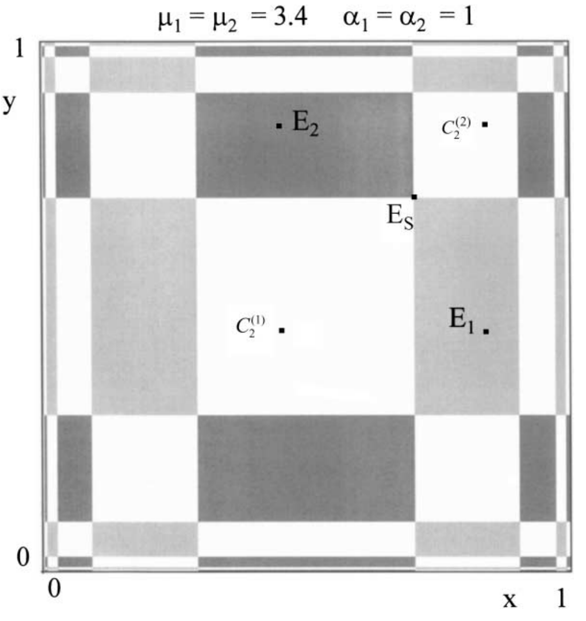

Fig. 1. For $\mu_1 = \mu_2 = 3.4$ the duopoly game with quadratic reaction functions and naive expectations has two stable steady-states (Nash equilibria) E_1 and E_2 and a stable cycle C_2 of period two. These attractors are represented in the strategy space $\mathcal{S} = \{[0, 1] \times [0, 1]\}$. The light and dark grey regions represent the basins of E_1 and E_2, respectively, the white region represents the basin of C_2.

it is not the distance from the initial belief to the equilibrium which determines the convergence properties: there are initial beliefs relatively close to a Nash equilibrium for which the learning process fails to converge. This becomes even more apparent if we consider not only the immediate basin, but the whole basins of attraction, i.e. the set of *all* white, dark grey and light grey points. Observe that the structure of the basins is quite complicated and these sets are nonconnected, despite the fact that the dynamics of the belief paths is simple.

Up to now, the literature on this kind of nonlinear Cournot games was mainly concerned with the analysis of the complex long-run behaviors, i.e. the existence of chaotic attracting sets (see Rand, 1978; Dana and Montrucchio, 1986; Kopel, 1996). Instead, here we focus on another kind of complexity which clearly appears in Fig. 1: it is related to the complex topological structure of the basins, even if the corresponding long-run dynamics are very simple, like the equilibria and the cycle of period 2. Indeed, it has been proven in Bischi et al. (2000b) that the coexistence of many attracting sets with rather complicated basins is an important feature of Cournot models with nonlinear reaction functions. Note that these properties can only be revealed by a global analysis of the dynamic economic model. Later on we will study the transition from simple to complicated basins, and we will reveal the mechanism which causes it.

4.2. Analysis of local stability

We first present a brief analysis of the local stability of the fixed points of the map (7) with the assumption of identical Best Replies (9). We further assume that the two firms are also homogeneous with regard to their expectations formation, i.e. they use the same adjustment coefficients

$$\alpha_1 = \alpha_2 = \alpha \tag{11}$$

in their adaptive expectations rule. Taken together, these two assumptions imply that the two competitors behave identically. In other words, if the firms start with equal beliefs $x(0) = y(0)$ and behave identically over time, then beliefs are equal for each $t \geq 0$, i.e. $x(t) = y(t)$ (and consequently $q_1(t) = q_2(t)$). Mathematically speaking, the map (7) has the following symmetry property: it remains the same if the variables x and y are swapped, i.e. $T(P(x, y)) = P(T(x, y))$, where $P : (x, y) \to (y, x)$ is the reflection through the diagonal $\Delta = \{(x, x), x \in \mathbb{R}\}$. This symmetry property implies that the diagonal Δ is a trapping subspace for the map T, i.e. $T(\Delta) \subseteq \Delta$. It is worth mentioning that the belief trajectories embedded in Δ are governed by the restriction of the two-dimensional map T to Δ, $f = T|_\Delta : \Delta \to \Delta$, where the map f, obtained by setting $x = y$ and $x' = y'$ in Eq. (7), is given by

$$x' = f(x) = (1 + \alpha(\mu - 1))x - \alpha\mu x^2. \tag{12}$$

This map may be interpreted as a simple one-dimensional model of a "representative firm", because the evolution of its beliefs reflects the common behavior of the two competitors.[6]

For the case of homogeneous players, a complete characterization of the stability of the equilibria can be easily obtained for values of the parameters μ and α in the set $\Omega = \{(\mu, \alpha)|\mu > 0, 0 \leq \alpha \leq 1\}$. The following result is presented in detail to make a precise statement about the regions of the parameter space for which equilibria exist and are stable. The proof of this statement is based on a standard analysis of the eigenvalues of the Jacobian matrix.

Proposition 2 (local stability). *Consider the case of homogeneous players, i.e. let* Eqs. (9) *and* (10) *hold. Then*

- *the equilibrium* $E_S = \{1 - 1/\mu, 1 - 1/\mu\}$ *exists for each* $(\mu, \alpha) \in \Omega$ *and it is locally asymptotically stable for* $(\mu, \alpha) \in \Omega^s(E_S)$ *, where*

$$\Omega^s(E_S) = \{(\mu, \alpha) \in \Omega \mid 1 < \mu < 3\}; \tag{13}$$

- *the equilibria* $E_1 = (\bar{x}, \bar{y})$ *and* $E_2 = (\bar{y}, \bar{x})$ *, given by Eq. (10), exist for* $\mu \geq 3$ *and are both locally asymptotically stable for* $(\mu, \alpha) \in \Omega^s(E_i)$*, where*

$$\Omega^s(E_i) = \left\{(\mu, \alpha) \in \Omega \mid \mu > 3,\ 0 < \alpha < \alpha_h(\mu) = \frac{2}{\mu^2 - 2\mu - 3}\right\}; \tag{14}$$

[6] The question when it makes sense to focus the analysis on the model of a representative player instead of the higher-dimensional model with two players is investigated in Kopel et al. (2000); Bischi et al. (1999).

- *in the subset of* $\Omega^s(E_i)$ *given by*

$$\Omega^s(E_i, C_2) = \left\{(\mu, \alpha) \in \Omega^s(E_i) \,|\, \alpha > \alpha_p(\mu) = \frac{6 - \sqrt{12\mu(\mu - 2)}}{3 + 2\mu - \mu^2}\right\} \tag{15}$$

the two stable equilibria $E_i, i = 1, 2$, *coexist with the stable cycle of period two*

$$C_2 = \{(p_1, p_1), (p_2, p_2)\} \in \Delta \tag{16}$$

where $p_1 = (\alpha(\mu - 1) + 2 - \sqrt{\alpha^2(\mu - 1)^2 - 4})/2\alpha\mu$ *and* $p_2 = (\alpha(\mu - 1) + 2 + \sqrt{\alpha^2(\mu - 1)^2 - 4})/2\alpha\mu$

The results given in Proposition 2 are illustrated in Fig. 2. It can be observed that a wide range of parameter values exists which gives coexistence of stable equilibria (the shaded region in Fig. 2). Moreover, for sufficiently high values of the adjustment coefficient α, namely for $\alpha > \alpha_p(\mu)$, also a stable cycle of period 2 coexists with the two stable equilibria. Hence, looking at these results two questions might be raised: first, if the criterion of local stability does not select a unique equilibrium, can we give a global analysis which provides us with more information about the link between initial beliefs and long run outcomes? It should be stressed again that in situations of multistability the local stability properties

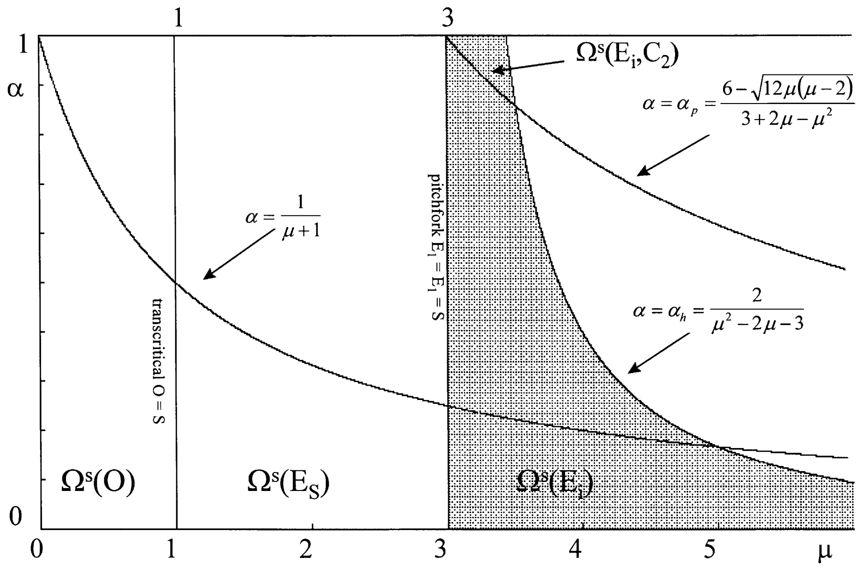

Fig. 2. Space of the parameters $\Omega = \{(\mu, \alpha) \,|\, \mu > 0, 0 \leq \alpha \leq 1\}$ for the map T under the assumption of homogeneous players. $\Omega^s(O)$ represent the set of parameters such that the fixed point O is asymptotically stable, $\Omega^s(E_S)$ represent the set of parameters such that the fixed point E_S is asymptotically stable, $\Omega^s(E_i)$ represents the common stability region of E_1 and E_2, $\Omega^s(E_i, C_2)$ represents the subset of $\Omega^s(E_i)$ where the stable cycle C_2 coexists with the two stable Nash equilibria E_1 and E_2. The portion of the curve of equation $\alpha = 1/(\mu + 1)$ included inside $\Omega^s(E_i)$ represents the set of parameters at which the transition between simply connected and nonconnected basins of the stable Nash equilibria E_1 and E_2 occurs.

are not sufficient to solve the question of equilibrium selection, and the delimitation of the basins of attraction becomes important in order to forecast the long-run evolution of any economic dynamic system. Second, if equilibria are only learned when initial beliefs are chosen from a certain subset of $\mathcal{S}$ and otherwise learning does not occur, it becomes crucial to get information on the relative size of the set of initial beliefs from which players can eventually coordinate their actions (see Mailath, 1998; Fudenberg and Levine, 1998). If initial beliefs from which learning does not occur dominate, then we should be concerned about the robustness of the model. Notice that the results of Section 4.1 are obtained in the limiting case, $\alpha = 1$.

5. Basins of coexisting equilibria in the case of homogeneous adaptive expectations

Stability arguments are often used to select among multiple equilibria (see, e.g. Cox and Walker, 1998). However, when several coexisting stable equilibria are present, each with its own basin of attraction, the initial conditions become crucial to deduce the outcome of the learning process. Recall that the *basin of attraction* of an attractor A is the (open) set of points which generate trajectories converging to A:

$$\mathcal{B}(A) = \{(x, y) | T^t(x, y) \to A \text{ as } t \to +\infty\}. \tag{17}$$

Fig. 3 shows two different structures of the basins of the two coexisting equilibria E_1 and E_2.

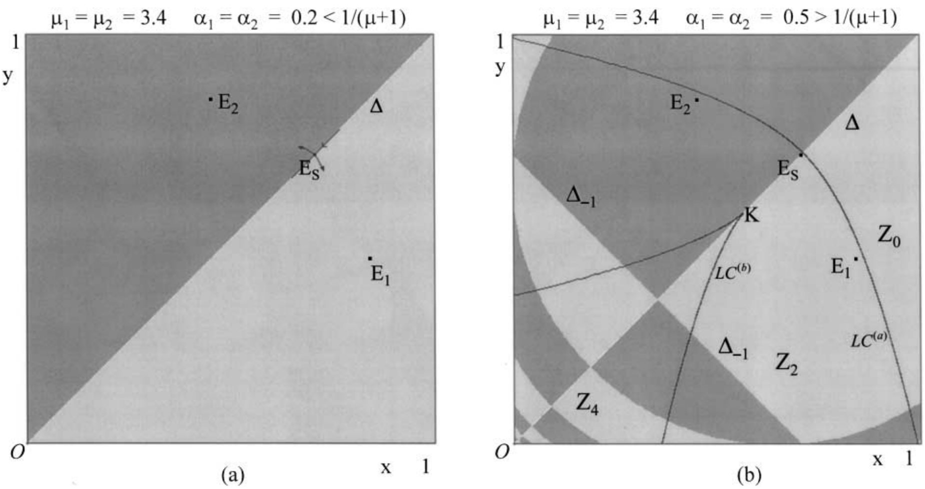

Fig. 3. Representation of the basins of the equilibria E_1 and E_2 in the case of homogeneous behavior. The meaning of the colors is the same as in Fig. 1. (a) With parameters $\mu_1 = \mu_2 = \mu = 3.4$ and $\alpha_1 = \alpha_2 = \alpha = 0.2 < 1/(\mu+1)$, the fixed point $O = (0, 0)$ belongs to the region Z_2 between the two branches $\mathrm{LC}^{(a)}$ and $\mathrm{LC}^{(b)}$ of the critical set LC, and the basins of E_1 and E_2 are simply connected sets. (b) With parameters $\mu = 3.5$ and $\alpha = 0.5 > 1/(\mu + 1)$, the fixed point $O = (0, 0)$ belongs to the region Z_4 bounded by $\mathrm{LC}^{(b)}$, and the basins of E_1 and E_2 are nonconnected sets.

In Fig. 3a, obtained with $\mu_1 = \mu_2 = \mu = 3.4$ and $\alpha_1 = \alpha_2 = \alpha = 0.2$, the basins have a simple structure. If we assume that initial beliefs are chosen randomly from the strategy set $\mathcal{S}$, then our analysis reveals that none of the equilibria is more likely to be observed, as the extent of both basins is equal. This is not a very surprising result, as we study a situation with homogeneous players. However, if we would have information which would allow us to infer that for some reason $q_1^e(0) > q_2^e(0)$ holds for the player's beliefs, then we could conclude that this property of the initial beliefs holds throughout and play converges to the equilibrium E_1. In this case, players learn the equilibrium quantities and eventually play the corresponding Nash equilibrium. On the other hand, if the inequality holds the other way round, then play converges to the equilibrium E_2. In economic terms this means that an initial difference in the expectations of the competitors uniquely determines which of the equilibria is selected in the long run. The expectations of the players become self-fulfilling: if $q_1^e(0) > q_2^e(0)(q_1^e(0) < q_2^e(0))$ then $q_1^e(t) > q_2^e(t)(q_1^e(t) < q_2^e(t))$ for any t and equilibrium E_1, where firm 1 dominates the market (equilibrium E_2 at which firm 2 dominates the market) is eventually selected.

In contrast to this, the situation shown in Fig. 3b, is quite different. It is obtained with the same value of the parameter μ, but with higher values of the adjustment coefficients, namely $\alpha_1 = \alpha_2 = 0.5$. In this case, the basins are no longer simply connected sets, and many portions of each basin are present both in the region above and below the diagonal. When initial beliefs are chosen randomly, we still can say that both equilibria are equally likely to be observed. However, as the basins are now nonconnected sets, the learning process of our dynamic game starting with initial beliefs $q_1^e(0) > q_2^e(0)$ (or $q_1^e(0) < q_2^e(0)$) may lead to convergence to either of the equilibria. Hence, the monotonicity property of beliefs described above is lost, and we would need far more detailed information about the initial beliefs of the players in order to decide which of the equilibria is reached.

The following proposition makes a precise statement about the global bifurcation which causes the change from simply connected to nonconnected basins. In the subsequent paragraphs we only give an intuitive explanation of the mechanism responsible for this structural change. For more rigorous arguments we refer to the Appendix A.

Proposition 3. *If the players are homogeneous with respect to their Best Replies and their expectation formation, i.e.* Eqs. (9) *and* (11) *hold, and* $(\mu, \alpha) \in \Omega^s(E_i)$ *with* $\alpha < \alpha_p(\mu)$, *then the belief trajectories of* Eq. (7) *converge to one of the stable equilibria* E_1 *or* E_2, *given by* Eq. (10). *The common boundary* $\partial\mathcal{B}(E_1) \cap \partial\mathcal{B}(E_2)$ *which separates the basin* $\mathcal{B}(E_1)$ *from the basin* $\mathcal{B}(E_2)$ *is given by the stable set* $W^s(E_S)$ *of the saddle point* E_S. *If* $\alpha(\mu + 1) < 1$ *then* $W^s(E_S) \subset \Delta$ *and the two basins are simply connected sets. If* $\alpha(\mu + 1) > 1$ *then the two basins are nonconnected sets, formed by many simply connected components. At* $\alpha(\mu + 1) = 1$ *a global bifurcation occurs caused by a contact between the cusp point* K *and the point* O.

In order to understand the structure of the basins, and the bifurcation that cause their qualitative changes as the parameter α in the adaptive expectation rule is varied, a study of the global dynamical properties of the dynamical system is necessary. In this context, 'global' refers to an analysis which is not based on the linear approximation of the map (7). In particular, the properties of the inverses of the map become important in order to understand the structure of the basins. This is due to the fact that if A is an attractor of the map and $U(A)$ is a neighborhood of A whose points generate trajectories converging to it, then the *basin of* A is the set of all points which are *eventually* mapped by T into $U(A)$ after a finite number of iterations. In the following, we shall see that the basins may have a complex topological structure if the map T is noninvertible (or "many-to-one"), that is, if distinct points $p_1 \neq p_2$ exist which have the same image, $T(p_1) = T(p_2) = p$. This can be equivalently stated by saying that points p exist which have several rank-1 preimages, i.e. the inverse relation $T^{-1}(p)$ is multivalued.

As the point (x', y') varies in the strategy set $\mathcal{S}$, the number of its rank-one preimages, computed by solving the algebraic system (7) with respect to the unknowns x and y, can change. According to the number of distinct rank-1 preimages associated with each point, the set $\mathcal{S}$ can be subdivided into regions whose points have k distinct preimages. These regions will be denoted by Z_k. Generally, pairs of real preimages appear or disappear as the point (x', y') crosses the boundary separating these regions. Accordingly, such boundaries are characterized by the presence of two coincident (merging) preimages. This leads to the definition of *critical curves* (see Gumowski and Mira, 1980; Mira et al., 1996), where the critical curve of rank-1, denoted by LC (from the French "Ligne Critique") is defined as the locus of points having two, or more, coincident rank-1 preimages. These preimages are located in a set called critical curve of rank-0, denoted by LC_{-1}. Geometrically, the action of a noninvertible map T can be expressed by saying that it "folds and pleats" the plane, so that two or more distinct points are mapped into the same point, or, equivalently, that several inverses are defined which "unfold" the plane. So, the backward iteration of a noninvertible map *repeatedly unfolds* the phase space and this implies that a basin may be nonconnected, i.e. formed by several disjoint portions. This becomes obvious when we consider the relation $\mathcal{B}(A) = \bigcup_{n=0}^{\infty} T^{-n}(\mathcal{B}_0(A))$, where $\mathcal{B}_0(A)$ is the *immediate basin* and $T^{-n}(\mathcal{B}_0(A))$ represents the set of rank-n preimages, which may include sets disjoint from $\mathcal{B}_0(A)$ due to the unfolding of the plane.

A thorough understanding of these properties is important because global bifurcations which give rise to complex topological structures of the basins can be explained in terms of contacts of basins boundaries and critical sets. In fact, if a parameter variation causes a crossing between a basin boundary and a critical set which separates different regions Z_k so that a portion of a basin enters a region where an higher number of inverses is defined, then new components of the basin may suddenly appear. This is the basic mechanism which causes the creation of more and more complex structures of the basins, as we shall see below.

We remark that the role of critical points (relative maxima and minima) in the dynamical properties of one-dimensional iterated maps, also in relation with the problem of the structure of one-dimensional basins, has been already stressed in the economic literature (see, e.g. Lorenz, 1992; Lorenz, 1993; Day, 1994). However, very rarely these methods have been extended to the study of economic models with dimension greater than one (some exceptions are Gardini, 1993; Delli Gatti et al., 1993; Bischi and Naimzada, 1999; Bischi et al., 2000a; Bischi et al., 2000b). Indeed, the set LC is the two-dimensional generalization of the notion of critical value (local minimum or maximum value) of a one-dimensional map, and LC_{-1} is the generalization of the notion of critical point (local extremum point). As in the case of differentiable one-dimensional maps, where the derivative necessarily vanishes at the local extremum points, for a two-dimensional continuously differentiable map the set LC_{-1} belongs to the set of points in which the Jacobian determinant vanishes:

$$LC_{-1} \subseteq \{(x, y) \in \mathbb{R}^2 | \det DT = 0\} \quad (18)$$

(see, e.g. Gumowski and Mira, 1980; Mira et al., 1996). Once the set LC_{-1} is determined, LC is simply obtained as the image of LC_{-1}, i.e. $LC = T(LC_{-1})$.

As we show in the Appendix A, the map T defined in Eq. (7) is a noninvertible map and the strategy set $\mathcal{S}$ can be subdivided into the regions Z_4, Z_2, and Z_0, separated by branches of critical curves LC. The branch $LC^{(a)}$ separates the region Z_0, whose points have no preimages, from the region Z_2, whose points have two distinct rank-1 preimages. The other branch, $LC^{(b)}$, separates the region Z_2 from Z_4, whose points have four distinct preimages. The cusp point K of $LC^{(b)}$, whose coordinates can be easily computed in the case of homogeneous players, plays a crucial role in the analysis for the following reason. When K enters the set $\mathcal{S}$ for $\alpha(\mu + 1) > 1$, suddenly points of $\mathcal{S}$ have a higher number of preimages then before. The unfolding process described above then causes the creation of nonconnected components of the basins. The bifurcation occurring at $\alpha(\mu + 1) = 1$ is a *global bifurcation*, i.e. it cannot be revealed by a study of the linear approximation of the dynamical system. It is characterized by a contact between the stable set of E_S and a critical curve. Accordingly, such a type of bifurcation has been called *contact* (or *nonclassical*) bifurcation in Mira et al., 1996.

With regard to these changes, two interesting issues have to be considered. First, the occurrence of the bifurcation which transforms the basins from simply connected to nonconnected sets causes a loss of predictability about the long-run outcome of the dynamic game. In fact, the presence of the many disjoint components of both basins causes a sensitivity with respect to the initial beliefs, in the sense that a small perturbation of the initial expectations may lead to a crossing of the boundary which separates the two basins and, consequently, the belief trajectory may converge to a different equilibrium. Second, for increasing values of the adjustment coefficient α, as the line Δ_{-1} in Fig. 3b moves upwards, certain connected parts of the basins of the equilibria come (relatively) close to the corresponding other equilibrium. That is, initial beliefs which eventually lead to convergence to E_i are located closely to equilibrium E_j, $i \neq j$. Note, however, that a local analysis would not have been able to provide us with information on the size of the neighborhood around the equilibria from which convergence to the corresponding equilibrium is achieved.

Our analysis so far has shown situations where, given their initial beliefs in $\mathcal{S}$, firms learn to coordinate their actions over time. Although, it can happen that the basins of the two equilibria are nonconnected sets and the equilibrium which is eventually chosen depends on the location of the initial beliefs of the players, their beliefs are eventually consistent and players choose their equilibrium assignments. This changes drastically for $(\mu, \alpha) \in \Omega^s(E_i, C_2)$, i.e. if we take the parameter values of (μ, α) inside the small set of the shaded region in Fig. 2 above the curve $\alpha_p(\mu)$. In this case, three coexisting attractors are present, and the boundaries which separate the basins of E_i and C_2 are given by the stable sets of the two (saddle) cycles which exist for $\alpha > \alpha_p$. The basins are similar to the regions presented in Fig. 1. However, the rectangles which make up the basins of E_i and C_2 are replaced by rounded shapes. If the adjustment coefficient α is further increased until the limiting value

$\alpha = 1$ is reached, the situation shown in Fig. 1 is obtained. Here the boundaries of the basins belong to horizontal and vertical lines, as proven in Bischi et al. (2000b).

6. Heterogeneous adaptive expectations

In this section, we relax the assumption of homogeneous beliefs of the players. We still assume that players have the same Best Replies, i.e. that Eq. (9) holds. However, we now consider a situation where the assumption (11) of identical adjustment coefficients does *not* hold. In this sense, the competitors are heterogeneous with respect to their belief formation process. The first point to note is that the steady-states for this generalized case are the same as those computed in Section 2, because they do not depend on the adjustment coefficients α_i. Analytical expressions of their stability conditions can still be obtained. In contrast to the previous section, it is no longer possible to obtain analytical expressions of the parameter values at which contact bifurcations occur that cause changes of the topological structure of the basins. However, such global bifurcations can be studied by geometrical and computer-assisted proofs based on the method of critical curves, as will be demonstrated below.

The stability regions of the equilibria in the 3-dimensional space of parameters $\Omega_3 = \{(\mu, \alpha_1, \alpha_2) | \mu > 0, 0 \leq \alpha_i \leq 1, i = 1, 2\}$ are defined by the following proposition. The proof is again based on a standard analysis of the eigenvalues of the Jacobian.

Proposition 4. *Let the players be heterogeneous with respect to their adaptive expectations, i.e. let* Eq. (9) *hold. Then*

1. *the steady-state* $E_S = (1 - 1/\mu, 1 - 1/\mu)$ *is stable for* $(\mu, \alpha_1, \alpha_2) \in \Omega_3^s(E_S)$, *where*
$$\Omega_3^s(E_S) = \{(\mu, \alpha_1, \alpha_2) \in \Omega_3 | 1 < \mu < 3\} \tag{19}$$
2. *the fixed points* $E_i, i = 1, 2$, *given in* Eq. (10), *are both stable for* $(\mu, \alpha_1, \alpha_2) \in \Omega_3^s(E_i)$, *where*
$$\Omega_3^s(E_i) = \left\{(\mu, \alpha_1, \alpha_2) \in \Omega_3 | 3 < \mu < 1 + \sqrt{4 + \frac{1}{\alpha_2} + \frac{1}{\alpha_1}}\right\}. \tag{20}$$

Proposition 4 shows that, as in the case of homogeneous beliefs, there exists a rather large set of parameter values for which two stable equilibria exist. Accordingly, like before, an equilibrium selection problem arises and the study of the corresponding basins becomes crucial. Note that the stability regions in Ω_3 must be such that their intersections with the two-dimensional submanifold of Ω_3 defined by the equation $\alpha_1 = \alpha_2$ give the corresponding stability regions for the homogeneous case as described in Proposition 2.

It is easy to see that slight differences between the two adjustment coefficients do not introduce significant changes in the local stability properties (i.e. in the modulus of the eigenvalues). In contrast to this, as will be demonstrated below, even small heterogeneities between the players may cause remarkable effects with regard to the structure of the basins. The main difference between the homogeneous and the heterogeneous case lies in the fact that the diagonal Δ is no longer invariant. Even if the fixed points remain the same, the basins are no longer symmetric with respect to Δ. Nevertheless, many of the arguments given in the previous section for the study of the boundaries of the basins and their global bifurcations continue to hold in the case of heterogeneous beliefs. For example, for $(\mu, \alpha_1, \alpha_2) \in \Omega_3^{\mathrm{s}}(E_i)$ the boundary which separates the basin of equilibrium E_1 from that of E_2 is still formed by the whole stable set $W^{\mathrm{s}}(E_{\mathrm{S}})$, but in the case $\alpha_1 \neq \alpha_2$ the local stable set $W^{\mathrm{s}}_{\mathrm{loc}}(E_{\mathrm{S}})$ is not along the diagonal Δ. The contact between $W^{\mathrm{s}}(E_{\mathrm{S}})$ and $\mathrm{LC}^{(b)}$, which causes the transition from simple to complex basins, does not occur at O (since now $O \notin W^{\mathrm{s}}(E_{\mathrm{S}})$) and no longer involves the cusp point of $\mathrm{LC}^{(b)}$. So, the parameter value at which such contact bifurcation occurs cannot be computed analytically. In what follows, we will demonstrate, however, that the occurrence of these bifurcations can be detected by computer-assisted proofs, based on the knowledge of the properties of the critical curves and their graphical representation (see, e.g. Mira et al. 1996). This "modus operandi", which is typical in the study of the global bifurcations of the two-dimensional maps, has been recently employed by Brock and Hommes (1997) in the analysis of dynamic economic models.

In Fig. 4a, obtained with $\mu = 3.6$, $\alpha_1 = 0.55$ and $\alpha_2 = 0.7$, the two equilibria E_1 and E_2 are stable, and their basins are connected sets. The introduction of an asymmetry in the expectation formation process has a negligible effect on the local stability properties of the equilibria, since the eigenvalues of the two fixed points are exactly the same and are very close to the ones obtained in the homogeneous case with the same value for μ and with $\alpha = (\alpha_1 + \alpha_2)/2$. However, it causes a significant asymmetry in the basins of attraction. As shown in Fig. 4a, when $\alpha_2 > \alpha_1$ the extension of $\mathcal{B}(E_2)$ is greater than the extension

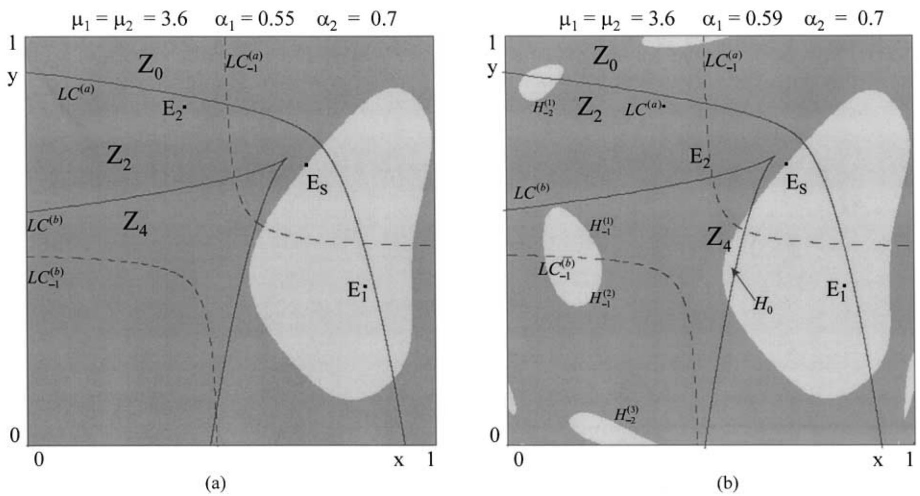

Fig. 4. Basins of E_1 and E_2 when players are heterogeneous with respect to their adaptive expectation rules. The colors have the same meaning as in the previous figures. (a) With $\mu_1 = \mu_2 = 3.6$ and $\alpha_1 = 0.55$, $\alpha_2 = 0.7$ the stable set of the saddle point E_{S}, which constitute the boundary between the two basins $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$, is entirely included in the regions Z_2 and Z_0. The basins are connected sets. (b) For $\mu_1 = \mu_2 = 3.6$ and $\alpha_1 = 0.59$, $\alpha_2 = 0.7$ a portion of the stable set of the saddle point E_{S} belongs to the region Z_4, hence the preimages of the portion H_0 of $\mathcal{B}(E_1)$ inside Z_4 constitute nonconnected portion of $\mathcal{B}(E_1)$ nested inside $\mathcal{B}(E_2)$.

of $\mathcal{B}(E_1)$. This property holds in general: numerical explorations show that *the equilibrium* E_i *dominates the equilibrium* E_j *in terms of the extension of the basin if* $\alpha_i > \alpha_j$. In economic terms, this means that in a situation with heterogeneous beliefs ($\alpha_2 > \alpha_1$) of the two players the set of initial beliefs such that E_2 is learned is larger than the set of initial beliefs for E_1 and asymmetric with respect to the diagonal. Since E_2 is the equilibrium where player 2 dominates in terms of market share, it follows that the player who puts more weight on the most recent observation when updating its belief, eventually "wins" the game in terms of market share in equilibrium. This property even holds when initially the beliefs are such that the firms expect firm 1 to dominate the market, where these initial beliefs are taken from a relatively large set below the diagonal (see Fig. 4a). This shows that even in situations characterized by a simple structure of the basins' boundaries where both basins are connected sets, the statement that the initial ordering of the expectations is maintained along the whole belief trajectory is no longer true. In fact, in the case of heterogenous adaptive learning with $\alpha_i > \alpha_j$, the typical occurrence is that the smaller basin $\mathcal{B}(E_j)$ is surrounded by points of $\mathcal{B}(E_i)$. Hence, the repeated Cournot game may lead to convergence to E_i in the long run, even if players start out with beliefs which are closer to the equilibrium E_j.

However, the situation is not always as simple as in Fig. 4a. The symmetric equilibrium E_S is a (saddle) fixed point which belongs to the boundary, given by the whole stable set $W^s(E_S)$, which separates the two basins. It can be noticed that in the simple situation shown in Fig. 4a, the whole stable set $W^s(E_S)$ is entirely included inside the regions Z_2 and Z_0. The fact that a portion of $W^s(E_S)$ is close to LC suggests that a contact bifurcation may occur if the adjustment coefficients are slightly changed. In fact, if a portion of $\mathcal{B}(E_1)$ enters Z_4 after a contact with $\mathrm{LC}^{(b)}$, new rank-1 preimages of that portion will appear near $\mathrm{LC}_{-1}^{(b)}$ and such preimages must belong to $\mathcal{B}(E_1)$. This is the situation illustrated by Fig. 4b, obtained after a small change of α_1. The portion of $\mathcal{B}(E_1)$ inside Z_4 is denoted by H_0. It has two rank-1 preimages, denoted by $H_{-1}^{(1)}$ and $H_{-1}^{(2)}$, which are located at opposite sides with respect to $\mathrm{LC}_{-1}^{(b)}$ and merge on it (the set H_0 is unfolded by the action of the inverses and, by definition, since it is bounded by a segment of $\mathrm{LC}^{(b)}$ its rank-1 preimages must merge along $\mathrm{LC}_{-1}^{(b)}$; see Appendix A). The set $H_{-1} = H_{-1}^{(1)} \cup H_{-1}^{(2)}$ constitute a nonconnected portion of $\mathcal{B}(E_1)$. Moreover, since H_{-1} belongs to the region Z_4, it has four rank-1 preimages, say $H_{-2}^{(j)}$, $j = 1, \ldots, 4$ (only two of them are entirely contained in the strategy space shown in Fig. 4b) which constitute other "islands"[7] of $\mathcal{B}(E_1)$. Points of these "islands" are mapped into H_0 after two iterations of the map T. Indeed, many higher rank preimages of H_0 exist, thus, giving smaller disjoint "islands" of $\mathcal{B}(E_1)$. Hence, again at the contact between $W^s(E_S)$ and LC the basin $\mathcal{B}(E_1)$ is transformed from a simply connected into a nonconnected set, constituted by many disjoint components. The whole basin is given by the union of the preimages of $\mathcal{B}_0(E_1)$ inside the strategy space $\mathcal{S} = [0, 1]^2$, where $\mathcal{B}_0(E_1)$ is the immediate basin. Our numerical results suggest that even if the introduction of small differences between the adjustment coefficients have, in general, small effects on the properties of the attractors, they may cause remarkable asymmetries in the structure of the basins, which can only be detected when the global properties of the economic model are studied.

[7] We follow the terminology introduced in Mira et al. (1994).

7. Discussion

In the last two decades the literature on dynamic modeling of economic and social systems was mainly concerned with the study of the attracting sets and the bifurcations which lead to more and more complex asymptotic dynamics. However, when several coexisting attractors are present — a situation often met in dynamic models of economic and social systems — another route to complexity is related to more and more complex boundaries which separate the basins of attraction. Generally, these two different routes to complexity are not correlated, in the sense that simple attractors may have complex basins and complex attractors may have simple basin boundaries. For Cournot duopoly games with nonmonotonic reaction curves many authors have shown that attracting sets which are more complex than Nash equilibria may occur, characterized by periodic or even chaotic oscillations around the Nash equilibria (see, e.g. Rand, 1978; Dana and Montrucchio, 1986; Puu, 1991; Puu, 1998; Kopel, 1996). On the other hand, the study of the complexity related to the structure of the basins has been rather neglected in the economic literature. In this paper we have analyzed a Cournot Duopoly game in which the players form their expectations adaptively and the Best Replies of the players are nonmonotonic. We have analyzed its dynamical properties in the belief space, and we have shown that for the nonlinear model multiple (locally) stable (expectation) equilibria can be observed for a large set of parameter values. Accordingly, an equilibrium selection problem and a situation of strategic uncertainty arises. Since stability arguments cannot be used to select one of the two equilibria, information about the basins of the equilibria becomes crucial (see Mailath, 1998; Fudenberg and Levine, 1998). In contrast to the existing work, we have studied global bifurcations that cause qualitative changes of the basins of attraction. We have shown that, despite the local stability of coexisting Nash equilibria, the situation might become quite complicated because basins with complicated topological structures (such as nonconnected sets formed by many disjoint portions) emerge as the adjustment speed (i.e. the extent of inertia of the players) changes.

There is a strong relationship between our work and Day (1994), in particular chapters 6 and 9. Day's multiple-phase dynamical systems are defined on regimes, which are subsets of the phase space. The trajectories of these systems switch from one regime to another if they enter so-called escape sets. On the other hand, if a regime is trapping then it is called stable. Similarly, in our study of the nonconnected basins of the Cournot model (see for examples, Figs. 3 and 4b) the immediate basins are trapping and, hence, stable. The nonconnected portions of the whole basins are the escape sets of properly defined other regimes. Trajectories switch from regime to regime until they finally converge to one of the equilibria and might exhibit various qualitative evolutions depending on the initial conditions. As we have demonstrated, it is important to notice that regime switching is often due to nonconnected basins, where the latter is caused by a global bifurcation. Here, we focused on the transition which creates nonconnected basins, whereas Day did not mention by which mechanism escape sets are created. Furthermore, Day studied only one-dimensional systems, whereas our example is two-dimensional. In this sense, the insights presented here are a continuation and extension of the results given in Day's work.

The final question which we would like to discuss here is "Are the steady states the only consistent belief equilibria?" The motivation for our interest in this topic stems from a recent paper by Hommes (1998), where he studies the consistency of backward-looking expectations like naive or adaptive expectation rules (see also Hommes and Sorger (1998)). The duopoly model considered in this paper is an expectations feedback system: the expectations of the competitors affect the actual outcomes, as can be seen in Eq. (1), and the actual outcomes affect the expected outcomes through the belief formation process (2). If the firms are assumed to be rational, this requires that (eventually) some kind of consistency between expected and actual outcomes emerges. As has been demonstrated above (see Eqs. (4) and (5)), this is the case when play converges to one of the equilibria. However, it is not the case when play converges to an emerging cycle of period two. Since the expectations of the players are consistently wrong and forecasting errors exhibit a systematic pattern, rational agents would recognize these (cyclic) patterns and revise their expectations accordingly. However, things might be different if the trajectory of expected outputs of the two firms (together with the realized quantities) evolves along a chaotic attractor. Although, we excluded these cases from the analysis above, we now consider exactly such a situation. Hommes' argument starts with the observation that chaotic time series can have zero autocorrelations at all lags (see, e.g. Bunow and Weiss, 1979; Sakai and Tomaru, 1980). The importance of this result for dynamic economic models is that expectational errors (i.e. the differences between the expected and the realized quantity) of simple backward-looking expectation rules may also have zero autocorrelations at all lags. Accordingly, agents using such predictors would not see any reason to change their beliefs, since their linear statistical techniques they employ cannot distinguish the error of their predictors from white noise. Hommes calls such predictors *consistent expectations*. Since emerging patterns are very hard to detect in a situation where expectational errors exhibit (close to) zero autocorrelation, he argues that simple expectation rules need not be inconsistent with rational behavior, in particular in the presence of noise.

We have used this idea to investigate if, in addition to the consistent equilibria given above, chaotic consistent expectations equilibria occur in our model. As a representative example of the situations we have analyzed we refer to Fig. 5 (with parameter values $\mu_1 = \mu_2 = 3.8, \alpha_1 = 0.85, \alpha_2 = 0.8$). As can be seen, two chaotic attractors A_1 and A_2 around the (unstable) equilibria E_1 and E_2 coexist, where the basins $\mathcal{B}(A_1)$ and $\mathcal{B}(A_2)$ are represented by light and dark grey regions, respectively. Starting from initial beliefs in the basin $\mathcal{B}(A_l), l = 1, 2$, a belief trajectory is generated by Eq. (3) which, after a short transient period, evolves along the chaotic attractor $A_l, l = 1, 2$. The expectational errors of firm j on the corresponding chaotic attractor can then be easily derived using (4): $e_{t+1} = q_i(t+1) - q_i^e(t+1) = r_i(q_j^e(t+1)) - q_i^e(t+1), \quad i, j = 1, 2, i \neq j$. In order to check if adaptive expectations are consistent in the sense of Hommes (1998), we have computed the (empirical) autocorrelation coefficients ρ_k of lag k of the expectational errors which are given by

$$\rho_k = \frac{\sum_{t=1}^{n-k}(e_t - \overline{e})(e_{t+k} - \overline{e})}{\sum_{t=1}^{n}(e_t - \overline{e})^2}$$

where $\overline{e}$ is the sample mean and n the length of the time series (in our experiments we have chosen $n = 250$).

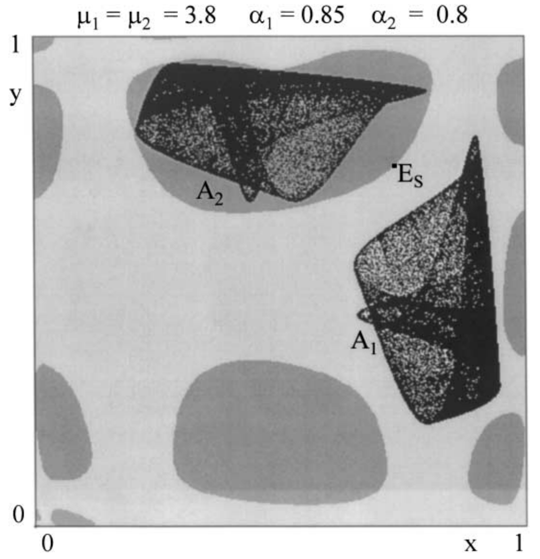

Fig. 5. With $\mu_1 = \mu_2 = \mu = 3.8$ and $\alpha_1 = 0.85$, $\alpha_2 = 0.8$, the equilibria E_1 and E_2 are unstable. Two chaotic attractors A_1 and A_2 coexist, with basins $\mathcal{B}(A_1)$ and $\mathcal{B}(A_2)$, represented by light and intermediate grey, respectively.

We were not able to find any evidence for the occurrence of chaotic consistent expectations equilibria. For all parameter settings we have tried, the autocorrelation coefficients were significantly different from zero (at a 5% confidence level), in particular for small lags. Even in the presence of (small additive or multiplicative) dynamic noise some autocorrelation of the expectational errors remained. Hence, given this evidence, in situations like the one depicted in Fig. 5 rational decision makers should realize that there is (linear) structure in the expectational errors and should revise their beliefs accordingly. We conclude from this that, in the model considered in this paper, the steady states are the only equilibria in which expectations are consistent.

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Appendix A. Proofs of Propositions 1, 2, 4

In order to prove Proposition 1, we need the following Lemma, which is a particular case of a more general statement given in Bischi et al. (2000b).

Lemma. *Let Φ be a map defined as $\Phi : (x, y) \to (r_1(y), r_2(x))$. If $P = (x_p, y_p)$ and $Q = (x_q, y_q)$ are fixed points of the map Φ then*

1. $C_2 = \{(x_p, y_q), (x_q, y_p)\}$ *is a cycle of period* 2 *of* Φ;
2. C_2 *is an attracting cycle if and only if* P *and* Q *are both stable.*

Proof of Proposition 1. Under the assumptions of the proposition the map (7) assumes the form $(x', y') = (\mu y(1-y), \mu x(1-x))$. If $0 < x < 1$ and $0 < y < 1$ then it is immediate to see that $0 < x' < \mu/4$ and $0 < y' < \mu/4$. If $x > 1$ then $y' < 0$ and all the images of higher rank are negative and divergent. The results on the stability of the fixed points follow from the study of the eigenvalues of the Jacobian matrix. As $\mathrm{Tr} = 0$, the modulus is given by $|z_1|^2 = |z_2|^2 = |\mathrm{Det}| = \mu^2(1-2x)(1-2y)$. Then, in O we have $|z_i| = \mu$, hence O is stable for $0 < \mu < 1$; in E_S we have $|z_i| = |2-\mu|$, hence E_S is stable for $1 < \mu < 3$. In E_1 and E_2 we have $|z_i| = |4 + 2\mu - \mu^2|$, so that E_i, $i = 1, 2$, are stable for $3 < \mu < 1 + \sqrt{6}$. The existence and the stability of C_2 follow from the Lemma given above. □

Proof of Proposition 2. Under the assumptions (9) and (11) the Jacobian matrix becomes

$$\mathrm{DT}(x, y) = \begin{bmatrix} 1-\alpha & \alpha\mu(1-2y) \\ \alpha\mu(1-2x) & 1-\alpha \end{bmatrix} \tag{A.1}$$

In the points of the diagonal Δ, DT assumes the structure

$$\mathrm{DT}(x, x) = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \tag{A.2}$$

with $A = 1 - \alpha$ and $B = \alpha\mu(1-2x)$. Such a matrix has real eigenvalues, given by

$$z_{\parallel} = A + B \text{ with eigenvector } \boldsymbol{r}_{\parallel} = (1, 1) \text{ along } \Delta$$

$$z_{\perp} = A - B \text{ with eigenvector } \boldsymbol{r}_{\perp} = (1, -1) \text{ perpendicular to } \Delta.$$

It is easy to see that the product of matrices with the structure (A.2) has the same structure. Hence, all the fixed points and the cycles embedded in the invariant diagonal Δ have real eigenvalues with eigenvectors along Δ and perpendicular to Δ, respectively.

For the symmetric equilibrium $E_S = (1 - 1/\mu, 1 - 1/\mu)$

$$z_{\parallel}(E_S) = 1 + \alpha(1-\mu) \text{ and } z_{\perp}(E_S) = 1 + \alpha(\mu - 3)$$

Being

$$-1 < z_{\parallel}(E_S) < 1 \text{ for } 0 < \alpha(\mu - 1) < 2;$$

$$-1 < z_{\perp}(E_S) < 1 \text{ for } -2 < \alpha(\mu - 3) < 0$$

E_S is a stable node in the region (13).

At $\mu = 3$, $z_{\perp}(E_S) = 1$, E_S loses stability in the direction transverse to Δ through a *supercritical pitchfork bifurcation* (see the remark below) at which the equilibria E_1 and E_2 are created for $\mu > 3$ and are stable just after the bifurcation. After such bifurcation E_S

becomes a saddle point with unstable set in the direction transverse to Δ and local stable set along the invariant diagonal Δ. At $\alpha(\mu-1)=2$, a flip bifurcation along Δ occurs at which E_S becomes a repelling node and a saddle cycle C_2 of period 2, whose periodic points (16) can be easily computed from the restriction of T to Δ, is created along the diagonal Δ, with stable set along Δ and unstable set transverse to it. The eigenvalues of C_2 are the eigenvalues of the matrix $\mathrm{DT}(p_1, p_1)\cdot \mathrm{DT}(p_2, p_2)$, given by

$$z_{\parallel}(C_2) = (1-\alpha+\alpha\mu(1-2p_1))(1-\alpha+\alpha\mu(1-2p_2)) = 5-\alpha^2(\mu-1)^2$$

and

$$\begin{aligned} z_{\perp}(C_2) &= (1-\alpha-\alpha\mu(1-2p_1))(1-\alpha-\alpha\mu(1-2p_2)) \\ &= (3+2\mu-\mu^2)\alpha^2 - 12\alpha + 13 \end{aligned}$$

We have $-1 < z_{\parallel}(C_2) < 1$ for $(2/(\mu-1)) < \alpha < (\sqrt{6}/(\mu-1))$, and $0 < z_{\perp}(C_2) < 1$ for $0 < \alpha < (6-\sqrt{12\mu(\mu-2)})/(3+2\mu-\mu^2)$.

Notice that $\sqrt{6}/(\mu-1) > 1$ for $\mu < 1+\sqrt{6}$. At $\alpha = \alpha_{\mathrm{p}}(\mu) = (6-\sqrt{12\mu(\mu-2)})/(3+2\mu-\mu^2)$ we have $z_{\perp}(C_2) = 1$ and at $\alpha = \alpha_{\mathrm{p}}$ it holds that $(\partial z_{\perp}(C_2))/(\partial\alpha) = -2\sqrt{12\mu(\mu-2)} < 0$.

So, C_2 becomes a stable node for $\alpha > \alpha_{\mathrm{p}}$ and two saddle cycles of period 2 are created through a subcritical pitchfork bifurcation (see the remark below).

In the fixed points E_1 and E_2 the Jacobian matrix (A.1) is given by

$$\mathrm{DT}(E_1) = \begin{bmatrix} A & B_1 \\ B_2 & A \end{bmatrix} \text{ and } \mathrm{DT}(E_2) = \begin{bmatrix} A & B_2 \\ B_1 & A \end{bmatrix},$$

respectively, with $B_1 = -\alpha(1-\sqrt{(\mu+1)(\mu-3)})$ and $B_2 = -\alpha(1+\sqrt{(\mu+1)(\mu-3)})$. It is easy to see that E_1 and E_2 have the same characteristic equation because the two matrices $\mathrm{DT}(E_i)$, $i=1,2$, have the same trace and determinant. Being $\mathrm{Tr}^2 - 4\mathrm{Det} = 4\alpha^2(4+2\mu-\mu^2)$ the eigenvalues are real for $\mu \le 1+\sqrt{5}$ and complex for $\mu > 1+\sqrt{5}$. It is easy to verify that at $\alpha(\mu^2-2\mu-3)=2$ the eigenvalues, exit the unit circle, so that the region of stability of both equilibria E_i, $i=1,2$, is Eq. (14). Furthermore, the two fixed points are transformed from stable to unstable foci through a supercritical Neimark–Hopf bifurcation at which two stable closed orbits are created around the two unstable Nash equilibria E_1 and E_2. □

Remark. A rigorous proof of the supercritical or subcritical nature of a Neimark–Hopf, or Pitchfork, bifurcation requires a center manifold reduction and the evaluation of higher order derivatives, up to the third order (see, e.g. Guckenheimer and Holmes, 1983). This is rather tedious in a two-dimensional map, and we claim numerical evidence in order to ascertain the nature of such bifurcations.

Proof of Proposition 4. Under assumption (9), the Jacobian matrix becomes

$$\mathrm{DT}(x, y) = \begin{bmatrix} 1-\alpha_1 & \alpha_1\mu(1-2y) \\ \alpha_2\mu(1-2x) & 1-\alpha_2 \end{bmatrix} \quad (\text{A.3})$$

The system of inequalities (see, e.g. Gumowski and Mira, 1980, p. 159)

$$P(1) = 1 - \mathrm{Tr} + \mathrm{Det} > 0;\; P(-1) = 1 + \mathrm{Tr} + \mathrm{Det} > 0;\; 1 - \mathrm{Det} > 0$$

gives necessary and sufficient conditions for the two eigenvalues to be inside the unit circle of the complex plane.

At $E_S = (1-1/\mu, 1-1/\mu)$, we have $\mathrm{Tr}^2 - 4\mathrm{Det} = \alpha_1^2+\alpha_2^2+14\alpha_1\alpha_2+4\alpha_1\alpha_2\mu(\mu-4) \geq (\alpha_1-\alpha_2)^2 \geq 0$, being $\mu(\mu-4) \geq -4$. So the eigenvalues are always real at the fixed point E_S, and the stability conditions reduce to

$$P(1) = \alpha_1\alpha_2(-\mu^2 + 4\mu - 3) > 0 \text{ for } 1 < \mu < 3;$$

$$P(-1) = \alpha_1\alpha_2\mu^2 - 4\alpha_1\alpha_2\mu + 3\alpha_1\alpha_2 + 2(\alpha_1+\alpha_2) - 4 > 0 \text{ for } \mu < 2 + \sqrt{1 + 2\frac{2-(\alpha_1+\alpha_2)}{\alpha_1\alpha_2}}.$$

Then, E_S is a stable node in the region (19).

At $\mu = 1$ a transcritical bifurcation occurs at which O and E_S exchange stability, and at $\mu = 3$ a pitchfork bifurcation of E_S occurs at which the fixed points E_1 and E_2 are created.

Since

$$\mathrm{DT}(E_1) = \begin{bmatrix} 1-\alpha_1 & -\alpha_1(1-\sqrt{(\mu+1)(\mu-3)}) \\ -\alpha_2(1+\sqrt{(\mu+1)(\mu-3)}) & 1-\alpha_2 \end{bmatrix}$$

and

$$\mathrm{DT}(E_2) = \begin{bmatrix} 1-\alpha_1 & -\alpha_1(1+\sqrt{(\mu+1)(\mu-3)}) \\ -\alpha_2(1-\sqrt{(\mu+1)(\mu-3)}) & 1-\alpha_2 \end{bmatrix}$$

it is easy to realize that E_1 and E_2 have the same characteristic equation. The fixed points E_i are transformed from stable nodes into stable foci when

$$\mathrm{Tr}^2 - 4\mathrm{Det} = -4\alpha_1\alpha_2\mu^2 + 8\alpha_1\alpha_2\mu + 14\alpha_1\alpha_2 + \alpha_1^2 + \alpha_2^2 = 0,$$

i.e. at $\mu = 1 + \sqrt{(9/2) + (\alpha_1/4\alpha_2) + (\alpha_2/4\alpha_1)}$. In this case, since $P(1) = \alpha_1\alpha_2(\mu+1)(\mu-3) > 0$ for $\mu > 3$ and $P(-1) = 4 - 2(\alpha_1+\alpha_2) + \alpha_1\alpha_2(\mu+1)(\mu-3) > 0$ for $\mu > 3$, the stability conditions for E_i, $i = 1, 2$, reduce to

$$\mathrm{Det} - 1 = \alpha_1\alpha_2\mu^2 - 2\alpha_1\alpha_2\mu - 3\alpha_1\alpha_2 - \alpha_1 - \alpha_2 < 0$$

which is satisfied in the region $\Omega_3^s(E_i)$ of the parameters space Ω_3. The equation

$$\mu = 1 + \sqrt{4 + \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}}$$

defines a bifurcation surface through which a supercritical Neimark–Hopf bifurcation occurs (see the remark given above). □

Appendix B. Critical curves and proof of Proposition 3

In this appendix, we first describe some properties of the critical curves of the map T and then we prove Proposition Eq. (3). The map T defined in Eq. (7) is a noninvertible map. In fact, given a point $(x', y') \in \mathbb{R}^2$ its preimages are computed by solving the following algebraic system obtained from Eq. (7) with respect to x and y:

$$\begin{cases} (1-\alpha_1)x + \alpha_1\mu_2 y(1-y) = x' \\ (1-\alpha_2)y + \alpha_2\mu_1 x(1-x) = y' \end{cases} \tag{B.1}$$

This is a fourth degree algebraic system, which may have four or two real solutions or no real solution at all. For example, let us consider the origin $O = (0, 0)$. Under the assumptions (9) and (11) its rank-1 preimages can be analytically computed by solving the algebraic system (B.1) with $x' = y' = 0$. If $0 < \alpha < 1/(\mu+1)$ there are just two rank-1 preimages of O, both belonging to Δ: one is O itself (being O a fixed point) the other one is

$$O_{-1}^{(1)} = \left(\frac{1+\alpha(\mu-1)}{\alpha\mu}, \frac{1+\alpha(\mu-1)}{\alpha\mu}\right) \tag{B.2}$$

which can be easily computed by using the one-dimensional restriction $f(x) = (1+\alpha(\mu-1))x - \alpha\mu x^2$ of T to the diagonal. If $\alpha > 1/(\mu+1)$ then other two rank-1 preimages exist, because $O \in Z_4$. These two further preimages, $O_{-1}^{(2)}$ and $O_{-1}^{(3)}$, are located on the line Δ_{-1} of equation [8]

$$x + y = 1 + \frac{1}{\mu}\left(1 - \frac{1}{\alpha}\right). \tag{B.3}$$

in symmetric positions with respect to Δ. Hence

$$O_{-1}^{(2)} = \left(\frac{\alpha(\mu+1) - 1 + \sqrt{\alpha^2\mu^2 + 2\alpha\mu(1-\alpha) - 3(\alpha^2+1) + 6\alpha}}{2\alpha\mu}, \frac{\alpha(\mu+1) - 1 - \sqrt{\alpha^2\mu^2 + 2\alpha\mu(1-\alpha) - 3(\alpha^2+1) + 6\alpha}}{2\alpha\mu}\right) \tag{B.4}$$

and the symmetric point $O_{-1}^{(3)}$ is obtained from $O_{-1}^{(2)}$ by swapping the two coordinates.

For the map (7) LC_{-1} coincides with the set of points at which $\det \mathrm{DT} = 0$, i.e.

$$\left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right) = \frac{(1-\alpha_1)(1-\alpha_2)}{4\alpha_1\alpha_2\mu_1\mu_2} \tag{B.5}$$

This equation represents an equilateral hyperbola, so LC_{-1} is formed by the union of two disjoint branches, say $\mathrm{LC}_{-1} = \mathrm{LC}_{-1}^{(a)} \cup \mathrm{LC}_{-1}^{(b)}$, see Fig. 6a. Also $\mathrm{LC} = T(\mathrm{LC}_{-1})$ is the union of two branches: $\mathrm{LC}^{(a)} = T(\mathrm{LC}_{-1}^{(a)})$ and $\mathrm{LC}^{(b)} = T(\mathrm{LC}_{-1}^{(b)})$. The branch $\mathrm{LC}^{(a)}$ separates the region Z_0, whose points have no preimages, from the region Z_2, whose points have two

[8] This can be seen by setting $x' = y'$ in Eq. (B.1) and adding or subtracting the two symmetric equations.

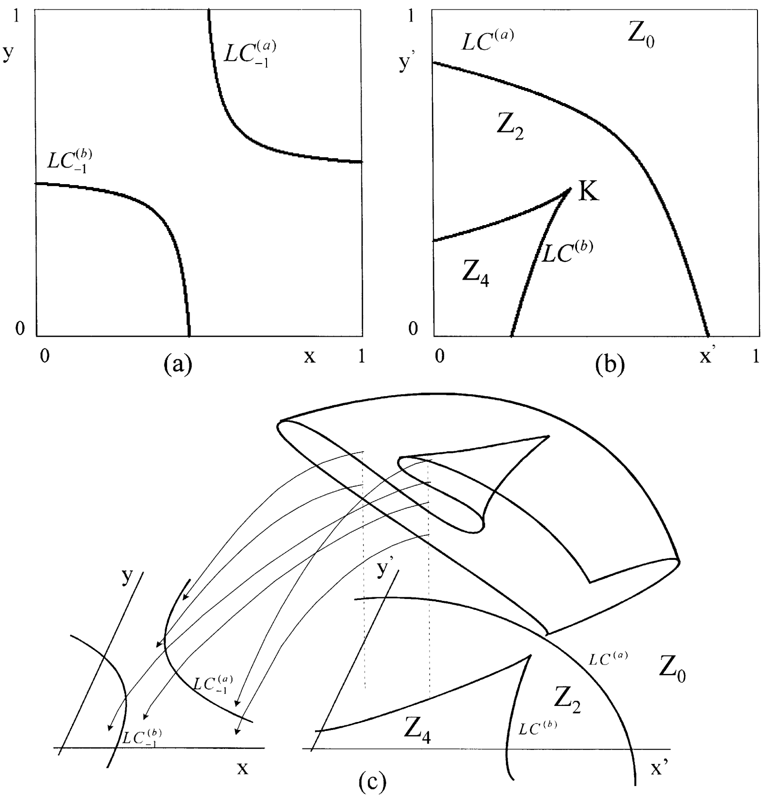

Fig. 6. (a) Critical curves of rank-0, obtained as the locus of points such that $\det(DT(x, y)) = 0$. (b) Critical curves of rank-1, obtained as $LC = T(LC_{-1})$. These curves separate the plane into three regions, denoted by Z_4, Z_2 and Z_0 whose points have four, two or no rank-1 preimages, respectively. (c) Riemann foliation of the plane. With each point of the region Z_4 four distinct inverses are associated, each defined on a different sheet of the foliation, whereas points of Z_2 are associated with two sheets. The projection on the phase plane of the folds connecting different sheets are the critical curves LC.

distinct rank-1 preimages. The other branch $LC^{(b)}$ separates the region Z_2 from Z_4, whose points have four distinct preimages (see Fig. 6b). Following the terminology of Mira et al. (1994), we say that the map (7) is a noninvertible map of $Z_4 > Z_2 - Z_0$ type, where the symbol ">" denotes the presence of a cusp point in the branch $LC^{(b)}$.

The coordinates of the cusp point of $LC^{(b)}$ can be easily computed in the symmetric case, i.e. when Eqs. (9) and (11) hold. In fact, in any point of LC_{-1} at least one eigenvalue of DT vanishes. In the point

$$C_{-1} = LC_{-1}^{(a)} \cap \Delta = (c_{-1}, c_{-1}) \text{ with } c_{-1} = \frac{\alpha(\mu - 1) + 1}{2\alpha\mu}$$

the eigenvalue $z_{\parallel}$ with eigendirection along Δ vanishes, and its image

$$C = LC^{(a)} \cap \Delta = (c, c) \text{ with } c = f(c_{-1}) = \frac{(\alpha(\mu - 1) + 1)^2}{4\alpha\mu}$$

is the point at which $LC^{(a)}$ intersects Δ. This point corresponds to the unique critical point of rank-1 (maximum value) of the restriction $f(x)$ of T to Δ. At the other intersection of LC_{-1} with Δ, given by

$$K_{-1} = LC_{-1}^{(b)} \cap \Delta = (k_{-1}, k_{-1}) \text{ with } k_{-1} = \frac{\alpha(\mu-1)-1}{2\alpha\mu}$$

the eigenvalue $z_\perp$ vanishes and the curve $LC^{(b)} = T(LC_{-1}^{(b)})$ has a *cusp point* (see, e.g. Arnold et al., 1986)

$$K = LC^{(b)} \cap \Delta = (k, k) \text{with } k = f(k_{-1}) = \frac{(\alpha(\mu+1)-1)(\alpha\mu+3(1-\alpha))}{4\alpha\mu} \tag{B.6}$$

In order to give a geometrical interpretation of the "unfolding action" of the multivalued inverse relation T^{-1}, it is useful to consider a region Z_k as the superposition of k sheets, each associated with a different inverse. Such a representation is known as Riemann foliation of the plane (see, e.g. Mira et al. 1996). Different sheets are connected by folds joining two sheets, and the projections of such folds on the phase plane are arcs of LC. The foliation associated with the map (7) is qualitatively represented in Fig. 6c. It can be noticed that the cusp point of LC is characterized by three merging preimages at the junction of two-folds.

Proof of Proposition 3. Let us assume that $(\mu, \alpha) \in \Omega^{s}(E_i)$ and $\alpha < \alpha_p(\mu)$, i.e. the two equilibria E_1 and E_2 are the only attractors. The boundary separating $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ contains the symmetric equilibrium E_S as well as its whole stable set $W^s(E_S)$. In fact, just after the bifurcation occurring at $\mu = 3$, at which the two stable fixed points E_1 and E_2 are created, the symmetric equilibrium $E_S \in \Delta$ is a saddle point, and the two branches of the unstable set $W^u(E_S)$ departing from it reach E_1 and E_2, respectively. Hence, since a basin boundary is backward invariant (see Mira et al., 1994; Mira et al., 1996), not only the local stable set $W^s_{loc}(E_S)$ belongs to the boundary that separates the two basins, but also its preimages of any rank: $W^s(E_S) = \bigcup_{k \geq 0} T^{-k}(W^s_{loc}(E_S))$. Because of the symmetry property of the system (7) with homogeneous players, the local stable set of E_S belongs to the invariant diagonal Δ. Indeed, as long as $\alpha(\mu+1) < 1$, the whole stable set $W^s(E_S)$ belongs to Δ and is given by $W^s(E_S) = OO_{-1}^{(1)}$, where $O_{-1}^{(1)}$ is the preimage of O located along Δ. In fact, if $\alpha(\mu+1) < 1$ holds, the cusp point K of the critical curve $LC^{(b)}$ has negative coordinates and, consequently, the whole segment $OO_{-1}^{(1)}$ belongs to the regions Z_0 and Z_2. This is the situation shown in Fig. 3a.

This implies that the two preimages of any point of $OO_{-1}^{(1)}$ belong to Δ (they can be computed by the restriction (12) of T to the invariant diagonal Δ). This proves that the segment $OO_{-1}^{(1)}$ is backward invariant, i.e. it includes all its preimages. The structure of the basins $\mathcal{B}(E_i)$, $i = 1, 2$, is very simple: $\mathcal{B}(E_1)$ is entirely located below the diagonal Δ and $\mathcal{B}(E_2)$ is entirely located above it (see Fig. 3a). Both of the basins $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ are simply connected sets.

Their structure becomes a lot more complex for $\alpha(\mu+1) > 1$. In order to understand the bifurcation occurring at $\alpha(\mu+1) = 1$, we consider the critical curves of the map (7): at $\alpha(\mu+1) = 1$ a contact between $LC^{(b)}$ and the fixed point O occurs, due to the merging between O and the cusp point K. For $\alpha(\mu+1) > 1$ the portion KO of $W^{s}_{loc}(E_S)$ belongs to the region Z_4 where four inverses of T exist. This implies that besides the two rank-1 preimages on Δ the points of KO have two further preimages, located on the segment $O^{(2)}_{-1}O^{(3)}_{-1}$ of the line Δ_{-1}. Since $OO^{(1)}_{-1} = W^{s}_{loc}(E_S) \subset \partial\mathcal{B}(E_1) \cap \partial\mathcal{B}(E_2)$, also its preimages of any rank belong to the boundary which separates $\mathcal{B}(E_1)$ from $\mathcal{B}(E_2)$. So the rank-1 preimages of the segment $O^{(2)}_{-1}O^{(3)}_{-1}$, which exist because portions of it are included in the regions Z_2 and Z_4, belong to $W^{s}(E_S)$ as well, being preimages of rank-2 of $OO^{(1)}_{-1}$. This repeated procedure, based on the iteration of the multivalued inverse of T, leads to the construction of the whole stable set $W^{s}(E_S)$. □

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