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Maps with a Vanishing Denominator. A Survey of some Results ⋆

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Abstract

This paper concerns some results on global dynamical properties and bifurcations of two-dimensional maps, invertible or noninvertible, presenting a vanishing denominator. This last characteristic may give rise to specific singularities of the phase plane, called *focal points* and *prefocal curves.* The presence of these sets may cause new types of bifurcations generated by contact between them and other singularities, which give rise to new dynamic phenomena and new structures of basin and invariant sets. Some of such behaviors can also be observed in maps without a vanishing denominators, but such that some of the inverses have a vanishing denominator.

Key words: Discrete dynamical systems, noninvertible maps, focal points, attractors, basins of attraction.

1 Introduction

This paper is devoted to a survey of results related to real two-dimensional maps, defined by $(x', y') = (F(x,y), G(x,y))$, with at least one of the components F or G containing a denominator which can vanish. This implies that the map is not defined in the whole plane. Such a characteristic is the source of some particular dynamical behaviors, related to the presence of new kinds of singularities and bifurcations, as recently evidenced in the references [1]–[4], where in particular the situation arising when $F(x,y)$ or $G(x,y)$ assumes the form $0/0$ in some points of $\mathbb{R}^2$ has been analyzed. In these papers new singularities, called *focal point* and *prefocal curve*, have been defined which permit

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to characterize specific geometric and dynamic properties, together with some new bifurcations. Roughly speaking, a *prefocal curve* is a set of points for which at least one inverse exists, which maps (or "focalizes") the whole set into a single point, called *focal point.* These singularities may also be important in the study of maps defined in the whole plane, but such that some of the inverses have a vanishing denominator and possess focal points (see [2], [4]).

This paper gives only a summarized presentation of some results obtained up to now, which can be found (with more details and proofs) in the references quoted above, and in recent working papers [5], [6]. Section 2 is devoted to definitions and some basic properties. Section 3 presents some geometric properties of the singular sets, with the general description of some contact bifurcations.

2 Definitions and basic properties

2.1 General properties

In order to simplify the exposition, it is assumed that only one of the two functions defining the map T has a denominator which can vanish

$$T: \begin{cases} x' = F(x,y) \\ y' = G(x,y) = N(x,y)/D(x,y) \end{cases} \tag{1}$$

where x and y are real variables, $F(x,y)$, $N(x,y)$ and $D(x,y)$ are continuously differentiable functions without common factors and defined in the whole plane $\mathbb{R}^2$. Hence, the *set of nondefinition* of the map T (which is given by the set of points where at least one denominator vanishes) reduces to

$$\delta_s = \{(x,y) \in \mathbb{R}^2 | D(x,y) = 0\}. \tag{2}$$

It is assumed that δ_s is given by the union of smooth curves of the plane. The two-dimensional recurrence obtained by the successive iterations of T is well defined provided that the initial condition belongs to the set E given by $E = \mathbb{R}^2 \backslash \bigcup_{k=0}^{\infty} T^{-k}(\delta_s)$, where $T^{-k}(\delta_s)$ denotes the set of the rank-k preimages of δ_s, i.e. the set of points which are mapped into δ_s after k applications of T ($T^0(\delta_s) \equiv \delta_s$). Indeed, the points of δ_s, as well as all their preimages of any rank constituting a set of zero Lebesgue measure, must be excluded from the set of initial conditions that generate non interrupted sequences by the iteration of the map T, so that $T: E \to E$.

Now consider a bounded and smooth simple arc γ, parametrized as $\gamma(\tau)$, transverse to δ_s, such that $\gamma(0) = (x_0, y_0)$ and $\gamma \cap \delta_s = \{(x_0, y_0)\}$. We are interested in its image $T(\gamma)$. As $(x_0, y_0) \in \delta_s$ we have, according to the definition of δ_s, $D(x_0, y_0) = 0$, but in general $N(x_0, y_0) \neq 0$. Hence

$$\lim_{\tau \to 0\pm} T(\gamma(\tau)) = (F(x_0, y_0), \infty) \tag{3}$$

where ∞ means either $+\infty$ or $-\infty$. This means that the image $T(\gamma)$ is made up of two disjoint unbounded arcs asymptotic to the line of equation $x = F(x_0, y_0)$. A different situation may occur if the point $(x_0,y_0) \in \delta_s$ is such that not only the denominator but also the numerator of (1) vanishes in it, i.e. $D(x_0, y_0) = N(x_0, y_0) = 0$. In this case, the second component of the limit (3) takes the form 0/0. This implies that this limit may give rise to a finite value, so that the image $T(\gamma)$ is a bounded arc crossing the line $x = F(x_0, y_0)$ in the point $(F(x_0, y_0), y)$, where

$$y = \lim_{\tau \to 0} G(x(\tau), y(\tau)) \tag{4}$$

It is clear that the value y in (4) must depend on the arc γ. Furthermore it may have a finite value along some arcs and be infinite along other ones. This leads us to the following definition of the singular sets of *focal point* and *prefocal curve* [2]:

Definition. *Consider the map T in (1). A point $Q=(x_0,y_0)$ is a focal point if at least one component of T takes the form 0/0 in Q and there exist smooth simple arcs $\gamma(\tau)$, with $\gamma(0)=Q$, such that $\lim_{\tau\to 0} T(\gamma(\tau))$ is finite. The set of all such finite values, obtained by taking different arcs $\gamma(\tau)$ through Q, is the prefocal set δ_Q, the equation of which is $x = F(Q)$.*

In this paper we shall only consider simple focal points, i.e. points which are simple roots of the algebraic system

$$N(x, y) = 0, \qquad D(x, y) = 0$$

Thus a focal point $Q = (x_0, y_0)$ is simple if $\overline{N}_x\overline{D}_y - \overline{N}_y\overline{D}_x \neq 0$, where $\overline{N}_x = \frac{\partial N}{\partial x}(x_0, y_0)$ and analogously for the other partial derivatives. In the case of a simple focal point there exists a one-to-one correspondence between the point $(F(Q), y)$, in which $T(\gamma)$ crosses δ_Q, and the slope m of $\overline{\gamma}$ in Q, as shown in [2]:

$$m \to (F(Q), y(m)), \qquad \text{with} \qquad y(m) = (\overline{N_x} + m\overline{N_y}) \;/\; (\overline{D_x} + m\overline{D_y}) \tag{5}$$

and

$$(F(Q), y) \to m(y) \quad \text{with} \qquad m(y) = (\overline{D_x}y - \overline{N_x}) \;/\; (\overline{N_y} - \overline{D_y}y). \tag{6}$$

These relations can be obtained by using a method either based on a series expansion of the functions $N(x,y)$ and $D(x,y)$ in a neighborhood of $Q = (x_0, y_0)$, or by considering the Jacobian determinant of the inverse map T^{-1} (or one of the inverses if the map is noninvertible). We note that focal points which are not simple can occur in bifurcations called of class two in [6]. From the definition of the prefocal curve, it follows that the Jacobian $\det(DT^{-1})$ must necessarily vanish in the points of δ_Q. Indeed, if the map T^{-1} is defined in δ_Q, then all the points of the line δ_Q are mapped by T^{-1} into the focal point Q. This means that T^{-1} is not locally invertible in the points of δ_Q, being it a many-to-one map, and this implies that its Jacobian cannot be different from zero in the points of δ_Q.

From the relations (5), (6) it results that different arcs γ_j, passing through a focal point Q with different slopes m_j, are mapped by T into bounded arcs $T(\gamma_j)$ crossing δ_Q in different points $(F(Q), y(m_j))$. Interesting properties are obtained if the inverse of T (or the inverses, if T is a noninvertible map) is (are) applied to a curve that crosses a prefocal curve.

2.2 Case of an invertible map

Let δ_Q be a prefocal curve whose corresponding focal point is Q[1]. The map T being invertible, each point sufficiently close to δ_Q has its rank-1 preimage in a neighborhood of the focal point Q. If the inverse T^{-1} is continuous along δ_Q then all the points of δ_Q are mapped by T^{-1} in the focal point Q. Roughly speaking we can say that the prefocal curve δ_Q is "*focalized*" by T^{-1} in the focal point Q, or, more concisely, that $T^{-1}(\delta_Q) = Q$. We note that the map T is not defined in Q, thus T^{-1} cannot to be strictly considered as an inverse of T in the points of δ_Q, even if T^{-1} is defined in δ_Q.

The relation (6) implies that the preimages of different arcs crossing the prefocal curve δ_Q in the same point $(F(Q), y)$ are given by arcs all crossing the singular set through Q, and all with the same slope $m(y)$ in Q. Indeed, consider different arcs ω_n, crossing δ_Q in the same point $(F(Q), y)$ with different slopes, then these arcs are mapped by the inverse T^{-1} into different arcs $T^{-1}(\omega_n)$ through Q, all with the same tangent, of slope $m(y)$, according to (6). They must differ by the curvature at the point Q.

[1] Several prefocal curves may exist, each having a corresponding focal point.

2.3 Case of a noninvertible map

2.3.1 General considerations

In the case of a continuous noninvertible map T, several focal points may be associated with a given prefocal curve δ_Q, each with its own one-to-one correspondence between slopes and points, as that defined by (5) and (6) [2]. The phase space of a noninvertible map is subdivided into open regions (or zones) Z_k, whose points have k distinct rank-1 preimages, obtained by the application of k distinct inverse maps (see e.g. [7], [8]). In other words, if $(x, y) \in Z_k$ then k distinct points (x_j, y_j) are mapped into (x, y), i.e. $T(x_j, y_j) = (x, y)$ for $j = 1, ..., k$, or, equivalently, k distinct inverse maps T_j^{-1} exist such that $T_j^{-1}(x, y) = (x_j, y_j)$, $j = 1, ..., k$. A specific feature of noninvertible maps is the existence of the *critical set* LC (from the French "Ligne Critique", see [7], [8]), defined as the locus of points having at least two coincident rank-1 preimages, located on the set of merging preimages denoted by LC_{-1}. In any neighborhood of a point of LC_{-1} there are at least two distinct points mapped by T into the same point, so that the map T is not locally invertible in the points of LC_{-1}, which implies that for differentiable maps the set LC_{-1} is included in the set J_0 of points in which the Jacobian of T vanishes:

$$J_0 = \left\{(x, y) \in \mathbb{R}^2 | \det DT = 0\right\} \tag{7}$$

and $LC_{-1} \subseteq J_0$.

Segments of the critical curve $LC = T(LC_{-1})$ are boundaries that separate different regions Z_k, but the converse is not generally true, that is boundaries of regions Z_k, which are not portions of LC, may exist (this happens, for example, in polynomial maps having an inverse function with a vanishing denominator, as shown in [2]). This fact is related to the existence of a set which is mapped by T in one point, such a set belongs to J_0 but is not critical, so that we have a strict inclusion: $LC_{-1} \subset J_0$.

Another distinguishing feature in many noninvertible maps is the existence of a set of points, which we shall denote by J_C, crossing which we have a change in the sign of the Jacobian of T, $\det DT$. From the geometric action of the foliation of the Riemann plane we can also say that the critical set LC_{-1} must belong to J_C. In fact, a plane region U which intersects LC_{-1} is "folded" along LC into the side with more preimages, and the two folded images have opposite orientation; this implies that the map has different sign of the Jacobian in the two portions of U separated by LC_{-1}. So, $LC_{-1} \subseteq J_C$.

Moreover, we have seen that often for a *smooth map without a vanishing*

[2] Several prefocal curves may exist, each associated with one or more focal points.

denominator, $LC_{-1} = J_C$ ($\subseteq J_0$ and the strict inclusion occurs when there are sets mapped into a point), whereas for a *smooth map with a vanishing denominator* the situation is more complex, because it can be seen that the set J_C may include also the singular set δ_S (although this can be considered as a bifurcation case), so that it occurs the strict inclusion $LC_{-1} \subset J_C$. However, from the geometric properties of T, the following relation seems to hold:

$$LC_{-1} = J_C \cap J_0 \tag{8}$$

With continuously differentiable maps without a denominator, one has $LC_{-1} = J_C \subset J_0$, even if $LC_{-1} = J_C = J_0$ often occurs in the simplest situations. In smooth maps with a vanishing denominator $LC_{-1} = J_0 \subset J_C$ may occur. All these are particular cases of (8).

From the properties of maps with a vanishing denominator it results (see [5]) that generally a focal point Q belongs to the set $\overline{LC_{-1}} \cap \delta_S$, where $\overline{LC_{-1}}$ denotes the closure of LC_{-1}, but in the particular bifurcation cases mentioned above, in which δ_S belongs to J_C, it happens that a focal point Q may not belong to $\overline{LC_{-1}}$. The geometric behavior and the plane's foliation are different in the two cases. This leads to two different situations, according to the fact that the focal points belong or not to the set $\overline{LC_{-1}}$.

2.3.2 The focal points do not belong to $\overline{LC_{-1}}$.

The following properties have been shown in [2]. **(a)** *For each prefocal curve δ_Q we have $LC \cap \delta_Q = \emptyset$.* **(b)** *If all the inverses are continuous along a prefocal curve δ_Q, then the whole prefocal set δ_Q belongs to a unique region Z_k in which k inverse maps T_j^{-1} , j=1,...,k, are defined.*

It is plain that for a prefocal δ_Q at least one inverse is defined that "focalizes" it into a focal point Q. However, other inverses may exist that "focalize" it into distinct focal points, all associated with the same prefocal curve δ_Q. These focal points are denoted as $Q_j = T_j^{-1}(\delta_Q)$, $j = 1, ..., n$, with $n \leq k$. For each focal point Q_j the same results given above can be obtained with T^{-1} replaced by T_j^{-1}, so that for each Q_j a one-to-one correspondence $m_j(y)$ in the form (6) is defined. Following arguments similar to those given above, it is easy to see that an arc ω crossing δ_Q in a point $(F(Q), y)$, where $F(Q) = F(Q_j)$ for any j, is mapped by each T_j^{-1} into an arc $T_j^{-1}(\omega)$, through the corresponding Q_j with the slope $m_j(y)$. If different arcs are considered, crossing δ_Q in the same point, then these are mapped by each inverse T_j^{-1} into different arcs through Q_j, all with the same tangent.

We note that property (a) given above implies that the critical curve LC is generally asymptotic to the prefocal curves, as shown in [2] through several examples.

2.3.3 The focal points belong to $\overline{LC_{-1}}$.

When the focal points belong to $\overline{LC_{-1}}$ (closure of LC_{-1}) the "geometrical" situations of the phase plane, and the bifurcation types, are more complex (see [5]) with respect to those of section 2.3.2. This is due to the fact that now LC has contact points at finite distance with the prefocal curves. The property $Q_j = T_j^{-1}(\delta_Q)$, $j = 1, ..., n$, with $n \leq k$, does not occur. Now in the generic case a given prefocal curve δ_Q is not associated with several focal points Q_j as in section 2.3.2. Only one of the inverses T_j^{-1} maps a non critical point of a given prefocal curve into its related focal point, so that we can write $Q = T_j^{-1}(F(Q), y)$ (or $Q = T_j^{-1}(\delta_Q)$ for short), but the index j depends on the non critical point $(F(Q), y)$ considered on δ_Q. For this reason the previous situation of δ_Q (focal points do not belong to $\overline{LC_{-1}}$) appears as non generic (indeed it may result from the merging of two prefocal curves δ_Q^r and δ_Q^s without merging of the corresponding focal points, as shown in [5]).

A qualitative illustration of the simplest case is given in fig. 1, where a situation with two prefocal curves is represented for a noninvertible map $T(x, y) \rightarrow (x', x')$ of type (Z_0-Z_2). The inverse relation $T^{-1}(x', y')$ has two components in the region Z_2, denoted by T_1^{-1} and T_2^{-1}, and no real components in the region Z_0. The set of nondefinition δ_s is a simple straight line, and there are two prefocal lines, δ_{Q_i}, of equation $x = F(Q_i)$, associated with the focal points Q_i, $i = 1, 2$, respectively, and $V_i = LC \cap \delta_{Q_i}$ are the points of tangency between LC and the two prefocal curves. Let δ'_{Q_i} be the segment of δ_{Q_i} such that $y < y(V_i)$ (continuous line in Fig.1), and δ''_{Q_i} the segment of δ_{Q_i} such that $y > y(V_i)$ (pecked line in Fig.1). The "focalization" occurs in the following way:

$$T_1^{-1}(\delta'_{Q_i}) = Q_i, \qquad T_2^{-1}(\delta''_{Q_i}) = Q_i \tag{9}$$

with $T_2^{-1}(\delta'_{Q_i}) \cup T_1^{-1}(\delta''_{Q_i}) = \pi_i$, $i = 1, 2$, being the two lines passing through the focal points Q_i and tangent to LC_{-1}at these points. When $\delta_{Q_1} \rightarrow \delta_{Q_2}$, due to a parameter variation, without merging of the focal points, the points V_i on the prefocal curves tend to infinity, i.e. $\delta_{Q_1} = \delta_{Q_2}$ becomes an asymptote for LC.

These situations can be easily observed, for example, by using the following map

$$T_e : \begin{cases} x' = y + \varepsilon x \\ y' = \frac{\alpha x^2 + \gamma x}{y - \beta + \sigma x} \end{cases} \tag{10}$$

not defined in the points of the line δ_s of equation $y - \beta + \sigma x = 0$, on which two focal points exist given by $Q_1 = (0,\ \beta)$, $Q_2 = (-\frac{\gamma}{\alpha},\ \beta + \frac{\gamma\delta}{\alpha})$ and the corresponding prefocal curves δ_{Q_i}, of equation $x = F(Q_i)$, $i = 1, 2$, are $x = \beta$

and $x = \beta - (\varepsilon - \sigma)\,\gamma/\alpha$ respectively.

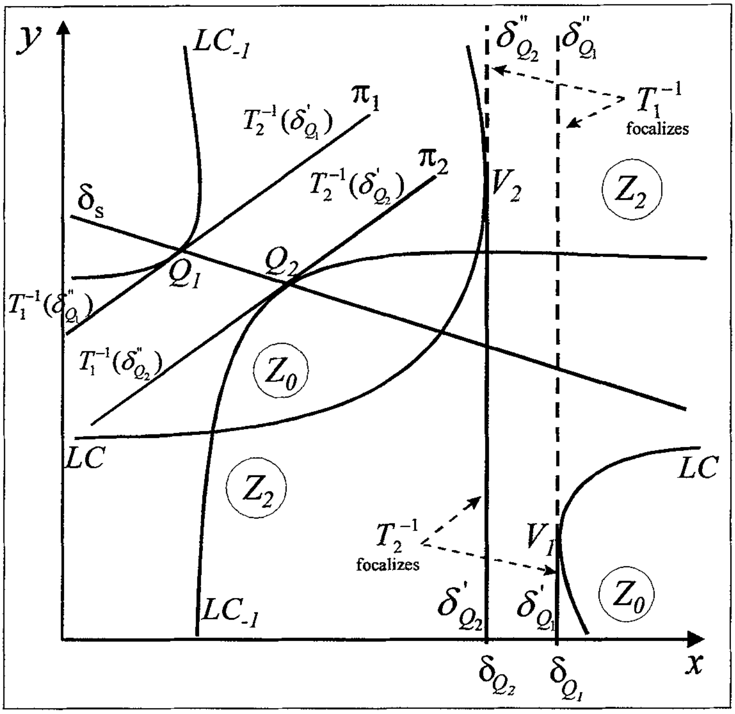

Figure 1

The map T_e is a noninvertible map of (Z_0-Z_2) type with inverses defined by

$$T_{e1,e2}^{-1}: \begin{cases} x = \frac{1}{2\alpha}\left((\sigma - \varepsilon)\,y' - \gamma\right) \mp \sqrt{\Delta(x', y')} \\ y = x' - \varepsilon x \end{cases} \tag{11}$$

where $\Delta(x', y') = (\gamma - \sigma y' + \varepsilon y')^2 - 4\alpha\,(\beta y' - x'y') > 0$ in the region Z_2 and $LC = \{(x, y)\,|\Delta(x', y') = 0\}$. In fig. 2a, obtained with parameters $\alpha = 0.5$, $\gamma = 0.5$, $\beta = \sqrt{2}$, $\sigma = 0.2$, $\varepsilon = -0.2$, a situation similar to the one shown in fig. 1 is obtained (the grey region represents the basin of the stable fixed point $O = (0, 0)$ and the complementary region the basin of infinity). In fig. 2b, obtained with $\varepsilon = \sigma = 0.1$ and the other parameters like in fig. 2a, the two focal points are still distinct, but the two prefocal lines merge and become an asymptote for LC. Note that we have $LC_{-1} = J_C = J_0$ before the bifurcation, while at the bifurcation the hyperbola LC_{-1} degenerates into two lines, the vertical branch gives the new critical set LC_{-1} and the other collapses into the singular set δ_s. At this bifurcation $LC_{-1} = J_0 \subset J_C$ (because J_C also includes the set of nondefinition δ_s), the resulting "double" prefocal curve is an asymptote of LC, the arcs π_1 and π_2 degenerate into the focal points, which now are not located on $\overline{LC_{-1}}$.

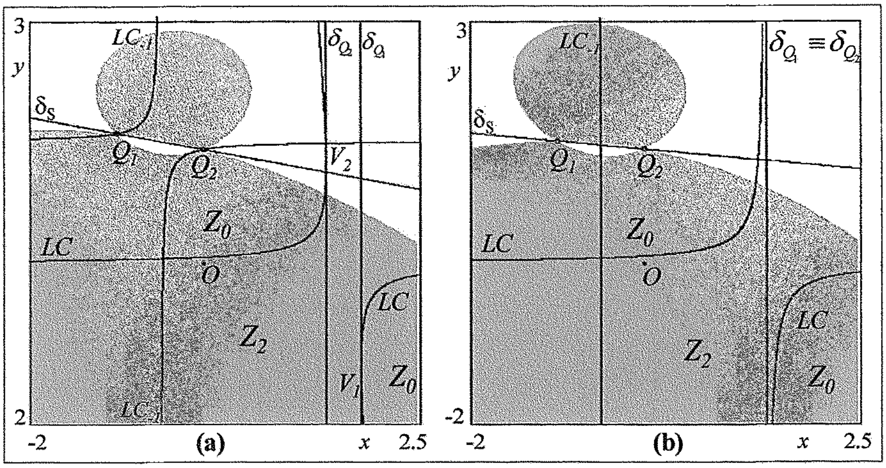

Figure 2

3 Some geometric properties of focal points and prefocal curves

3.1 Generalities

When the presence of a vanishing denominator induce the existence of focal points, important effects on the geometrical and dynamical properties of the map T can be observed. Indeed, a contact between a curve segment γ and the singular set δ_s causes noticeable qualitative changes in the shape of the image $T(\gamma)$. Moreover, a contact of an arc ω with a prefocal curve δ_Q, gives rise to important qualitative changes in the shape of the preimages $T_j^{-1}(\omega)$. When the arcs ω are portions of phase curves of the map T, such as *invariant closed curves*, *stable or unstable sets* of saddles, *basin boundaries*, one has that contacts between singularities of different nature generally induce important qualitative changes, which constitute *new types of global bifurcations* that change the structure of the attracting sets, or of their basins.

In order to understand the geometric and dynamic properties of maps with vanishing denominator, and their particular global bifurcations, it is assumed that δ_s and δ_Q are made up of branches of simple curves of the plane. In the following we describe what happens to the images of a small curve segment γ when it has a tangential contact with δ_s and then crosses it in two points, and what happens to the preimages of a small curve segment ω when it has a contact with a prefocal curve δ_Q and then crosses it in two points.

3.2 Action of the map

Consider first a bounded curve segment γ that lies entirely in a region in which no denominator of the map T vanishes, so that the map is continuous in all the points of γ. As the arc γ is a compact subset of $\mathbb{R}^2$, also its image $T(\gamma)$ is compact (see the upper qualitative sketch in fig.3). Suppose now to move γ towards δ_s, until it becomes tangent to it in a point $A_0 = (x_0, y_0)$ which is not a focal point. This implies that the image $T(\gamma)$ is given by the union of two disjoint and unbounded branches, both asymptotic to the line σ of equation $x = F(x_0, y_0)$. Indeed, $T(\gamma) = T(\gamma_a) \cup T(\gamma_b)$, where γ_a and γ_b are the two arcs of γ separated by the point $A_0 = \gamma \cap \delta_s$. The map T is not defined in A_0 and the limit of $T(x, y)$ assumes the form (3) as $(x, y) \to A_0$ along γ_a, as well as along γ_b. In such a situation any image of γ of rank $k > 1$, given by $T^k(\gamma)$, includes two disjoint unbounded branches, asymptotic to the rank-k image of the line σ, $T^k(\sigma)$. When γ crosses through δ_s in two points, say $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$, both different from focal points, then the asymptote σ splits into two disjoint asymptotes σ_1 and σ_2 of equations $x = F(x_1, y_1)$ and $x = F(x_2, y_2)$ respectively, and the image $T(\gamma)$ is given by the union of three disjoint unbounded branches (see the lower sketch in fig.3).

When γ is, for example, the local unstable manifold W^u of a saddle point or saddle cycle, the qualitative change of $T(\gamma)$, due to a contact between γ and δ_s, as described above, may represent an important contact bifurcation of the map T. Indeed the creation of a new unbounded branch of W^u, due to a contact with δ_s, may cause the *creation of homoclinic points*, from new transverse intersections between the stable and unstable sets, W^s and W^u, of the same saddle point (or cycle). In such a case it is worth noting that the corresponding *homoclinic bifurcation* does not come from a tangential contact between W^u and W^s. For maps with a vanishing denominator, this implies that homoclinic points can be created without a homoclinic tangency between W^u and W^s, from the sudden creation of unbounded branches of W^u when it crosses through δ_s (see [2]). If before the bifurcation W^u is associated with a chaotic attractor, the *homoclinic bifurcation* resulting from the contact between W^u and δ_s may gives rise to an *unbounded chaotic attractive set made up of unbounded, but not diverging, chaotic trajectories* (see [3]). If before the bifurcation W^u is not associated with a chaotic attractor, the *homoclinic bifurcation* resulting from the contact between W^u and δ_s may gives rise to a *basin explosion* as described in [2].

If the map is noninvertible, a direct consequence of the above arguments concerns the action of the curve of nondefinition δ_s on LC_{-1}. *If* $\overline{LC_{-1}}$ *has n transverse intersections with the set* δ_s *in non focal points* $P_i = (x_i, y_i)$, $i = 1, .., n$, *then the critical set* $LC = T(LC_{-1})$ *includes* $(n + 1)$ *disjoint unbounded branches, separated by the n asymptotes* σ_i *of equation* $x = F(x_i, y_i)$,

$i = 1, .., n.$

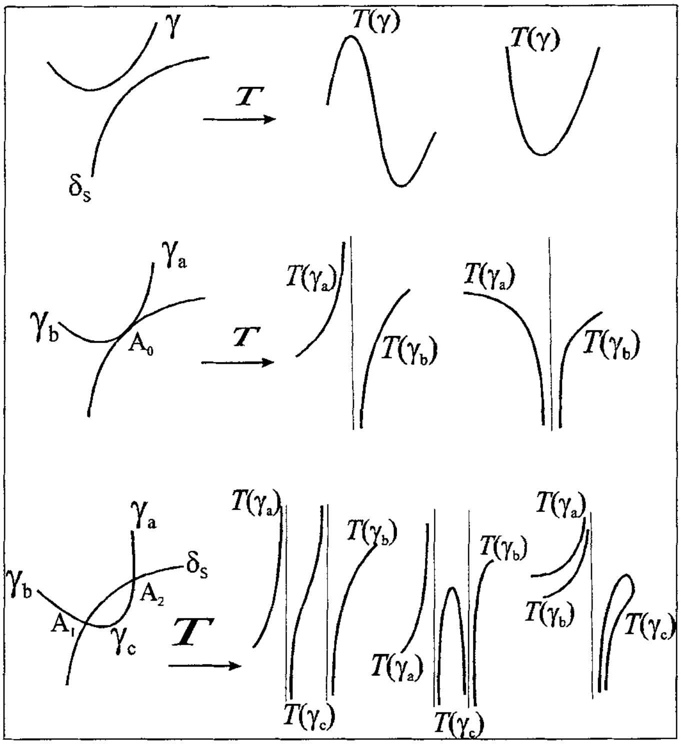

Figure 3

3.3 Action of the inverses

- (a) Let T be the map (1), which is assumed to be an *invertible map.* Consider a smooth curve segment ω that moves towards a prefocal curve δ_Q until it crosses through δ_Q (see fig.4) so that only a focal point $Q = T^{-1}(\delta_Q)$ is associated with δ_Q. The prefocal set δ_Q belongs to the line of equation $x = F(Q)$, and the one-to-one correspondence defined by (5) and (6) holds. When ω moves toward δ_Q, its preimage $\omega_{-1} = T^{-1}(\omega)$ moves towards Q. If ω becomes tangent to δ_Q in a point $C = (F(Q), y_c)$, then ω_{-1} has a cusp point at Q. The slope of the common tangent to the two arcs, that join at Q, is given by $m(y_c)$, according to (6). If the curve segment ω moves further, so that it crosses δ_Q at two points $(F(Q), y_1)$ and $(F(Q), y_2)$, then ω_{-1} forms a *loop with a double point* at the focal point Q. Indeed, the two portions of ω that intersect δ_Q are both mapped by T^{-1} into arcs through Q, and the tangents to these two arcs of ω_{-1}, issuing from the focal point, have different slopes, $m(y_1)$ and $m(y_2)$ respectively, according to (6).

- (b) Now let T be a *noninvertible map with focal points not located on* $\overline{LC_{-1}}$. In this case $k \geq 1$ distinct focal points Q_j, $j = 1, ..., k$, may be associated with a prefocal curve δ_Q. Then each inverse T_j^{-1}, $j = 1, ..., k$, gives a distinct preimage $\omega_{-1}^j = T_j^{-1}(\omega)$ which has a cusp point in Q_j, $j = 1, ..., k$, when the arc ω is tangent to δ_Q. Each preimage ω_{-1}^j gives rise to a loop in Q_j when the arc ω intersects δ_Q in two points (see fig.5, concerning the case $k = 2$).

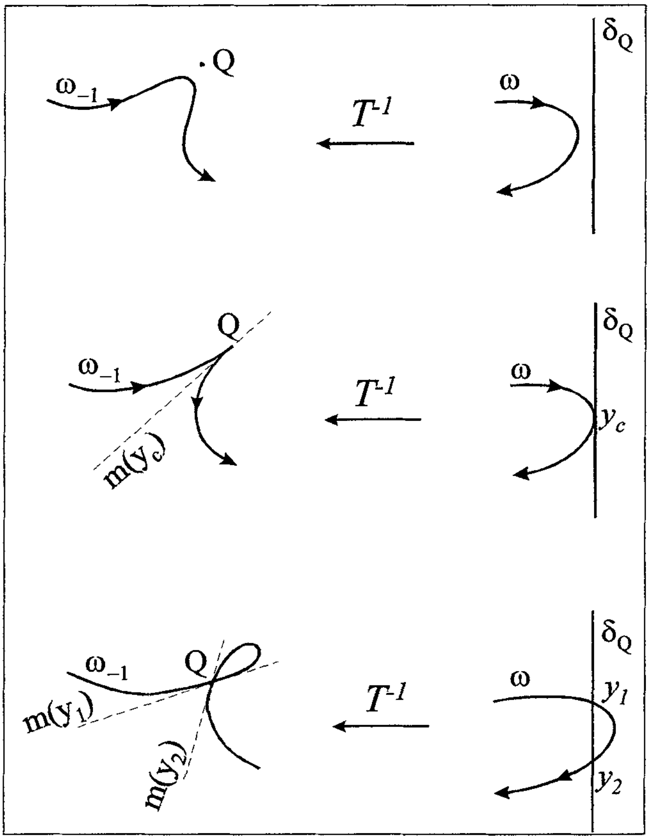

Figure 4

When ω is an arc belonging to a basin boundary $\mathcal{F}$, the qualitative modifications of the preimages $T_j^{-1}(\omega)$ of ω, due to a tangential contact of ω with the prefocal curve, can be particularly important for the global dynamical properties of the map T. As a frontier $\mathcal{F}$, generally is backward invariant, i.e. $T^{-1}(\mathcal{F}) = \mathcal{F}$, if ω is an arc belonging to $\mathcal{F}$, then all its preimages of any rank must belong to $\mathcal{F}$. This implies that if a portion ω of $\mathcal{F}$ has a tangential contact with a prefocal curve δ_Q, then necessarily at least k cusp points, located in the focal points Q_j, are included in the boundary $\mathcal{F}$. Moreover, if the focal points Q_j have preimages, then also they belong to $\mathcal{F}$, so that further cusps exist on $\mathcal{F}$, with tips at each of such preimages. It results that if the basin boundary $\mathcal{F}$ was smooth before the contact with the prefocal curve δ_Q, such a contact gives rise to points of non smoothness, which may be infinitely many if some focal point Q_j has preimages of any rank, with possibility of fractalization of $\mathcal{F}$ when it is nowhere smooth. When $\mathcal{F}$ crosses through δ_Q in two points, after the contact $\mathcal{F}$ must contain at least k loops with double points in Q_j. Also in this case, if some focal point Q_j has preimages, other loops appear (even infinitely many, with possibility of fractalization) with double points in the preimages of any rank of Q_j, $j = 1, ..., n$.

- (c) Whatever be the map T (invertible, or not, with focal points on $\overline{LC_{-1}}$ or not) *a contact of a basin boundary with a prefocal curve gives rise to a new type of basin bifurcation that causes the creation of cusp points and, after the crossing, of loops (called "lobes"), along the basin boundary.* This may give rise to a very particular fractalization of the basin boundary [1].

- (d) Let T be a *noninvertible map with focal points not located on* $\overline{LC_{-1}}$. In this case the contact of two *lobes* on LC_{-1} (related to a contact of LC with the basin boundary) gives rise to a *crescent* bounded by the two focal points, from which lobes appeared. *The creation of "crescents" [2], resulting from the contact of lobes, is specific to noninvertible maps with denominator,when the focal points are not located on* $\overline{LC_{-1}}$. It requires the intersection of the boundary with a prefocal curve (located in a region with more than one inverse), at which the lobes are created, followed by a contact with a critical curve, causing the contact and merging of the lobes. At the contact the lobes are not tangent to LC_{-1}. After the contact, they merge creating the crescent.

- (e) If T is a *noninvertible map with focal points located on* $\overline{LC_{-1}}$, then in the generic case we have a behavior similar to that of the invertible case, in which only one focal point is associated with δ_Q, but in a more complex situation with respect to the role of the components of the inverse map on δ_Q, and the presence of the arcs denoted π_i, in Fig.1. Details on this situation are given in [5]. Now a *crescent* does not results from the contact of two lobes, but from *the contact of a lobe (issuing from a focal point) with another focal point.* This situation is specific to noninvertible maps with denominator, when the focal points are located on $\overline{LC_{-1}}$. It requires the intersection of a basin boundary with a prefocal curve, followed by the contact of the resulting lobe with a focal point.

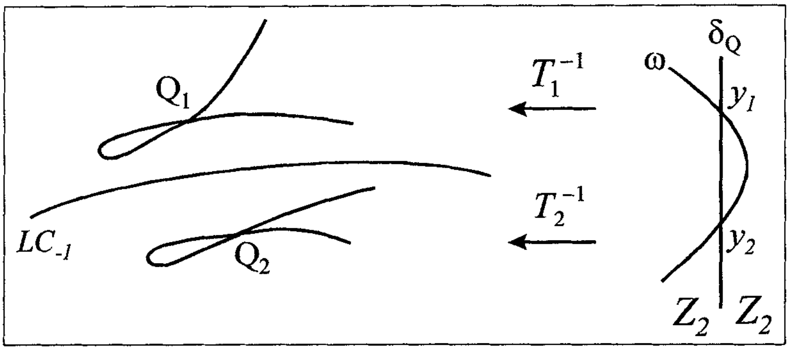

Figure 5

4 Further remarks and conclusions.

The theory of focal points and prefocal curves is also useful to understand some properties of maps defined in the whole plane $\mathbb{R}^2$, having at least one inverse map with vanishing denominator. Such maps may have the property that, among the points at which the Jacobian vanishes, there exists a curve which is mapped into a single point (see [2]). Another noticeable property of these maps is that a curve, at which the denominator of some inverse vanishes may separate regions of the phase plane characterized by a different number of preimages, even if it is not a critical curve of rank-1 (a critical curve of rank-1 is defined as a set of points having at least two merging rank-1 preimages). At least one inverse is not defined on these non-critical boundary curves, due to the vanishing of some denominator. In a two-dimensional map, the role of such a curve is the analogue of an horizontal asymptote in a one-dimensional map, separating the range into intervals with different numbers of rank-1 preimages [2]. The existence of focal points of an inverse map can also cause the creation of particular attracting sets. Indeed a focal point, generated by the inverse map, may behave like a "*knot*", where infinitely many invariant curves of an attracting set shrink into a set of isolated points ([2], [4]).

Maps with focal points and prefocal sets naturally arise in discrete dynamical systems of the plane found in several applications, such as economic modeling (see [9], [10]) or numerical iterative methods (see [11], [12], [13]). In such dynamic models, peculiar structures of the basins, characterized by the presence of lobes and crescents, have been observed, which can be explained in terms of contacts of two sets of different nature, such as prefocal sets with stable and unstable sets of saddles. These bifurcations are called of *first class* (see [5]). Bifurcations of *second class*, given by contacts of two sets of the same nature (such as merging of focal points) are described in [6].

References

[1] Bischi, G.I. and L. Gardini "Basin fractalization due to focal points in a class of triangular maps", *International Journal of Bifurcation and Chaos*, 7(7), pp. 1555-1577 (1997).

[2] Bischi, G.I. , L. Gardini and C. Mira "Maps with denominator. Part 1: some generic properties", *International Journal of Bifurcation & Chaos*, 9(1), 119-153 (1999).

[3] Bischi, G.I. , L. Gardini and C. Mira. "Unbounded sets of attraction", *International Journal of Bifurcation & Chaos*, 10(9), pp. 1437-1470 (2000).

[4] Bischi , G.I. , L. Gardini and C. Mira "New phenomena related to the presence of focal points in two-dimensional maps", *Journal of Annales Mathematicae Salesiane* (special issue Proceedings ECIT98), Vol.13 pp.81-90 (1999).

[5] Bischi G.I., L. Gardini and C. Mira. "Maps with denominator. Part 2: Simple focal points and critical curves", Working paper University of Urbino, 2000.

[6] Bischi, G.I. , L. Gardini and C. Mira "Maps with denominator. Part 3: Nonsimple focal points and related bifurcations", Working paper University of Urbino, 2000.

[7] Gumowski, I. and C. Mira. *Dynamique Chaotique. Transition ordre-désordre*, Cepadues Editions, Toulouse (1980).

[8] Mira C., L. Gardini, A. Barugola and J.C. Cathala. *Chaotic dynamics in two-dimensional noninvertible maps.* World Scientific, Singapore, Series on Nonlinear Science, Series A, vol. 20 (1996).

[9] Bischi, G.I. and A. Naimzada. "Global analysis of a nonlinear model with learning" *Economic Notes*, vol. 26, n.3, pp. 143-174 (1997).

[10] Bischi G.I., M. Kopel, and A. Naimzada. "On a rent-seeking game described by a non-invertible iterated map with denominator". Nonlinear Analysis, T.M.&A., Proceedings of WCNA 2000, Catania, to appear.

[11] Billings, L. and Curry, J.H. "On noninvertible maps of the plane: Eruptions", *Chaos*, 6, 108-119 (1996)

[12] Yee, H.C. and P.K. Sweby "Global asymptotic behavior of iterative implicit schemes", *International Journal of Bifurcation & Chaos*, vol.4, n.6, 1579-1611 (1994).

[13] Gardini, L., G.I. Bischi and D. Fournier-Prunaret "Basin boundaries and focal points in a map coming from Bairstow's method", *Chaos,* 9(2), pp. 367-380 (1999).