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Chaos, Solitons and Fractals 18 (2003) 561–576

CHAOS
SOLITONS & FRACTALS

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Multistability and path dependence in a dynamic brand competition model

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Abstract

We introduce a dynamic market share attraction model where agents are boundedly rational. They follow a simple rule of thumb which is based on marginal profits to determine their actions over time. We show that multistability arises, i.e. several attractors coexist. In such a situation the selected long run state becomes path-dependent, and a thorough knowledge of the basins and their structure becomes crucial for the researcher to be able to predict the long run outcome of the economic system. We show that the basins of the coexisting attracting sets might have quite complicated structure. Furthermore, we give insights into the mechanism which is responsible for the creation of complex basins of attraction.

1. Introduction

For a long time game-theoretic models focused on the question of outcomes in static games, where static in this context refers to the fact that players meet only once and by some kind of process of introspection simultaneously and immediately choose strategies which correspond to a (Nash) equilibrium. Players were assumed to be fully rational, i.e. have perfect knowledge about every detail of the game they are playing (and also know that the same is true for the other players). In the more recent literature on dynamic games, these assumptions have been relaxed in two ways. First, situations are considered where players interact with each other repeatedly over time. Second, these players choose their actions (or strategies) by trial-and-error methods which require less information with respect to fully rational decision making. Agents behave adaptively and adjust their strategies to changes in and responses from the environment they are living in; see e.g. [4,14,18,28]. The resulting models more closely resemble real world situations, where complexities and difficulties arise when agents are faced when making decisions. For example, it is obvious that the ability of agents to compute optimal solutions is limited, that it is difficult to foresee all contingencies in the future and that also oftentimes it is prohibitively costly to calculate and implement an optimal plan of action.

On the other hand, in such a framework of boundedly rational behavior, new research questions arise: Does a reasonable adaptive process (e.g. based on the best responses of the players) converge to anything? If so, to what does it converge in the long run? A related problem which often arises in the study of dynamic games concerns the coexistence of several attractors, each with its own basin of attraction. In such a situation where multiple attractors coexist, which of these attractors characterizes the long run dynamics of the game? Here, an adaptive mechanism might clarify which long run outcome one might expect (see [6,27]). Of course, if *multistability* prevails, the selected long run outcome is path-dependent and the choices of the initial actions (the initial condition) are of crucial importance [5]. Stable Nash

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0960-0779/03/$ - see front matter
doi:10.1016/S0960-0779(02)00677-X

equilibria might coexist with other kinds of attractors and the boundedly rational players may in the long run learn to play Nash equilibrium strategies or they may continue to play a different set of strategies that are not part of any equilibrium selected by fully rational players. To get more insights into these issues one is quite naturally led to study the basins of attraction, which requires a global analysis of the underlying dynamical system. In fact, a local stability analysis, based on the linear approximation of the dynamical system around the attractors, is not enough to characterize the structure of the basins and their qualitative changes. Local stability means that the game converges to a particular attracting set in the long run, provided that the initial strategies are sufficiently close to it. On the other hand, interesting phenomena may occur when the game starts far away from an equilibrium (or, in general, from an attracting set), since global dynamic properties may influence the time path. However, this question and, related to it, the study of the complex structure of the basins has been rather neglected in the economics and game theory literature.

An investigation of global bifurcations that change the qualitative structure of the basins is particularly challenging in the case of discrete time dynamical systems governed by the iteration of noninvertible maps. Indeed, in this case the basins may have complicated topological structures, since they may be multiply-connected or nonconnected sets, often formed by the union of infinitely many disjoint portions. With the help of recent results on basin bifurcations in noninvertible maps (see e.g. [1,2,6,7,22]), insights into the structure of the basins and into the creation of complex basin boundaries can be obtained. As some parameter is varied, such changes in the structure can be characterized by global bifurcations: they are the consequence of *contact bifurcations*, i.e. due to contacts between critical sets and invariant sets (such as fixed points or cycles or their stable sets). For two-dimensional maps, such kinds of bifurcations can be very rarely studied by analytical methods, since the equations of such singularities are not known in general. Hence these global bifurcations are mainly studied by geometrical and numerical methods. For recent applications of this approach to models arising in economics, game theory or finance, see e.g. [2,6,8,24,25].

Finally, before we introduce our model in the next section, we would like to point out that in the literature on dynamic games two different routes to complexity have been studied. The first one is related to the complexity of the attracting sets which characterize the long run evolution of the dynamic process and describe the evolution of players' actions over time (see, e.g. [20,23]). The second one focuses on the complexity of the boundaries which separate the basins when several coexisting attractors are present. It is important to realize that these two different kinds of complexity are not related. Very complex attractors may have simple basin boundaries, whereas boundaries which separate the basins of simple attractors, such as coexisting stable equilibria, may have very complex structure. Since we feel that for game theoretic considerations the second line of research is more important, here we will mainly focus on the second kind of complexity and consider a global analysis of dynamic games. In our model for market share competition where two boundedly rational agents interact repeatedly, we will study the basins of attraction of long run outcomes, the basin boundaries in situations of multistability, and the corresponding changes when structural parameters of the games are changed.

The paper is organized as follows. In Section 2 we introduce the model and characterize the fixed points of the map and in Section 3 we briefly focus on the properties of the symmetric model of homogeneous brands. A more general analysis of the two different cases of homogeneous and heterogeneous brands is given in Section 4 (linear speed of adjustment) and Section 5 (quadratic speed of adjustment). Section 6 concludes.

2. The model

Market share attraction models have been used in a variety of contexts to describe the behavior of competitors in a market. Not only have they been employed frequently in empirical applications, they are also prevalent in the economics, game theory and operations research literature. Market share attraction models have also received increasing attention among marketing researchers in recent years. This type of model specifies that the market share of a competitor is equal to its attraction divided by the total attraction of all the competitors in the market, where each competitors' attraction is given in terms of competitive effort allocations. It has the theoretically appealing property that it is logically consistent: it yields market shares that are between zero and one, and sum to one across all the competitors in the market. For simplicity, let us consider the case of two brands, which compete against each other on the basis of both the quality and the magnitude of the marketing effort expended for each brand. Let B denote the sales potential of the market (in terms of customer market expenditures). If x units of marketing effort are expended for brand 1 and y units of marketing effort are expended for brand 2, then the resulting shares of the market in monetary terms accruing to brand 1 and to brand 2 are Bs_1 and $Bs_2 = B - Bs_1$, respectively, where

$$s_1 = \frac{ax^{\beta_1}}{ax^{\beta_1} + by^{\beta_2}}; \quad s_2 = \frac{by^{\beta_2}}{ax^{\beta_1} + by^{\beta_2}} \tag{1}$$

The expressions $A_1 = ax^{\beta_1}$ and $A_2 = by^{\beta_2}$ represent the attractions of customers to brand 1 and 2, respectively, given the expenditures of x and y units of effort. [1] The parameters a and b denote the relative effectiveness of the efforts. Since

$$\frac{dA_1}{dx}\frac{x}{A_1} = \beta_1 \quad \text{and} \quad \frac{dA_2}{dx}\frac{x}{A_2} = \beta_2$$

the parameters β_1 and β_2 denote the elasticity of the attraction of brand i with respect to the effort expended for brand i. We refer to these parameters as response parameters. Note that the payoffs for each of the brands depend on the efforts expended for both brands. In the existing literature market share attraction models are predominantly used in a *static* framework. Here the emphasis of the investigation lies on demonstrating the existence and uniqueness of (Nash) equilibria; see [13,21,26].

A dynamic version of a market share attraction model has been considered in Bischi et al. [8], where it is assumed that brand managers choose the next period's marketing effort according to the current profits. In the present paper, we propose a different decision rule, namely one that is based on the perceived marginal profits. At time t the marketing efforts for the next period, $x(t+1)$ and $y(t+1)$ are determined according to the following adjustment process:

$$\begin{aligned} x(t+1) &= x(t) + \lambda_1(x)\left[\frac{\partial \pi_1(x(t), y(t))}{\partial x}\right] \\ y(t+1) &= y(t) + \lambda_2(x)\left[\frac{\partial \pi_2(x(t), y(t))}{\partial y}\right] \end{aligned} \tag{2}$$

where π_1 and π_2 represent the one-period profits of firm 1 and 2 respectively:

$$\begin{aligned} \pi_1 &= m_1 B s_1 - c_1 x \\ \pi_2 &= m_2 B s_2 - c_2 y \end{aligned} \tag{3}$$

with Bs_i = share of firm i in terms of the total sales potential; m_i = profit margin per unit sold by firm i (gross profit margin); c_i = marginal cost of firm i. The expressions $\lambda(\cdot)$ determine how much effort allocations can vary from period to period following a given profit signal. It can be interpreted as the "speed of reaction" of the brand managers.

If we use the expressions for the market shares s_1 and s_2 in (1) and substitute them into (3), the resulting dynamic market share attraction model (2) becomes:

$$T : \begin{cases} x(t+1) = x(t) + \lambda_1(x(t))[m_1\beta_1 B F_1(x(t), y(t)) - c_1] \\ y(t+1) = y(t) + \lambda_2(y(t))[m_2\beta_2 B F_2(x(t), y(t)) - c_2] \end{cases} \tag{4}$$

where

$$F_1(x,y) = \frac{s_1 s_2}{x} = k\frac{x^{\beta_1} y^{\beta_2}}{x(x^{\beta_1} + k y^{\beta_2})^2}; \quad F_2(x,y) = \frac{s_1 s_2}{y} = k\frac{x^{\beta_1} y^{\beta_2}}{y(x^{\beta_1} + k y^{\beta_2})^2}$$

with $k = b/a$. The symbol T will be used in the following to denote the two-dimensional map whose iteration defines the time evolution of the marketing efforts for the two competing brands, according to $T : (x(t), y(t)) \to (x(t+1), y(t+1))$. The corresponding time evolution of market shares is obtained by (1).

Obviously, the map (4) is defined only for positive values of the dynamic variables x and y. Hence, the first question to study is under which conditions the positivity of the trajectories is obtained. In fact, starting from a given initial state (or initial condition) x_0, y_0, a *feasible* time evolution of the system is obtained only if the trajectory $(x_t, y_t) = T^t(x_0, y_0)$, $t = 0, 1, 2, \dots$ is entirely contained in the positive orthant $\mathbb{R}^2_+ = \{(x,y) | x > 0 \text{ and } y > 0\}$. We call a trajectory feasible if it has the property stated above, and we will call feasible set the subset of $\mathbb{R}^2_+$ whose points generate feasible trajectories. In the study of a dynamical model used to simulate the time evolution of an economic system, another important property to check is that of boundedness of the trajectories. However, for the model (4) it is easy to prove that, the infinity is repelling, as stated by the following proposition:

Proposition 1. *Any feasible trajectory of the discrete dynamical system* (4) *is bounded.*

The proof of this proposition is straightforward. In fact, as $0 < s_1 s_2 < 1$, we have $x(t+1) < x(t)$ whenever $x(t) > m\beta_1 Bk/c_1$, and $y(t+1) < y(t)$ whenever $y(t) > m\beta_2 Bk/c_2$.

[1] In the marketing literature market share attraction models are often times used to describe the competition between several brands of a product in the market; see, e.g., [12,17]. The models are then oftentimes referred to as brand competition models.

2.1. Fixed points and Nash equilibria

The steady states of the dynamical model (4), are the fixed points of the map T, defined by the condition $T(x,y) = (x,y)$, i.e.

$$\begin{cases} \lambda_1(x)\dfrac{\partial \pi_1}{\partial x} = 0 \\ \lambda_2(y)\dfrac{\partial \pi_2}{\partial y} = 0 \end{cases}$$

The positive solutions of the system

$$\frac{\partial \pi_1}{\partial x} = 0; \quad \frac{\partial \pi_2}{\partial y} = 0, \tag{5}$$

which yield maximum profits, are Nash equilibria, and it is easy to see that these solutions are also fixed points of the map (4). In other words, if effort allocations coincide with a Nash equilibrium profile, the system remains there forever. Note, however, that the converse is not true in general, since fixed points which are not Nash equilibria may exists. They are given by the solutions of one of the systems

$$\begin{cases} \lambda_1(x) = 0 \\ \lambda_2(y) = 0 \end{cases} \quad \text{or} \quad \begin{cases} \lambda_1(x) = 0 \\ \dfrac{\partial \pi_2}{\partial y} = 0 \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{\partial \pi_1}{\partial x} \\ \lambda_2(y) = 0 \end{cases} \tag{6}$$

The following result holds:

Proposition 2. *If $\beta_1 - \beta_2 + 1 \geqslant 0$ and $\beta_2 - \beta_1 + 1 \geqslant 0$ then a unique Nash equilibrium exists.*

The proof of this proposition is a constructive one, i.e. it also gives a method to compute the Nash equilibrium.

Proof. Being $x > 0$ and $y > 0$, the system (5) can be written as

$$\begin{cases} km_1B\beta_1 x^{\beta_1} y^{\beta_2} - c_1 x(x^{\beta_1} + ky^{\beta_2})^2 = 0 \\ km_2B\beta_2 x^{\beta_1} y^{\beta_2} - c_2 y(x^{\beta_1} + ky^{\beta_2})^2 = 0 \end{cases} \tag{7}$$

whose solutions must belong to the line

$$y = \frac{c_1 m_2 \beta_2}{c_2 m_1 \beta_1} x \tag{8}$$

After the inclusion of this equation in the first of (7) we obtain $x^{\beta_1+\beta_2}G(x) = 0$, where

$$G(x) = k\left(\frac{c_1 m_2 \beta_2}{c_2 m_1 \beta_1}\right)^{\beta_2}(m_1B\beta_1 - 2c_1x) - c_1 x^{\beta_1-\beta_2+1} - c_1k^2\left(\frac{c_1 m_2 \beta_2}{c_2 m_1 \beta_1}\right)^{2\beta_2} x^{\beta_2-\beta_1+1} \tag{9}$$

If x^* is a zero of the function G, then the point $E^* = (x^*, y^*)$, with y^* computed according to (8), is a Nash equilibrium. As

$$G(0) = k\left(\frac{c_1 m_2 \beta_2}{c_2 m_1 \beta_1}\right)^{\beta_2} m_1B\beta_1 > 0$$

and

$$G\left(\frac{m_1B\beta_1}{2c_1}\right) = -c_1\left[x^{\beta_1-\beta_2+1} + k^2\left(\frac{c_1 m_2 \beta_2}{c_2 m_1 \beta_1}\right)^{2\beta_2} x^{\beta_2-\beta_1+1}\right] < 0$$

then a solution $x^* \in \left(0, \dfrac{m_1B\beta_1}{2c_1}\right)$ exists. Uniqueness of such solution is ensured if G is a decreasing function. Since

$$G'(x) = -c_1\left[2k\left(\frac{c_1 m_2 \beta_2}{c_2 m_1 \beta_1}\right)^{\beta_2} + (\beta_1 - \beta_2 + 1)x^{\beta_1-\beta_2} + k^2\left(\frac{c_1 m_2 \beta_2}{c_2 m_1 \beta_1}\right)^{2\beta_2}(\beta_2 - \beta_1 + 1)x^{\beta_2-\beta_1}\right]$$

sufficient conditions for $G'(x) < 0$ are $\beta_1 - \beta_2 + 1 \geqslant 0$, $\beta_2 - \beta_1 + 1 \geqslant 0$, i.e. the strip of the parameter plane (β_1, β_2) between the two parallel lines $\beta_2 = \beta_1 - 1$ and $\beta_2 = \beta_1 + 1$. □

Observe that the restrictions on the elasticities given in Proposition 1 are usually satisfied in applications, because usually $(\beta_1, \beta_2) \in (0,1) \times (0,1)$ which is a weaker condition. Furthermore, note that the inequalities in Proposition 1 are trivially satisfied if $\beta_1 = \beta_2$. Finally, we would like to point out that the particular functional form of the speeds of reaction $\lambda_i(\cdot)$ are inessential for the computation of the Nash Equilibrium.

Of course, the existence of a unique Nash equilibrium does not imply that any trajectory converges to it. It may be locally (and not globally) stable, i.e. only trajectories starting close to the Nash equilibrium may converge to it, or it may be unstable, i.e. even trajectories starting arbitrarily close to it converge to other attracting sets. Moreover, several coexisting attractors, such as periodic cycles, quasi-periodic orbits or chaotic sets, may be present, each with its own basin of attraction.

In order to study the stability of the steady states and, in particular, of the Nash equilibrium, we need to consider its Jacobian matrix

$$DT(x,y) = \begin{bmatrix} 1 + \lambda_1'(x)(m_1\beta_1 BF_1 - c_1) + m_1\beta_1 B\lambda_1(x)\dfrac{\partial F_1}{\partial x} & m_1\beta_1 B\lambda_1(x)\dfrac{\partial F_1}{\partial y} \\ m_2\beta_2 B\lambda_2(y)\dfrac{\partial F_2}{\partial x} & 1 + \lambda_2'(x)(m_2\beta_2 BF_2 - c_2) + m_2\beta_2 B\lambda_2(x)\dfrac{\partial F_2}{\partial y} \end{bmatrix} \tag{10}$$

evaluated at the fixed points of the map T. From this conditions for the localization of the eigenvalues inside the unit circle of the complex plane can be obtained. [2] However, a study of the stability property of the Nash equilibrium is not an easy task, due to the fact that we do not have an analytic expression of the equilibrium effort allocations. Oftentimes, a good strategy to get some insights is to consider first the simpler problem obtained under the assumption that the competitors behave identically, i.e. to study the properties of the symmetric map obtained under the assumption that structural parameters of the model are equal for both firms. This situation is often considered in the literature on brand competition models, and in our case it is a useful starting point for a more rigorous analysis of the more complicated model of heterogenous brands.

3. The symmetric case of homogeneous brands

In the case of homogeneous brands, that is, when the two competing firms react in the same way to profit signals, $\lambda_1(\cdot) = \lambda_2(\cdot) = \lambda(\cdot)$, and are characterized by identical parameters

$$\beta_1 = \beta_2 = \beta; \quad c_1 = c_2 = c; \quad m_1 = m_2 = m \quad \text{and} \quad \alpha_1 = \alpha_2 \quad (\text{i.e. } k = 1) \tag{11}$$

the map T in (4) assumes a symmetric form T_s. 'Symmetry' in this case refers to the fact that the map remains the same if the variables x and y are swapped. More formally, $T_s \circ S = S \circ T_s$, where $S : (x,y) \to (y,x)$ is the reflection through the diagonal $\Delta = \{(x,x), x \in \mathbb{R}\}$.

This symmetry property implies that given an orbit G of T_s, either $S(G) = G$, i.e. G is symmetric with respect to Δ, or $S(G) \neq G$ is an orbit of T_s as well, where $S(G)$ is the symmetric orbit of G with respect to Δ. For example, if G is a cycle of period k, say $G = C_k = \{(x_1, y_1), \ldots, (x_k, y_k)\}$, then either C_k is symmetric with respect to Δ, i.e. for each periodic point $(x_i, y_i) \in C_k$ also $S(x_i, y_i) = (y_i, x_i) \in C_k$, or the symmetric k-cycle $C_k' = \{(y_1, x_1), \ldots, (y_k, x_k)\}$ exist, with the same stability property.

In particular, the symmetry property implies that the diagonal Δ is a trapping subspace for the map T_s, i.e. $T_s(\Delta) \subseteq \Delta$. This corresponds to the trivial statement that, in a deterministic framework, identical firms that start from the same initial efforts $x(0) = y(0)$, behave identically over time, i.e. $x(t) = y(t)$ for each $t \geqslant 0$. The trajectories trapped into Δ are governed by a one-dimensional map, given by the restriction of T_s to Δ, say $f(x) = T_s|_\Delta : \Delta \to \Delta$, where the map f results from setting $y = x$ in (4), and is given by

$$x(t+1) = f(x(t)) = x(t) + \lambda(x)\left[\frac{m\beta B}{4x} - c\right]. \tag{12}$$

[2] It is worth noticing that, even if the Nash equilibrium does not depend on the speeds of reaction λ_i, the stability properties may nevertheless be influenced by it. This will be further stressed in the following.

In fact, under assumption (11), $F_1(x,x) = F_2(x,x) = 1/(4x)$. The map (12) may be interpreted as a one-dimensional model describing the dynamics of a "representative firm" that summarizes the common behavior of the two identical and synchronized firms. [3] In this case, the following problem can be considered: Let A_s be an attractor of the one-dimensional map (12). Is it also an attractor, embedded into Δ, for the two-dimensional map T? Of course, A_s is stable with respect to perturbations along Δ, so an answer to the question raised above can be given through a study of the stability of A_s with respect to perturbations transverse to Δ (*transverse stability*). In general, although stable along Δ, A_s can be transversely unstable (see e.g. [3,9,11]). However, this cannot occur for the model we are considering, because from (10) with (11) one easily gets for any point $(x,x) \in \Delta$ that

$$DT_s(x,x) = f'(x)\mathbf{I}, \tag{13}$$

where $\mathbf{I}$ is the identity matrix and

$$f'(x) = 1 + \lambda'(x)\left(\frac{m\beta B}{4x} - c\right) - \lambda(x)\frac{m\beta B}{4x^2} \tag{14}$$

is the derivative of (12). So, any periodic point located along Δ has two identical eigenvalues and any bifurcation that occurs along the invariant line Δ has an identical bifurcation occurring in the direction orthogonal to Δ. As we shall see in what follows, this property has important consequences for the creation of coexisting cyclic attractors of the symmetric map, both located along Δ (characterized by identical decisions of the firms at each time period) or symmetric with respect to Δ (characterized by oscillations where the decisions of the two firms alternate, by exchanging x and y as time goes on, a sort of cyclic imitative behavior).

We end this section by stressing that under assumption (11) the computation of the Nash equilibrium becomes straightforward. In this case the equilibrium is embedded inside Δ and according to (8) it is characterized by identical efforts of the two brands:

$$E^* = (x^*, x^*) \quad \text{with} \quad x^* = \frac{mB\beta}{4c} \tag{15}$$

A sufficient condition for the stability of E^* is $|f'(x)| < 1$, where $f'(x)$ is given in (14). Of course, this is strongly influenced by the choice of $\lambda_i(\cdot)$. So, in the following we will consider different choices of $\lambda_i(\cdot)$ in order to obtain some insight into the wide range of asymptotic behaviors of the brand competition model under consideration.

4. Linear speed of adjustment

In this section we consider the following form for the speeds of reaction

$$\lambda_1(x) = v_1 x \quad \text{and} \quad \lambda_2(y) = v_2 y. \tag{16}$$

That is, for both firms the decision about next period's marketing efforts is characterized by a reaction speed which is proportional to the previous-period effort, where v_1 and v_2 are the constants of proportionality (*relative speeds*). Bischi et al. [8] make the same assumption about the linearity of the speeds of reaction, but in their paper the adjustment process is based on profits rather than on marginal profits as in our paper. Nevertheless, the same choice of the functional form gives us an opportunity to compare the results obtained for these two different models.

With (16) the map (4) governing the time evolution of the dynamical system becomes

$$T_L : \begin{cases} x(t+1) = x(t) + v_1\left[m_1\beta_1 Bk\dfrac{x^{\beta_1}(t)y^{\beta_2}(t)}{(x^{\beta_1}(t) + ky^{\beta_2}(t))^2} - c_1 x(t)\right] \\ y(t+1) = y(t) + v_2\left[m_2\beta_2 Bk\dfrac{x^{\beta_1}(t)y^{\beta_2}(t)}{(x^{\beta_1}(t) + ky^{\beta_2}(t))^2} - c_2 y(t)\right] \end{cases} \tag{17}$$

The following proposition gives a sufficient condition under which all the points of $\mathbb{R}^2_+$ generate feasible and bounded trajectories

[3] This point of view is proposed in Bischi et al. [10] and Kopel et al. [19].

Proposition 3. *If* $v_1c_1 < 1$ *and* $v_2c_2 < 1$, *then all the points of* $\mathbb{R}^2_+$ *generate feasible and bounded trajectories.*

Proof. The map (17) can be rewritten as

$$T_L : \begin{cases} x(t+1) = (1 - v_1c_1)x(t) + v_1m_1\beta_1Bk\dfrac{x^{\beta_1}(t)y^{\beta_2}(t)}{(x^{\beta_1}(t) + ky^{\beta_2}(t))^2} \\ y(t+1) = (1 - v_2c_2)y(t) + v_2m_2\beta_2Bk\dfrac{x^{\beta_1}(t)y^{\beta_2}(t)}{(x^{\beta_1}(t) + ky^{\beta_2}(t))^2} \end{cases}$$

Hence from $1 - v_ic_i > 0$, $i = 1, 2$, it follows that $(x(t), y(t)) \in \mathbb{R}^2_+ \Rightarrow (x(t+1), y(t+1)) \in \mathbb{R}^2_+$. The fact that trajectories of (17) cannot be positively divergent follows from the Proposition 1.

Indeed, the conditions $1 - v_ic_i > 0$, $i = 1, 2$, turn out to be also necessary for the feasibility of the whole phase space $\mathbb{R}^2_+$, i.e. if at least one of these two inequalities is reversed then points of $\mathbb{R}^2_+$ exist that generate unfeasible trajectories. In order to prove this, we start from an important property of the map (17), namely that the coordinate axes are invariant submanifolds. That is, $x(t) = 0$ implies $x(t+1) = 0$ and, analogously, $y(t) = 0$ implies $y(t+1) = 0$. The dynamics along the invariant x-axis is governed by the one-dimensional restriction $f_1 = T_L|_{y=0}$ of (17), where f_1 is given by the one-dimensional linear map

$$x(t+1) = f_1(x(t)) = (1 - v_1c_1)x(t) \tag{18}$$

Analogously, the dynamics on the invariant y-axis is governed by the one-dimensional restriction $f_2 = T_L|_{x=0}$ given by

$$x(t+1) = f_2(y(t)) = (1 - v_2c_2)y(t) \tag{19}$$

If $v_1c_1 < 1$, then a point $(x, 0)$, with $x > 0$, generates a positive sequence of points on the x-axis according to (18). By continuity the same holds for points (x, y) with arbitrarily small y. Hence, in this case the feasible region extends up to the x-axis. Instead, if $v_1c_1 > 1$, then a point $(x, 0)$, with $x > 0$, generates a negative point after the first iteration of (18). Hence, in this case the whole x-axis must be out of the feasible region (or, equivalently, it belongs to the set of unfeasible points). The same reasoning applies to the y-axis.

In order to obtain an exact delimitation of the boundary that separates the feasible region from the unfeasible one we must consider the invariant coordinate axes and their preimages inside $\mathbb{R}^2_+$. In fact, the map T_L is a noninvertible map. This means that even if it maps a point $(x(t), y(t))$ into a unique image $(x(t+1), y(t+1))$, the inverse relation $(x(t+1), y(t+1)) \to (x(t), y(t))$ is not necessarily unique. In other words, considering positive solutions of the system (17) with respect to x and y, where x' and y' are given, there may be more than one or they may not exist. This can be expressed by saying that a point may have more than one preimages or no preimage. Applying this idea to points of the coordinate axes we notice that these points can be obtained not only as the image of a point belonging to the same axis (computed according to the restrictions (18) and (19)), but also as the image of a point away from the axes. Let us e.g. consider the generic point $(0, y')$, $y' > 0$, of the y-axis. Its preimages are the positive solutions of the system

$$\begin{cases} (1 - v_1c_1)x(x^{\beta_1} + ky^{\beta_2})^2 + v_1m_1\beta_1Bkx^{\beta_1}y^{\beta_2} = 0 \\ ((1 - v_2c_2)y - y')(x^{\beta_1} + ky^{\beta_2})^2 + v_2m_2\beta_2Bkx^{\beta_1}y^{\beta_2} = 0 \end{cases}$$

If $c_2v_2 < 1$ one solution always exists on the y-axis, given by $x = 0$, $y = (y'/(1 - v_2c_2))$. Solutions with $x > 0$ cannot exist if $v_1c_1 < 1$, because the first equation is never satisfied in this case. On the other hand, if $v_1c_1 > 1$ two preimages with $x > 0$ exist, located on the curves of the equation

$$y = \left\{ \frac{x^{\beta_1 - 1}}{2k(v_1c_1 - 1)} \left[v_1m_1\beta_1B - 2(v_1c_1 - 1)x \pm \sqrt{v_1^2m_1^2\beta_1^2B^2 - 4v_1m_1\beta_1B(v_1c_1 - 1)x} \right] \right\}^{1/\beta_2} \tag{20}$$

The same arguments, applied to the preimages of a generic point of the x-axis $(x', 0)$, can be used to prove that points of the invariant x-axis have preimages in the positive quadrant $\mathbb{R}^2_+$ only if $v_1c_1 > 1$, and such preimages are located on the curves of the equation

$$x = \left\{ \frac{ky^{\beta_2 - 1}}{2(v_2c_2 - 1)} \left[v_2m_2\beta_2B - 2(v_2c_2 - 1)y \pm \sqrt{v_2^2m_2^2\beta_2^2B^2 - 4v_2m_2\beta_2B(v_2c_2 - 1)y} \right] \right\}^{1/\beta_1} \tag{21}$$

These results on the preimages of the invariant axes will prove to be important tools for determining the boundaries of the feasible region.

The unique positive fixed point of the map (17) represents the Nash equilibrium of the game (computed according to the procedure outlined in the proof of Proposition 2). The only solution of the system (6) is the point (0, 0) in which the map is not defined. Of course, the stability properties and the local bifurcations of the Nash equilibrium can be easily studied numerically, using the solution of the equation $F(x) = 0$ and the Jacobian matrix. In the special case where brands are homogeneous (see (11)) even an analytical study of the stability properties and local bifurcations of the Nash equilibrium is possible.

4.1. Homogeneous brands

Under the assumption (11) of homogeneous brands, also the diagonal $x = y$ is an invariant line (line of synchronized dynamics). The dynamics along the invariant diagonal are governed by the one-dimensional linear map

$$x(t+1) = (1 - vc)x(t) + \frac{vm\beta B}{4}$$

Of course, the fixed point of such a restriction coincides with the Nash equilibrium (15), and the Jacobian matrix computed at any point of the diagonal (and, in particular, at the Nash equilibrium) becomes $DT(x,x) = (1 - vc)\mathbf{I}$. So, the following result holds:

Proposition 4. *In the homogeneous case* (11) *the Nash equilibrium* (15) *is stable for* $0 < vc < 2$. *For* $0 < vc < 1$ *its basin of attraction is given by the whole phase space* R^2, *i.e. it is globally asymptotically stable. For* $1 < vc < 2$ *its basin* $B(E^*)$ *is a proper subset of* R^2, *bounded by the preimages of the coordinate axes, belonging to the curves of equation*

$$y = \left\{ \frac{x^{\beta-1}}{2(vc-1)} \left[vm\beta B - 2(vc-1)x + \sqrt{v^2m^2\beta^2B^2 - 4vm\beta B(vc-1)x} \right] \right\}^{1/\beta}$$

and

$$x = \left\{ \frac{y^{\beta-1}}{2(vc-1)} \left[vm\beta B - 2(vc-1)y + \sqrt{v^2m^2\beta^2B^2 - 4vm\beta B(vc-1)y} \right] \right\}^{1/\beta}$$

The proof of this proposition is a straightforward consequence of the arguments given above.

The rank-1 preimages of the axes are curves issuing from the origin, symmetric with respect to the diagonal $y = x$ and joining at the rank-one preimage of the origin

$$O_{-1} = \left(\frac{vmB\beta}{4(vc-1)}, \frac{vmB\beta}{4(vc-1)} \right)$$

Thus, for $vc > 1$, the length of the segment OO_{-1}, which is proportional to the parameters m and β and inversely proportional to the aggregate parameter $(vc - 1)$, gives a rough idea of the extension of the feasible region. At $vc = 2$ the Nash equilibrium (15) looses stability and the generic trajectory is unfeasible.

The results stated above are illustrated by the numerical computation shown in Fig. 1. The white region represents the basin of attraction of the Nash equilibrium, denoted by E^*, and the points in the black region generate unfeasible trajectories. Fig. 1 is obtained with a set of parameters such that $vc = 1.05 > 1$, so that $O_{-1} = (10.5, 10.5)$. If vc is increased, with the other parameters held constant, the feasible region shrinks. If vc is decreased below 1, then global stability of E^* is obtained, i.e. every positive initial condition generates a trajectory converging to E^*. If vc is increased beyond the value $vc = 2$, then the Nash equilibrium loses its stability and becomes a repelling star node. In the case of homogeneous firms this implies that the unfeasible region covers the whole phase space and, consequently, any meaningful dynamics is lost.

4.2. Heterogeneous brands

We now relax the assumption of homogeneous firms. Also in this case, if $v_ic_i < 1$, $i = 1, 2$, the feasible region coincides with the whole phase space $\mathbb{R}^2_+$, because no preimages of the coordinate axes exist inside $\mathbb{R}^2_+$. The numerical simulations show that the Nash equilibrium is globally asymptotically stable, i.e. every initial condition in $\mathbb{R}^2_+$ generates a trajectory which converges to it. Like in the case of homogeneous brands, also in this case a wide range of parameters exist such that the Nash equilibrium is stable (a stable node or a stable focus). If one (or both) of the above inequalities

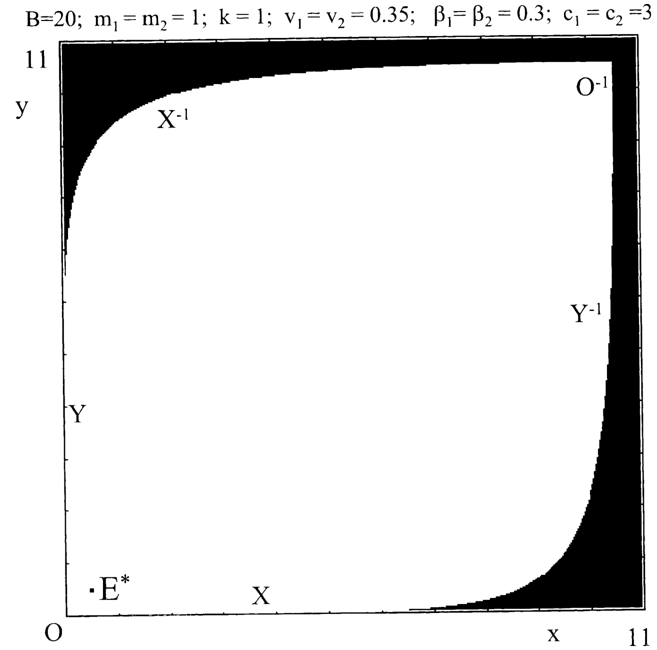

Fig. 1. Brand competition model with linear speeds of reaction and homogeneous brands. The white region represents the basin of attraction of the stable Nash equilibrium E^*, the black region represents the unfeasible set. This figure is obtained with parameters $B = 20$, $m = 1$, $v = 0.35$, $\beta = 0.3$, $c = 3$.

are reversed, the Nash equilibrium becomes only locally stable, since it only attracts the points of the feasible set, which no longer covers the whole phase space.

If $v_1c_1 < 1$ and $v_2c_2 > 1$, then the feasible set is an unbounded region (extending for arbitrarily large x) with the upper boundary formed by the rank-1 preimage of the x-axis, say X_{-1} (see Fig. 2(a)) whose equation is given by (21) with the "+" sign. The curve X_{-1} is tangent to the x-axis at the origin. Analogously, if $v_1c_1 > 1$ and $v_2c_2 < 1$, then the feasible set is an unbounded region (extending for arbitrarily large y) with right boundary formed by the rank-1 preimage of the y-axis, say Y_{-1}, whose equation is given by (20) with the "+" sign.

If $v_1c_1 > 1$ and $v_2c_2 > 1$, then the feasible set is a *bounded region*, whose boundary is formed by the curves X_{-1} and Y_{-1}, issuing from the origin O tangent to the axes and intersecting at the preimage of the origin O_{-1} (see Fig. 2(b)). Hence, the conditions $v_ic_i = 1$ and $v_jc_j > 1$, $i \neq j$, denote the occurrence of a global bifurcation, at which the feasible region is changed from unbounded to bounded. Other bifurcations that change the topological structure of the boundaries of the feasible region may occur, due to the fact that higher order preimages of the coordinate axes appear inside $\mathbb{R}^2_+$. In fact, in the case considered, a preimage of rank-k of a coordinate axis bounds a region of the phase space whose points are unfeasible, since points in this set are mapped into points with a negative coordinate after k iterations. Two of such regions are visible in Fig. 2(b), and have the shape of small lobes issuing from O and O_{-1}. They are bounded by preimages of rank-2 and rank-3 of the x-axis, say X_{-2} and X_{-3}.

Also in this case, the Nash equilibrium loses stability as one or both of the expressions v_ic_i are increased. For example, in the presence of small heterogeneities, the local bifurcation at which the Nash equilibrium loses stability is a flip bifurcation occurring approximately when one of the products v_ic_i becomes greater than 2. Starting from the situation shown in Fig. 2(a), an increase of v_1c_1 causes the occurrence of a flip (or period doubling) bifurcation, through which the Nash equilibrium becomes a saddle point and a stable cycle of period two appears close to it. It is worth noticing that in this case, differently from the homogeneous case, only one eigenvalue of $DT_L(E^*)$ exits the unit circle of the complex plane (since in general v_1c_1 is different from v_2c_2 in the heterogeneous case) so that a standard flip bifurcation occurs, at which the Nash equilibrium becomes a saddle point and a stable cycle of period 2 appears near E^* along its unstable set $W^u(E^*)$. Hence, differently from the homogeneous case, more complex bounded attractors (such as periodic cycles) may exist around the unstable Nash equilibrium, so that the long-run dynamics may be characterized by bounded periodic (or even aperiodic) oscillations around the Nash equilibrium. However, the occurrence of such local bifurcations, at which new bounded attracting sets appear inside the feasible region, is not related to the global

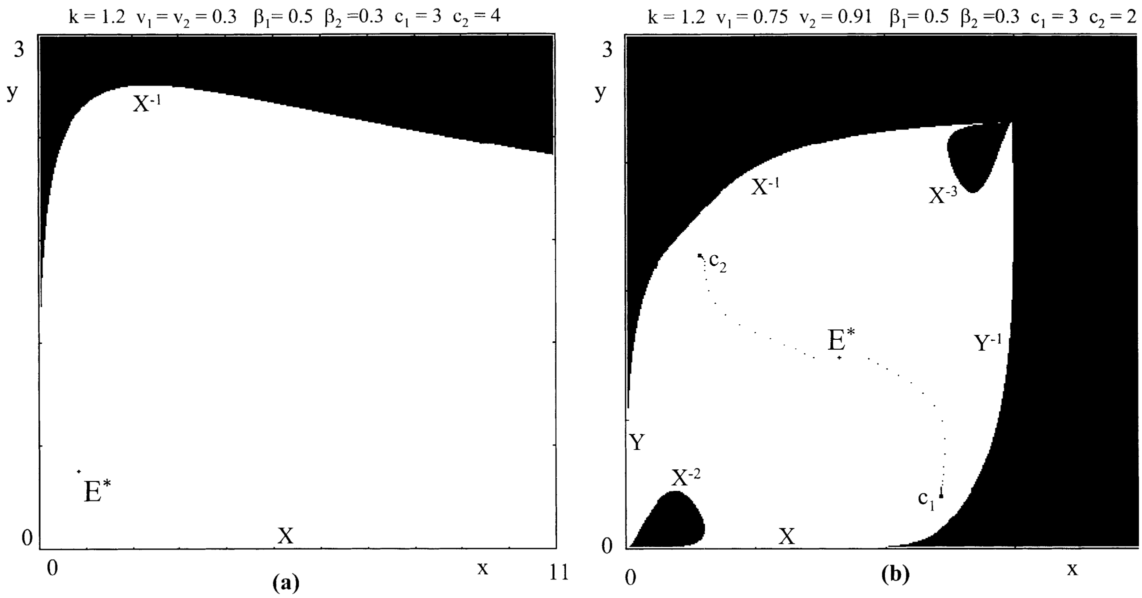

Fig. 2. Brand competition model with linear speeds of reaction and heterogeneous brands. (a) The parameters are $B = 20$, $m_1 = m_2 = 1, k = 1.2$, $v_1 = v_2 = 0.3$, $\beta_1 = 0.5$, $\beta_2 = 0.3$, $c_1 = 3$ and $c_2 = 4$. The basin of attraction of the Nash equilibrium E^* is unbounded along the x direction, since $v_1c_1 < 1$. (b) With $v_1 = 0.75$, $v_2 = 0.91$, $c_1 = 3$ and $c_2 = 2$ the Nash equilibrium E^* is unstable and a stable cycle of period two attracts the trajectories that start in the white region: one of these trajectories, starting from an initial condition close to E^*, is represented by a sequence of dots.

bifurcations that change the shape of the boundaries of the feasible region. As it can be seen in Fig. 2(b), if v_1 and/or v_2 are further increased after the flip bifurcation, the periodic points, denoted by c_1 and c_2 in the figure, move far away from the Nash equilibrium until they reach the boundary of the feasible region. When this contact occurs, a typical global bifurcation happens, called *final bifurcation* (or *boundary crisis*), that marks the disappearance of the bounded attractor. After this bifurcation the generic trajectory will be unfeasible.

5. Quadratic speeds of reaction

In this section we assume that

$$\lambda_1(x) = v_1x^2 \quad \text{and} \quad \lambda_2(y) = v_2y^2 \tag{22}$$

i.e. the *relative speed* of adjustment of both firms is proportional to the efforts of the previous period. With (22) the map (4) governing the time evolution of the dynamical system becomes

$$T_Q : \begin{cases} x(t+1) = x(t) + v_1x(t)\left[m_1\beta_1Bk\dfrac{x^{\beta_1}y^{\beta_2}(t)}{(x^{\beta_1}(t) + ky^{\beta_2}(t))^2} - c_1x(t)\right] \\ y(t+1) = y(t) + v_2y(t)\left[m_2\beta_2Bk\dfrac{x^{\beta_1}(t)y^{\beta_2}(t)}{(x^{\beta_1}(t) + ky^{\beta_2}(t))^2} - c_2y(t)\right] \end{cases} \tag{23}$$

Also in this case the coordinate axes are invariant, but now the restrictions of the two-dimensional map to the axes are given by one-dimensional quadratic maps

$$x' = x - c_1v_1x^2 \quad \text{and} \quad y' = y - c_2v_2y^2. \tag{24}$$

Both of these quadratic maps are conjugate to the standard logistic map $z' = \mu z(1 - z)$ with $\mu = 1$ through the linear homeomorphisms $z = c_1v_1x$ and $z = c_2v_2y$ respectively. From these one-dimensional restrictions to the invariant axes we can easily deduce that the portion $X = (O, O^1_{-1}) = (0, \frac{1}{c_1v_1})$ of the x-axis and the portion $Y = (O, O^2_{-1}) = (0, \frac{1}{c_2v_2})$ of the

y-axis are part of the boundary of the feasible region, whereas the remaining portions belong to the unfeasible region. The complete boundaries of the feasible region of the two-dimensional phase space are formed by the preimages X_{-k} and Y_{-k} of these segments. Such preimages are always included inside the phase space $\mathbb{R}^2_+$. In fact, a rank-1 preimage Y_{-1} of the y-axis belongs to the curve

$$y = \left\{ \frac{x^{\beta_1}}{2k(v_1c_1x-1)} \left[v_1m_1\beta_1B - 2(v_1c_1x-1) + \sqrt{v_1^2m_1^2\beta_1^2B^2 - 4v_1m_1\beta_1B(v_1c_1x-1)} \right] \right\}^{1/\beta_2} \tag{25}$$

which intersects the x-axis in the point $O^1_{-1} = (\frac{1}{v_1c_1}, 0)$. Analogously, a rank-1 preimage X_{-1} of the x-axis belongs to the curve

$$x = \left\{ \frac{ky^{\beta_2}}{2(v_2c_2y-1)} \left[v_2m_2\beta_2B - 2(v_2c_2y-1) + \sqrt{v_2^2m_2^2\beta_2^2B^2 - 4v_2m_2\beta_2B(v_2c_2y-1)} \right] \right\}^{1/\beta_1} \tag{26}$$

which intersects the y-axis in the point $O^2_{-1} = (0, \frac{1}{v_2c_2})$. The two curves X_{-1} and Y_{-1} intersect in a point O^3_{-1} interior to $\mathbb{R}^2_+$. The quadrilateral region $OO^2_{-1}O^3_{-1}O^1_{-1}$ gives a rough estimate of the feasible region, as we shall see in the examples below. Its sides OO^2_{-1} and OO^1_{-1} belong to the x-axis and y-axis, $O^2_{-1}O^3_{-1}$ and $O^3_{-1}O^1_{-1}$ belong to X_{-1} and Y_{-1} respectively. The vertexes O, O^2_{-1}, O^3_{-1}, O^1_{-1} are the four rank-1 preimages of the origin. In fact the map (23) is a noninvertible map of type $Z_4 > Z_2 - Z_0$, according to the terminology introduced in Mira et al. [22] (see also [2]). The origin $O = (0,0)$ always belong to the region Z_4, because it always has four rank-1 preimages. These can be obtained as solutions of the system (23) with $x(t+1) = 0$ and $y(t+1) = 0$. Also the existence of the Z_0 region, whose points have no preimages, can be easily proved. In fact, the system (23), solved with respect to the unknowns $x(t+1)$ and $y(t+1)$, has no solutions if

$$x(t+1) > \frac{1 + v_1m_1\beta_1B}{2} \quad \text{or} \quad y(t+1) > \frac{1 + v_2m_2\beta_2B}{2}$$

From

$$\frac{kx^{\beta_1}y^{\beta_2}}{(x^{\beta_1} + ky^{\beta_2})^2} < 1$$

we have

$$x\left[1 - v_1c_1x + v_1m_1\beta_1Bk\frac{x^{\beta_1}y^{\beta_2}}{(x^{\beta_1} + ky^{\beta_2})^2}\right] < x[1 - v_1c_1x + v_1m_1\beta_1B] < \frac{1 + v_1m_1\beta_1B}{2}$$

where the last inequality follows from the fact that $(1 + v_1m_1\beta_1B)/2$ is the maximum value of the concave parabola $x(1 - v_1c_1x + v_1m_1\beta_1B)$. Also in this case, the map (17) has a unique fixed point which coincides with the Nash equilibrium, because the only solution of (6) is the point (0,0) in which the map is not defined.

5.1. Homogeneous brands

In the case (11) of homogeneous (or identical) brands also the diagonal $x = y$ is an invariant line, i.e. $x_0 = y_0$ implies $x_t = y_t$ for each $t \geqslant 0$ (line of synchronized dynamics). The dynamics along the invariant diagonal are governed by the one-dimensional linear map

$$x(t+1) = \left(1 + \frac{vm\beta B}{4}\right)x(t) - vcx(t)^2 \tag{27}$$

which is conjugate to the standard logistic $z(t+1) = \mu z(t)(1 - z(t))$ with parameter

$$\mu = 1 + \frac{vm\beta B}{4} \tag{28}$$

by the linear transformation $z = 4vc/(4 + vm\beta B)$. Of course the positive fixed point of such a restriction coincides with the Nash equilibrium (15), and the rank-1 preimage of the origin computed by (27) gives the coordinates of O^3_{-1}, that in this case belongs to the diagonal

$$O^3_{-1} = \left(\frac{4 + vm\beta B}{4vc}, \frac{4 + vm\beta B}{4vc}\right). \tag{29}$$

This is the intersection of the two curves X_{-1} and Y_{-1}, that now are symmetric with respect to the diagonal, and intersect the axes in the symmetric points $O^1_{-1} = (\frac{1}{vc}, 0)$ and $O^2_{-1} = (0, \frac{1}{vc})$. The Jacobian matrix, computed at any point of the diagonal (and, in particular, at the Nash equilibrium) assumes the form (10), that becomes

$$\mathbf{DT}(x,x) = \left(1 + \frac{vm\beta B}{4} - 2vcx\right)\mathbf{I}$$

Hence, the Nash equilibrium is an attracting star node for $vm\beta B < 8$, then loses stability via a degenerate flip bifurcation (i.e. both eigenvalues are -1) at $vm\beta B = 8$, and becomes a repelling star node for $vm\beta B > 8$. Differently from the case studied for the model with linear speed of adjustment, due to the nonlinearity of (27), this bifurcation does not cause divergence in this case, but it creates two attracting cycles of period 2, one along the diagonal (which coincides with the two-cycle of the one-dimensional restriction (27)) and the other one with periodic points out of the diagonal and in symmetric position with respect to it.

As it is well-known from the bifurcation diagram of the logistic map, this first flip bifurcation is followed by a sequence of flip bifurcations at which attracting cycles of period $2, 4, \ldots, 2^k \ldots$ are created. In our two-dimensional symmetric model this gives rise to a very particular dynamic scenario: in fact, whenever a period doubling bifurcation of a k-cycle flip bifurcates and creates a stable $2k$-cycle along the diagonal, also a coexisting stable $2k$-cycle is simultaneously created out of the diagonal. This is due to the fact that the Jacobian matrix of any periodic point along the diagonal is a multiple of the identity matrix, so the cycles embedded into the diagonal have always identical eigenvalues, associated to the invariant line Δ and to the direction orthogonal to Δ. Consequently, any period doubling bifurcation along Δ is associated with a period doubling bifurcation in the symmetric direction. This leads to the creation of many coexisting attracting cycles, each with its own basin of attraction, a particularly complex phenomenon of *multistability*.

Fig. 3 illustrates this situation. In Fig. 3(a) the Nash equilibrium is stable, and its basin of attraction coincides with the feasible region (the white region) bounded by the quadrilateral whose vertexes are the origin of the coordinate axes $O = (0, 0)$ and its rank-1 preimages, O^k_{-1}, $k = 1, 2, 3$. If the parameter v is increased, at $vm\beta B = 8$ the first period doubling bifurcation occurs at which the Nash equilibrium loses stability and becomes a repelling star node. This implies that not only a stable cycle $\{c_1, c_2\}$ of period 2 is created along the invariant diagonal $x = y$, but also a stable cycle $\{p_1, p_2\}$ of period 2 is created out of the diagonal, with periodic points symmetric with respect to it (see Fig. 3(b)). The basins of the two stable cycles, represented by light and dark grey regions in Fig. 3(b), are separated by the unstable sets of the star node E^*. The complicated topological structure of the basins, formed by many non connected portions that accumulate along the outer boundary that separates them from the unfeasible set, is typical of noninvertible maps (see [2,6,22]).

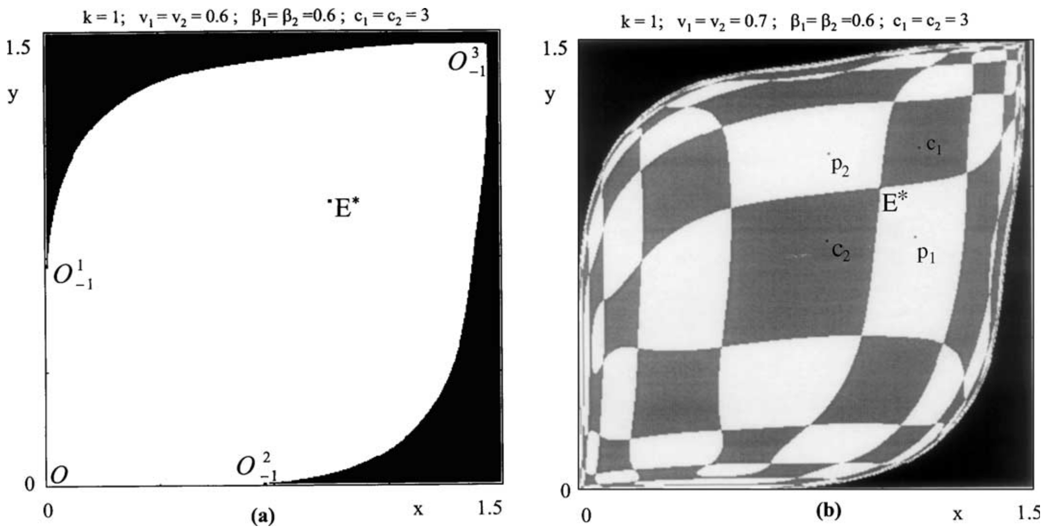

Fig. 3. Brand competition model with quadratic speeds of reaction and homogeneous brands. (a) With parameters $B = 20$, $m = 1$, $v = 0.6$, $\beta = 0.6$, $c = 3$ the Nash equilibrium E^* is stable with a bounded basin, represented by the white region. (b) With $v = 0.7$ the Nash equilibrium is unstable, and two stable cycles of period two denoted by $\{c_1, c_2\}$ and $\{p_1, p_2\}$ co-exist, whose basins of attraction are represented by the dark grey and light grey, respectively.

Analogously, when the two-cycle along the diagonal undergoes the second period doubling bifurcation, two stable four-cycles are created, one along the diagonal and one with periodic points symmetric to it. These two stable cycles coexist with the two-cyclic attractor $\{p_1, p_2\}$ previously created out of the diagonal. As the parameter v is increased, also the cycle $\{p_1, p_2\}$ is transformed: it becomes a stable focus, and then undergoes a Neimark-Hopf bifurcation at which it becomes a two-cyclic attractor formed by a 2-piece quasi-periodic attractor, then a two-piece chaotic attractor. Then also the cycle of period 4 along the diagonal undergoes the degenerate flip bifurcation at which two stable eight-cycles are obtained and so on. Such a sequence of bifurcations leads to the creation of several coexisting attractors. For example, in Fig. 4(a), obtained with $v = 0.851$, four coexisting attractors are present: a stable cycle of period 8 along the diagonal $x = y$ (green basin), a stable cycle of period 8 with periodic points symmetric with respect to the diagonal (white basin), a quasi-periodic 4 pieces attractor formed by four-cyclic closed invariant curves (red basin) and a two-cyclic chaotic attractor (yellow basin). This figure reveals the complexity of the basins of attraction which we referred to in the introduction. A small displacement of the initial conditions (or, equivalently, a small exogenous shock) may lead to a very distinct long run evolution of the trajectory. This kind of sensitivity with respect to small changes of the initial conditions is different from the sensitive dependence on initial conditions along a chaotic attractor (also known as the "butterfly effect"). When we consider the dynamics along a chaotic attractor, even if two initially very close chaotic trajectories depart fast as time increases (at an exponential rate), such trajectories are finally trapped inside the same compact invariant set in the phase space (the chaotic attractor). In contrast to this, when we consider the situation of complex basins, for points close to a basin boundary a small change in the coordinates will cause the trajectory to converge to a different attractor. Considering Fig. 4 this attractor may be quite far from the previous one and characterized by a qualitatively different asymptotic motion. Some authors call this kind of sensitivity "final state sensitivity" (see [16]).

The attractors that are created by local bifurcations and evolve through sequences of flip and Neimark-Hopf bifurcations, are destroyed by contact bifurcations, due to contacts between their boundaries and the boundary of their basin, as described in the previous section. These contact bifurcations are called *final bifurcation* in Mira et al. [22] and Abraham et al. [1] or *boundary crisis* in Grebogi et al. [15]. For example, the two-cyclic attractor shown in Fig. 4(a) is very close to a contact with the boundary of its basin of attraction. This means that a slight change of a parameter, such as a small increase of v from 0.851 to 0.852, can result in the disappearance of the chaotic attractor. This situation is depicted in Fig. 4(b). In this case the complexity of the basin boundaries, and consequently the final state sensitivity, suddenly becomes even more striking. This is related to the fact that the destruction of the chaotic attractor means that it is transformed into a chaotic repeller, whose skeleton is formed by the densely distributed repelling periodic cycles

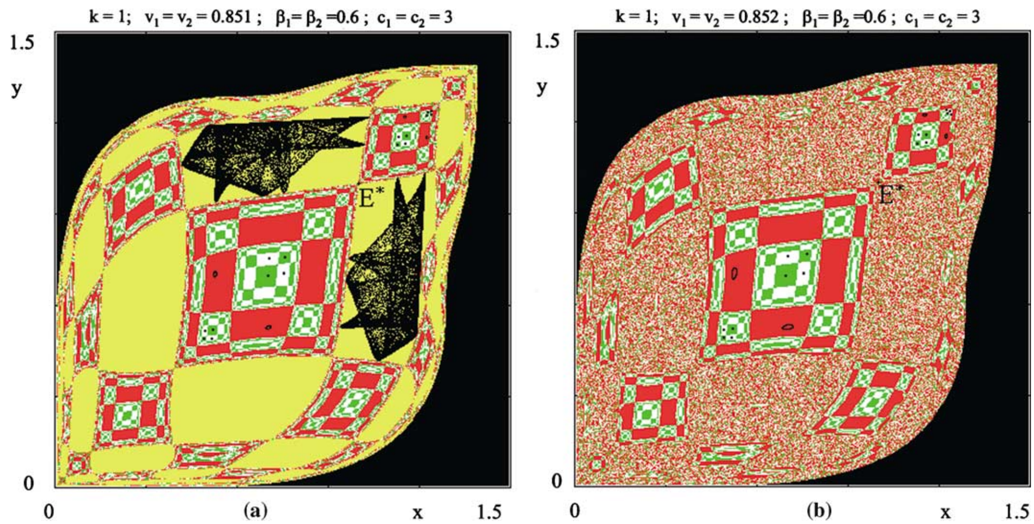

Fig. 4. Brand competition model with quadratic speeds of reaction and homogeneous brands. (a) With parameters $B = 20$, $m = 1$, $v = 0.851$, $\beta = 0.6$, $c = 3$ the Nash equilibrium E^* is unstable and four coexisting attractors are present: a stable cycle of period 8 along the diagonal $x = y$ (green basin), a stable cycle of period 8 with periodic points symmetric with respect to the diagonal (white basin), a quasi-periodic four pieces attractor formed by four-cyclic closed invariant curves (red basin) and a two-cyclic chaotic attractor (yellow basin). (b) After a slight increase of the parameter v, i.e. $v = 0.852$, the chaotic attractor disappears, and its basin is filled up by a very intermingled set of point from the white, green and red basin.

that where embedded into the chaotic attractor which just disappeared. These periodic points are located on the very intermingled boundaries that separate the green, white and red basins in the region that was occupied by the yellow basin. This "ghost" of the former chaotic attractor gives rise to chaotic transients before a trajectory will reach one of the surviving attractors.

For $\mu > \mu_\infty = 3.5699\ldots$ (the Feigenbaum point), corresponding to $vm\beta B > 10.2796\ldots$ according to (28), the restriction (27) of T_Q to the diagonal has chaotic attractors with infinitely many "windows" of attracting cycles. This leads to a very particular (and complex) situation: any chaotic attractor $\mathscr{A}$ belonging to the diagonal is a chaotic saddle (i.e. it is totally transversely repelling, because all the densely distributed unstable periodic points embedded inside it are also transversely repelling due to the particular structure of the Jacobian matrix, whereas every "periodic window" of the logistic, created via a fold bifurcation of (27), creates a stable star node for the two-dimensional map T_Q. Moreover, the period-doubling cascade that follows the creation of a stable cycle also creates attracting cycles out of the diagonal through the mechanism of the degenerate flips described above. This causes the creation of several coexisting attracting cycles, both embedded into the invariant diagonal Δ (synchronized cycles) and out of Δ, with periodic points symmetric with respect to Δ (imitation cycles). The creation of so many coexisting periodic attractors give rise to very particular situations of multistability, characterized by basins of attraction with quite complicated topological structures. This implies a strong path dependence, i.e. the convergence to a particular attractor is very much influenced by historical accidents, because even a very small change in the initial condition of a trajectory may cause the convergence to a different attractor, and consequently to completely different long run dynamics. Finally, bounded attractors cannot be observed for $vm\beta B > 12$ (corresponding to $\mu > 4$ according to (28).

5.2. Heterogeneous brands

If the two brands are heterogeneous (i.e. characterized by different parameter values), a rigorous analysis of the local bifurcations of the Nash equilibrium is not possible. This is due to the fact that its coordinates can be only numerically computed. However, guided by the results obtained in the simpler case of homogeneous brands, we can introduce small heterogeneities and study how this changes the dynamics.

Starting from a benchmark situation obtained in the symmetric case of homogeneous brands, we can slightly change some parameters in order to break the symmetry, and study the effects of the introduction of such small heterogeneities. For example, starting from the set of parameters used in Fig. 4(a), we introduce the following parameter variations: $k = 1.02$, $v_1 = 0.82$, $c_2 = 2.8$. The result of such a modification is shown in Fig. 5(a), where two coexisting attractors are

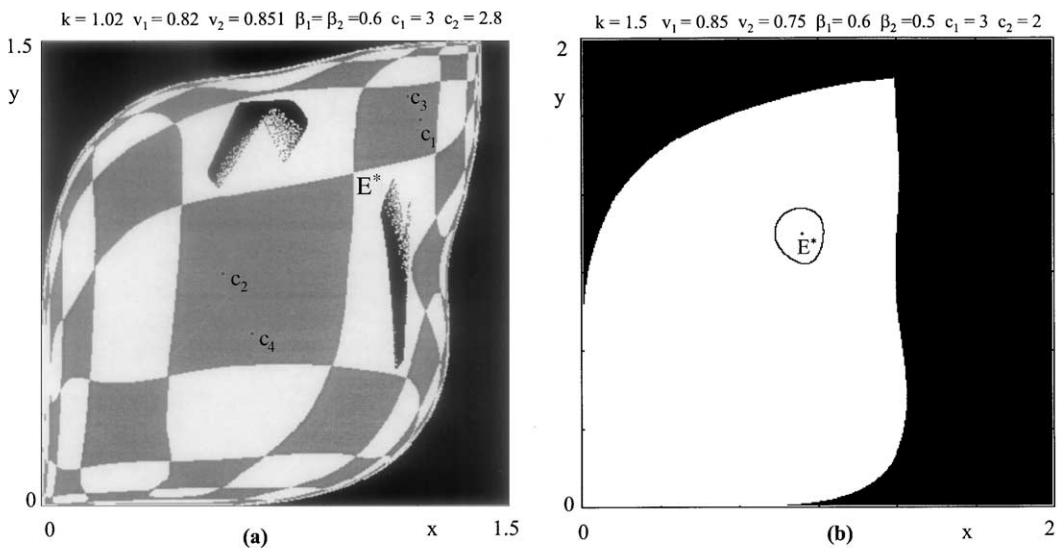

Fig. 5. Brand competition model with quadratic speeds of reaction and heterogeneous brands. (a) Some parameters are slightly changed with respect to Fig. 4(a): $k = 1.02$, $v_1 = 0.82$, $c_2 = 2.8$. Two coexisting attractors are present: a cycle of period 4 and a two-cyclic chaotic attractor. (b) With parameters $k = 1.5$, $v_1 = 0.85$, $v_2 = 0.75$, $\beta_1 = 0.6$, $\beta_2 = 0.5$, $c_1 = 3$, $c_2 = 2$ the Nash equilibrium E^* is an unstable focus, and a stable closed invariant curve exists around it, on which the long run behavior of the system is characterized by quasi-periodic oscillations.

present, a cycle of period 4 and a two-cyclic chaotic attractor. So, the property of multistability persists even in the presence of heterogeneities. Of course, stronger heterogeneity may result in quite different dynamic scenarios which cannot be observed in the case of homogeneous brands. To show this, we can consider the situation shown in Fig. 5(b), obtained with parameters $k = 1.5$, $v_1 = 0.85$, $v_2 = 0.75$, $\beta_1 = 0.6$, $\beta_2 = 0.5$, $c_1 = 3$, $c_2 = 2$. In this case, the Nash equilibrium E^* is an unstable focus, and the unique attractor is a stable closed invariant curve located around it, on which the long run behavior of the system is characterized by quasi-periodic oscillations. Due to the heterogeneities, the shape of the boundary of the feasible region is no longer symmetric with respect to the diagonal $x = y$. Of course, it may be interesting to study (at least numerically) the influence of the introduction of asymmetries in other parameters, such as the profit margins m_1 and m_2 or the marginal costs c_1 and c_2, both on the kind of attractors and on the shape of the basin boundaries.

6. Conclusions

In this paper we have introduced a dynamic market share attraction model. Brand managers in firms are assumed to be boundedly rational and over time adapt their marketing efforts for their brand in correspondence to the marginal profits of the previous period. In contrast to existing studies which oftentimes focus only on the local dynamics and maybe demonstrate the possibility of cyclic and erratic fluctuations, we have provided some insight into the properties of the global dynamical behavior, i.e. we tried to characterize the attractors and their corresponding basins. We have demonstrated that situations of multistability may arise, i.e. several stable attracting sets may coexist and the initial conditions determine the long run fate of the trajectories of effort allocations. Furthermore, our results show that these basins can have very complicated topological structure and, therefore, some kind of final state sensitivity plays an important role.

In the study of the model we have first tried to understand for which initial efforts meaningful long run outcomes are obtained. We have shown that by using an invariance property of the coordinate axes it is possible to delimitate the feasible set of points for which trajectories are bounded and e.g. converge to the Nash equilibrium. Realizing that a rigorous analysis of the general model of heterogeneous brands is impossible, we have focused—as a starting point—on the symmetric model, which captures a situation where brand managers behave in the same way and marginal costs, profit margins, elasticities and response parameters of the two brands are equal. We have provided a fairly complete study of the attracting sets and the basins in this case. Using these insights as guidelines we could then move to the general case of heterogeneous brands. The combination of rigorous arguments and numerical experiments give an insightful characterization of the wide array of possible dynamics which can arise in this model of brand competition.

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