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Financial conditions, strategic interaction and complex dynamics: a game-theoretic model of financially driven fluctuations

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Abstract

We propose a game-theoretic model in which each firm chooses the level of economic activity on the basis of its own financial conditions and of the financial conditions of rival firms. The model generates the laws of motion of firms' net worth which may determine convergence to a symmetric steady state or more complex dynamical behaviors, periodic or chaotic, depending on the values of the parameters.

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1. Introduction

In the theoretical literature, financial fragility and instability have been associated with models of (sometimes vicious) interaction between financial and goods markets. This literature has been pioneered by Minsky (see Minsky, 1982) and recently revived in the new view of the relationship between imperfect financial markets and the macroeconomy (Bernanke et al., 1999; Greenwald and Stiglitz, 1993; Kiyotaki and Moore, 1997). In the presence of asymmetric information, in fact, financing constraints are important in investment and production decisions.

The more recent literature, however, is essentially concerned with the emergence of financial fragility in a perfect competition setting. A remarkable example is the theoretical

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framework put forward by Greenwald and Stiglitz (GS hereafter; see Greenwald and Stiglitz, 1993). They assume that each firm faces an infinitely elastic demand function subject to a random idiosyncratic shock. Firms are unable to raise external finance on the Stock market because of equity rationing (Greenwald et al., 1984; Myers and Majluf, 1984). Therefore, they rely first and foremost on internal funds in order to finance production and resort to bank credit if internal funds are insufficient. As a consequence, firms run the risk of bankruptcy. By assumption the probability of bankruptcy is a decreasing function of net worth (or equity base), which is a measure of financial robustness: the higher net worth (the lower financial fragility), the lower the probability of bankruptcy will be. Therefore, if bankruptcy is costly the scale of production is increasing with net worth.

Assuming perfect competition (price taking firms), GS rule out strategic interaction. In this paper we follow a different route, allowing for imperfect competition and strategic interaction among firms which take financial conditions into account when deciding their scale of production. In our framework, each firm faces a negatively sloped demand function subject to a random idiosyncratic shock. The selling price of the firm, however, depends also on the quantity produced by the competitors. In an oligopolistic setting, we can show that in (Nash) equilibrium the scale of production of each firm is a function not only of its own net worth but also of the net worth of rival firms.

In principle a firm can be either financially constrained (incomplete collateralization regime) or unconstrained (full collateralization). If internal funds are insufficient to pay for the wage bill, the firm is financially constrained, goes into debt and incurs bankruptcy costs. If net worth is more than enough to fund the wage bill, the firms is unconstrained and benefits from a financial solidity bonus which plays a role symmetrical to that of bankruptcy costs for the constrained firm.

Each firm accumulates its own equity base according to a law of motion which can be conceived of as an accounting identity: the absolute change in net worth is equal to (expected) retained profits. This law captures the simple idea according to which each firm accumulates net worth in as much as it retains profits instead of distributing them to shareholders as dividends.

The Nash equilibrium level of output of each firm is driven by endogenous fluctuations in the equity bases of the firm itself and of its competitors. Therefore the game-theoretic framework is ideal for the study of composition and cascade effects which are crucial in the development of financially driven fluctuations.

It is worth noting that the endogenous dynamics implicit in our game-theoretic framework are a consequence of the evolution over time of net worth. So far, endogenous dynamics in a game-theoretic framework have been explored in the context of evolutionary game theory where they depend on myopic learning processes, a controversial assumption in a context of rationality and widespread information.

The paper is organized as follows. In Section 2, we discuss the background assumptions. We borrow some of them from Greenwald and Stiglitz (1993), but we give up the representative agent-perfect competition hypothesis on which their framework is based. In our framework, firms operate in a simple oligopolistic set-up. The strategic variable is the quantity produced (Cournot competition).

In Section 3, we examine the benchmark case in which constrained firms do not incur bankruptcy costs and unconstrained firms do not benefit from the solidity bonus. In this case, the quantity produced in the Cournot–Nash equilibrium is the same for each oligopolist (Symmetric Nash Equilibrium (SNE)) and is independent of the financial condition (i.e. the level of the equity base) of the firm and of its rivals. As a consequence, the law of motion of the equity base of each firm is independent of the accumulation of net worth on the part of rival firms. In the benchmark case, therefore, there are two types of *irrelevance* of financial conditions. First, financial conditions of the firms are irrelevant for the determination of equilibrium output. Second, the accumulation of net worth on the part of rival firms does not affect the accumulation of net worth on the part of the individual firm.

In the general case, when firms incur bankruptcy costs/solidity bonuses, these irrelevance results do not hold true any more, as we show in Section 4. First, the quantity produced in Nash equilibrium depends on the financial conditions (that is the equity bases) of the firm and of its rivals. Second, the Nash equilibrium is not symmetric, i.e. the quantity is different from one firm to the other. Third, it is only temporary, because the equity bases of the firms are changing over time. Fourth, the accumulation of the equity base of each firm is affected by the accumulation of net worth on the part of rival firms.

The evolution of the firms' equity bases is represented by a discrete-time dynamical system which can generate a wide range of dynamic patterns: convergence to a steady state, periodic orbits or more complex evolutions, even chaotic. In other words, we obtain endogenous fluctuations of the equity bases of the firms, which drive the dynamic pattern of Cournot–Nash equilibrium. By analytical and numerical arguments we show that if the retention ratios are not uniform across firms and "sufficiently low" the equity bases of the firms will converge to the Symmetric Steady state Nash Equilibrium (SSNE), i.e. in the long run firms become homogeneous as far as the equity ratio and the level of output are concerned. Increasing the value of at least one retention ratio, the equity base of each firm oscillates in a range which is different from one firm to the other. Further increases in at least one of the retention ratios yield more complex, and consequently less predictable, dynamics of the equity bases.

Section 5 is devoted to the analysis of the impact of changes in the retention ratios on the long run dynamical properties of the system by means of bifurcation diagrams. Section 6 concludes.

2. Background assumptions

We focus on the behavior of firms. The price $\tilde{p}_i$ at which the ith firm sells its good is uncertain.[1] Price uncertainty is captured by assuming that $\tilde{p}_i$ differs from the general (average) price level P because of a random idiosyncratic shock $\tilde{u}_i$, with support $(u_{\min}, u_{\max})$ distributed according to a density function $f(\tilde{u}_i)$ with expected value $E(\tilde{u}_i) = \int \tilde{u}_i f(\tilde{u}_i)\,d\tilde{u}_i = 0$. In symbols:

$$\tilde{p}_i = \tilde{u}_i + P$$

Therefore, the expected value of $\tilde{p}_i$ will be $E(\tilde{p}_i) = P$.

[1] Following a widely adopted convention, a tilde on a variable means that the variable is stochastic. For the sake of notational simplicity, undated variables are referred to the current period (period t).

The average price level, in turn, is a function of total production: $P = P\left(\sum_i q_i\right)$, where q_i is the quantity produced by the ith firm and $P(\cdot)$ is a decreasing function. In the simplest case in which the corporate sector consists only of two firms, assuming a linear functional form, the inverse aggregate demand function is:

$$P = a - b(q_1 + q_2) \tag{2.1}$$

where a and b are positive parameters.

In order to simplify the argument, we assume that production is carried out by means of one-to-one technology: $q_i = n_i$ where n_i is employment. Firms finance production costs, i.e. the wage bill (wq_i), at least partially by means of internally generated funds, which will be referred to as *net worth or equity base* (A_i). In the following we will keep the nominal wage constant.

We can envisage two financial regimes according to the relative magnitude of the wage bill and the equity base. The first regime—which we will label *incomplete collateralization*—occurs when net worth is not sufficient to pay the wage bill, i.e. $A_i < wq_i$. In this case the firm is financially constrained and has to resort to credit. The demand for loans is $B_i = wq_i - A_i$. The (gross) interest rate $R = 1+r$ is exogenous. At that interest rate, banks extend credit on demand. Moreover, debt must be repaid completely in one period (there is no accumulation of debt). In this case, RB_i represent debt commitments for the firm and is a cost component.

Bankruptcy occurs if the firm is unable to service its debt. In order to simplify the analysis, in the following we will assume that the probability of bankruptcy is captured by the ratio of debt to the wage bill:

$$PB_i = \frac{B_i}{wq_i} = 1 - \frac{A_i}{wq_i} \tag{2.2}$$

According to (2.2) the probability of bankruptcy is increasing with output and decreasing with net worth. (2.2) is a very simple way of linking the probability of bankruptcy to a measure of financial fragility of the firm.[2]

Finally, we assume that bankruptcy is costly and that bankruptcy costs are a quadratic function of the scale of production:[3]

$$CB_i = cq_i^2 \tag{2.3}$$

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[2] Defining leverage as the debt to equity ratio, we get

$$l_i = \frac{B_i}{A_i} = \frac{wq_i}{A_i} - 1 = \frac{PB_i}{1 - PB_i}$$

and rearranging

$$PB_i = \frac{l_i}{1 + l_i}.$$

The probability of bankruptcy therefore is an increasing concave function of leverage such that $\lim_{l_i \to 0} PB_i = 0$ and $\lim_{l_i \to +\infty} PB_i = 1$ (as one would expect).

[3] On bankruptcy cost, see (Altman, 1984; Gilson, 1990; Kaplan and Reishus, 1990).

In this regime, the objective function of the firm is equal to expected profit less bankruptcy cost in case of default:

$$V_i = E(\tilde{\Pi}_i) - CB_i \cdot PB_i \tag{2.4}$$

The individual price being stochastic, also profit is a random variable:

$$\tilde{\Pi}_i = \tilde{p}_i q_i - RB_i = \tilde{u}_i q_i + Pq_i - R(wq_i - A_i) = (\tilde{u}_i + P - Rw)q_i + RA_i \tag{2.5}$$

Therefore, expected profit is:

$$E(\tilde{\Pi}_i) = Pq_i - RB_i = Pq_i - R(wq_i - A_i) = (P - Rw)q_i + RA_i \tag{2.6}$$

where P is given by (2.1). Substituting (2.2), (2.3) and (2.6) into (2.4) we end up with:

$$V_i = (P - Rw)q_i + RA_i - cq_i^2 + cq_i \frac{A_i}{w} \tag{2.7}$$

The second regime, characterized by *full collateralization*, occurs when net worth is more than enough to fund the wage bill, i.e. $A_i > wq_i$. In this case the firm has financial slack $S_i = A_i - wq_i$ which it can invest at the going interest rate and get a return of RS_i. This is a revenue component.

The ratio of the financial slack to the wage bill captures the degree of financial robustness of the firm $FR_i = (A_i - wq_i)/wq_i$. We assume that in this regime the firm is granted a *solidity bonus* equal to cq_i^2 times the degree of financial robustness: $cq_i^2 FR_i$.

In this regime, the objective function of the firm is equal to expected profit plus the solidity bonus:

$$V_i = E(\tilde{\Pi}_i) + cq_i^2 FR_i \tag{2.8}$$

Expected profit is

$$E(\tilde{\Pi}_i) = Pq_i + RS_i = Pq_i + R(A_i - wq_i) = (P - Rw)q_i + RA_i$$

where P is given by (2.1). Substituting this expression into (2.8) and taking into account the definition of financial robustness we get:

$$V_i = (P - Rw)q_i + RA_i - cq_i^2 + cq_i \frac{A_i}{w}$$

which is identical to the objective function of the firm in the incomplete collateralization case (see (2.7) above).

In the end, the firm has the same quadratic objective function regardless of the financial regime it is experiencing. This conclusion greatly simplifies the analytic structure of the model. Of course this is a consequence of the assumptions made above, in particular of the introduction of a solidity bonus in the full collateralization regime. The rationale for the solidity bonus is symmetrical to that of the bankruptcy cost in the incomplete collateralization regime.

Besides the legal and administrative costs of bankruptcy, according to Davis (1992, p. 46) there are indirect costs due to the fact that "imminent bankruptcy may change the firm's stream of cash flow, owing to various factors, such as the inability to obtain trade credit, inability to retain key employees, declining faith among the customers in the product, etc." and wider costs, such as the "loss of reputation" that managers of a bankrupt firm face. Symmetrically, we assume that financial solidity makes the firm's cash flow less susceptible to sudden changes and boosts the reputation of the managers, showing up as a revenue component in the objective function of the firm.

3. The benchmark case: $c = 0$

Let's assume, as a convenient special case, the absence of bankruptcy costs for the financially constrained firm and of solidity bonuses for the unconstrained firm ($c = 0$). In this case, the objective function of the ith firm is equal to the expected profit.

$$E(\tilde{\Pi}_i) = Pq_i - R(wq_i - A_i) = (P - Rw)q_i + RA_i$$

where P is given by (2.1).

The output level is decided according to the following optimization problem

$$\max_{q_i} E(\tilde{\Pi}_i) \tag{3.1}$$

The first-order conditions are:

$$\begin{aligned} \frac{\partial E(\tilde{\Pi}_1)}{\partial q_1} &= a - 2bq_1 - bq_2 - Rw = 0 \\ \frac{\partial E(\tilde{\Pi}_2)}{\partial q_2} &= a - 2bq_2 - bq_1 - Rw = 0. \end{aligned} \tag{3.2}$$

The Eq. (3.2) represent the Best Reply Functions (BRF) of the two firms. They are both linear and negatively sloped, and they only depend on the parameters a, b of the demand function (2.1) and on the marginal cost Rw. These parameters are uniform across firms, hence the players are symmetric.

From (3.2) we obtain a unique equilibrium $E^* = (q_1^*, q_2^*)$, where q_1^* and q_2^* are given by:

$$q_1^* = q_2^* = \frac{1}{3b}(a - Rw) \tag{3.3}$$

We will assume

$$a > Rw \tag{3.4}$$

i.e. the maximum level of the average price is greater than the marginal cost in order to assure that q_1^* and q_2^* positive quantities. With such assumption (3.3) defines the Symmetric Nash Equilibrium, at which the equilibrium level of output is identical for the two firms.

It is worth noting that the equilibrium level of output does not depend on the individual financial conditions. In principle, the two firms can differ in their degree of financial fragility/robustness as captured by the individual net worth. This difference, however, plays no role in the determination of the equilibrium level of output. In other words, the symmetry between the players, which is evident from (3.2), leads to identical equilibrium levels of output and makes the individual financial conditions "irrelevant" for output determination.

The different degrees of financial robustness, however, play a role in the accumulation of net worth. For each firm, in fact, the law of motion of net worth is described by the difference equation:

$$A_{i,t+1} = A_{i,t} + v_i E(\tilde{\Pi}_{i,t}) \quad i = 1, 2 \tag{3.5}$$

where $v_i \in [0, 1]$ is the *retention ratio*, and (expected) profit $E(\tilde{\Pi}_{i,t})$ in period *t* is given by (2.6), i.e.

$$E(\tilde{\Pi}_{1,t}) = (a - Rw)q_{1,t} - bq_{1,t}^2 - bq_{1,t}q_{2,t} + RA_{1,t}$$

and

$$E(\tilde{\Pi}_{2,t}) = (a - Rw)q_{2,t} - bq_{2,t}^2 - bq_{1,t}q_{2,t} + RA_{2,t}$$

According to (3.5), the absolute change of net worth is equal to (expected) retained profits. This captures the simple idea that each firm accumulates net worth in as much as it retains profits instead of distributing them to shareholders as dividends.

At the Nash equilibrium $E^* = (q_1^*, q_2^*)$ with q_1^* and q_2^* given by (3.3), we have

$$E(\tilde{\Pi}_i) = \frac{1}{9b}(a - Rw)^2 + RA_i \tag{3.6}$$

For each firm, profit "today" (and therefore net worth "tomorrow", (see (3.5))) depends on the individual net worth (the product RA_i is a scale factor for the level of profit).

Assuming that both producers instantaneously move to the Nash equilibrium (3.3) in each time period, so that (expected) profits are given by (3.6), the equation that governs the evolution of the equity base of each firm becomes:

$$A_{i,t+1} = \frac{v_i}{9b}(a - Rw)^2 + (1 + v_i R)A_{i,t} \tag{3.7}$$

Therefore, the law of motion of the equity base of the *i*th firm is independent of the accumulation of its rivals' net worth, and is expressed by a linear first-order difference equation whose dynamical behavior is trivial. In fact, the graph of (3.7) on the $(A_{i,t}, A_{i,t+1})$ plane is a straight line with intercept $(v_i/9b)(a - Rw)^2 > 0$ and slope $(1 + v_i R) > 1$, so that the time evolution of each net worth A_i, $i = 1, 2$, is always characterized by an increasing sequence $\{A_{i,t}, t \geq 0\}$. Notice that, according to (3.5), the condition $A_{i,t+1} = A_{i,t}$, which characterizes the steady states, is equivalent to the condition $E(\tilde{\Pi}_i) = 0$, that is, the steady states are points of zero expected profit for each firm. From (3.7), however, it is clear that this cannot happen at the Nash equilibrium. In other words, at the Nash equilibrium each firm accumulates net worth at a pace equal to:

$$A_{i,t+1} - A_{i,t} = \frac{v_i}{9b}(a - Rw)^2 + v_i RA_{i,t}$$

The rate of net worth accumulation is:

$$gA_i := \frac{A_{i,t+1} - A_{i,t}}{A_{i,t}} = \frac{v_i}{9bA_{i,t}}(a - Rw)^2 + v_i R$$

The "long run" rate of net worth accumulation therefore is:

$$\hat{g}A_i := \lim_{A_{i,t}\to\infty} \frac{A_{i,t+1} - A_{i,t}}{A_{i,t}} = v_i R$$

Notice that the expected profit of each firm is always positive. The long run rate of equity accumulation is increasing with the individual retention ratio and the interest rate.

4. The general case: $c \neq 0$

4.1. Nash equilibrium

If there are bankruptcy costs and solidity bonuses, the objective function of the ith firm is:

$$V_i = (P - Rw)q_i + RA_i - cq_i^2 + cq_i \frac{A_i}{w} \tag{4.1}$$

where P is given by (2.1).

The output level is decided according to the following optimization problem:

$$\max_{q_i} V_i \tag{4.2}$$

The first-order conditions are:

$$\frac{\partial V_1}{\partial q_1} = a - 2bq_1 - bq_2 - Rw - 2cq_1 + c\frac{A_1}{w} = 0 \tag{4.3}$$

$$\frac{\partial V_2}{\partial q_2} = a - 2bq_2 - bq_1 - Rw - 2cq_2 + c\frac{A_2}{w} = 0 \tag{4.4}$$

These equations represent the Best Reply Functions of the two firms. They are linear and negatively sloped, as in the case analyzed in the previous section. However, now the BRF of each player does not depend only on the parameters of the demand function (2.1) and on the marginal cost Rw, which are uniform across firms, but also on the equity base of the player. In as much as the equity bases are different, the symmetry between players which characterized the benchmark case is lost. In fact, the optimal level of output of each firm is an increasing function of its own equity base in both regimes. In the incomplete collateralization regime, the higher is net worth, the lower the probability of bankruptcy and the associated cost for the firm and the higher the volume of output. In the full collateralization regime, the higher is net worth, the higher the degree of financial robustness and the solidity bonus for the firm and the higher the volume of output.

Solving (4.3) and (4.4) we obtain a unique Cournot–Nash equilibrium $E^* = (q_1^*, q_2^*)$. where q_1^* and q_2^* are given by the following linear functions of the equity bases A_1 and A_2:

$$\begin{aligned} q_1^* &= \frac{1}{3b+2c}\left[a - Rw + \frac{2c(b+c)}{(b+2c)w}A_1 - \frac{bc}{(b+2c)w}A_2\right] \\ q_2^* &= \frac{1}{3b+2c}\left[a - Rw - \frac{bc}{(b+2c)w}A_1 + \frac{2c(b+c)}{(b+2c)w}A_2\right]. \end{aligned} \tag{4.5}$$

For each firm, the Nash equilibrium level of output is an increasing function of its own equity base and a decreasing function of the equity base of the rival. This result is an obvious consequence of the *strategic substitutability* implicit in Cournot competition: the higher the net worth of the second firm, the higher its output, the lower the average price level and the output of the first firm.

As expected, other things being equal, the firm with lower equity base (the *smaller firm*, for short) produces less, at the equilibrium, than the firm with higher equity base (the *bigger firm*). Therefore the Nash equilibrium (4.5) is not symmetric, unless $A_1 = A_2$ by a fluke.

As in the benchmark case, we assume that both firms reach the Nash equilibrium (4.5) at each time period. In other words, we assume that the payoffs are known with certainty and they are common knowledge. This is tantamount to assuming that in each time period both players know the parameters characterizing the demand function, the bankruptcy cost/solidity bonus, the marginal cost and the equity bases. We rule out, therefore, the learning process typical of evolutionary games.

Notice, however, that the equity base of each player is changing over time—as we will see in the following section—so that (4.5) is temporary, i.e. bound to change with the passing of time. Summing up, *the Nash equilibrium* (4.5) *is* (*generally*) *non-symmetric*, *instantaneous and temporary*.

4.2. The equity base motion

For each firm, the accumulation of net worth is described by (3.5) and the expected profit $E(\tilde{\Pi}_i)$ in period *t* is given by (2.6) as in the previous section.

At the Nash equilibrium $E^* = (q_1^*, q_2^*)$, the expected profit of each firm is:

$$E(\tilde{\Pi}_i) = (a - bQ^* - Rw)q_1^* + RA_i \tag{4.6}$$

where $Q^* = q_1^* + q_2^*$. Substituting q_1^* and q_2^* given by (4.5) into (4.6) we can specify the expected profit function of the two firms as follows:

$$E(\tilde{\Pi}_1) = h_0 + h_1A_1 + h_2A_2 + h_3A_1^2 + h_4A_2^2 + h_5A_1A_2 \tag{4.7}$$

where

$$h_0 = \frac{2c+b}{(3b+2c)^2}(a - Rw)^2 \tag{4.8}$$

$$h_1 = R + \frac{c(2c+b)}{(3b+2c)^2 w}(a - Rw) \tag{4.9}$$

$$h_2 = -\frac{2bc}{(3b+2c)^2 w}(a - Rw) \tag{4.10}$$

$$h_3 = -\frac{2bc^2(b+c)}{(b+2c)(3b+2c)^2 w^2} \tag{4.11}$$

$$h_4 = \frac{b^2c^2}{(b+2c)(3b+2c)^2 w^2} \tag{4.12}$$

$$h_5 = h_3 + h_4 = -\frac{bc^2}{(3b+2c)^2 w^2} \tag{4.13}$$

and

$$E(\tilde{\Pi}_2) = h_0 + h_1 A_2 + h_2 A_1 + h_4 A_1^2 + h_3 A_2^2 + h_5 A_1 A_2 \tag{4.14}$$

h_0 and h_4 are positive parameters, whereas h_3 and h_5 are negative. Furthermore, if (3.4) holds, then $h_1 > 0$ and $h_2 < 0$.

Therefore, the expected profit of each firm depends in a complicated way on its own equity base and on the equity base of the rival firm.

For instance, an increase in the equity base of firm 1 may increase or decrease its own expected profit because:

$$\text{sign}\left(\frac{\partial E(\tilde{\Pi}_1)}{\partial A_1}\right) = \text{sign}\,(h_1 + 2h_3 A_1 + h_5 A_2) \tag{4.15}$$

is undecided. More precisely,

$$\frac{\partial E(\tilde{\Pi}_1)}{\partial A_1} > 0 \quad \text{if} \quad h_1 > -(2h_3 A_1 + h_5 A_2) \tag{4.16}$$

In this case, we have a positive feedback of an increase in the equity base of the firm on the accumulation of the equity base, which is consistent with our intuition. If (4.16) is not satisfied, on the contrary, an increase of the equity base of the firm will have a negative feedback on the accumulation of the equity base, which is a counterintuitive but perfectly possible result.

Analogously, an increase in the equity base of firm 2 may increase or decrease the expected profit of firm 1, because:

$$\text{sign}\left(\frac{\partial E(\tilde{\Pi}_1)}{\partial A_2}\right) = \text{sign}\,(h_2 + 2h_4 A_2 + h_5 A_1) \tag{4.17}$$

is undecided. More precisely,

$$\frac{\partial E(\tilde{\Pi}_1)}{\partial A_2} < 0 \quad \text{if} \quad h_2 + 2h_4 A_2 < -h_5 A_1 \tag{4.18}$$

In this case, we have a negative feedback of an increase in the equity base of firm 2 on the accumulation of equity base of firm 1. If (4.18) is violated, however, an increase of the equity base of firm 2 will have a positive feedback on the accumulation of the equity base of firm 1.

Assuming that the change in the equity base is equal to expected profit time the retention ratio, the evolution of the equity bases of the two firms, governed by Eq. (3.5), can be obtained by the iteration of a two-dimensional map $T : (A_{1,t}, A_{2,t}) \to (A_{1,t+1}, A_{2,t+1})$ given by

$$T : \begin{cases} A_1' = v_1 h_0 + (1 + v_1 h_1) A_1 + v_1 h_2 A_2 + v_1 h_3 A_1^2 + v_1 h_4 A_2^2 + v_1 h_5 A_1 A_2 \\ A_2' = v_2 h_0 + (1 + v_2 h_1) A_2 + v_2 h_2 A_1 + v_2 h_3 A_2^2 + v_2 h_4 A_1^2 + v_2 h_5 A_1 A_2 \end{cases} \tag{4.19}$$

where $'$ denotes the unit-time advancement operator. Starting from a given initial condition

$$\boldsymbol{A}_0 = (A_{1,0}, A_{2,0}) \in \mathbb{R}^2_+ \tag{4.20}$$

the iteration of map (4.19) generates a trajectory

$$\tau(\boldsymbol{A}_0) = \{(A_{1,t}, A_{2,t}) = T^t(A_{1,0}, A_{2,0}), t \geq 0\}$$

which may converge to a fixed point or to a periodic cycle or to a more complex attracting set, such as a strange (or chaotic) attractor, as we shall see in the following. Endogenous fluctuations of the equity bases $A_1(t)$ and $A_2(t)$ can occur. Therefore also the Cournot–Nash equilibrium E^* fluctuates. In fact, the dynamic behavior of A_1 and A_2 determines, through (4.5), a sequence of Cournot–Nash equilibria. As we have emphasized above, each equilibrium point E^* can be thought of as a temporary equilibrium, i.e. an equilibrium point whose position is driven by the dynamical behavior of net worth according to the law of motion (4.19).

4.3. *Characterization of the state space*

Each point of the state space—i.e. the positive orthant of the (A_1, A_2) plane—can be characterized according to the financial regime of the firms. The first firm is incompletely collateralized if $A_1 < wq_1$. Recalling that

$$q_1^* = \frac{1}{3b+2c}\left[a - Rw + \frac{2c(b+c)}{(b+2c)w}A_1 - \frac{bc}{(b+2c)w}A_2\right]$$

(see (4.5)) the first firm is in the regime of incomplete collateralization if $A_1 < h_6 - h_7 A_2$ where

$$h_6 = \frac{(b+2c)w}{3b(b+2c)+2c^2}(a-Rw), \qquad h_7 = \frac{bc}{3b(b+2c)+2c^2}$$

Measuring A_1 on the horizontal axis and A_2 on the vertical axis, if the point (A_1, A_2) in the space of equity bases is below (above) the negatively sloped line of equation

$$A_1 + h_7 A_2 = h_6 \tag{4.21}$$

the first firm is incompletely (fully) collateralized. Following a symmetrical reasoning, if the point (A_1, A_2) lies below (above) the straight line of equation

$$h_7 A_1 + A_2 = h_6 \tag{4.22}$$

the second firm is incompletely (fully) collateralized.

In the end, tracing the two straight lines (4.21) and (4.22) we can partition the space of equity bases in four regions (see Fig. 1). When the equity base is “small” (“large”) for both firms—i.e. when the point of the state space lies below (above) both lines—both firms are incompletely (fully) collateralized. In these two cases the firms happen to be in the same financial regime. The two “mixed” cases where one firm is financially constrained and the other is not can be derived straightforwardly.

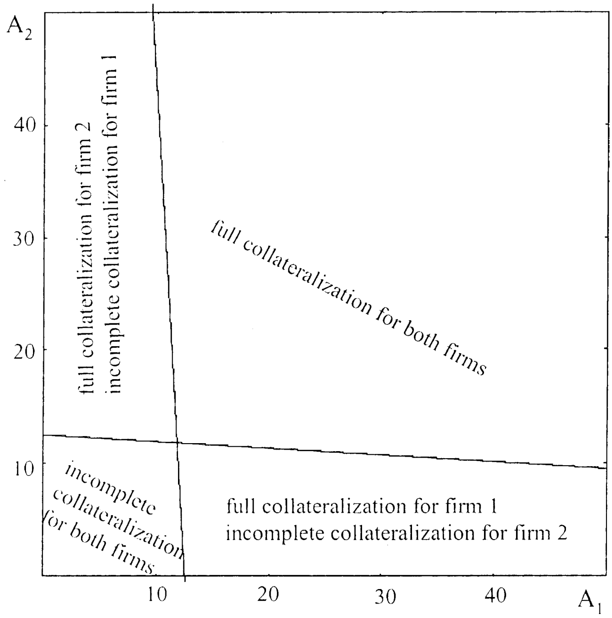


Fig. 1. The two straight lines (4.21) and (4.22) divide the space of equity bases (A_1, A_2) into four regions, according to the firm(s) which are incompletely or fully collaterized. This figure is obtained with parameters' values $a = 20$, $b = 0.2$, $R = 1.05$, $w = 1$, $c = 1$.

Each point on the state space can be characterized also by the total quantity produced in the economy $Q = q_1 + q_2$ and consequently by the price level $P = a - bQ$. The price level is non-negative if $Q \leq \hat{Q}$ where $\hat{Q} \equiv a/b$ is the maximum amount which can be absorbed by market demand. In the Nash equilibrium, total quantity is

$$Q^* = q_1^* + q_2^* = \frac{1}{3b + 2c}\left[2(a - Rw) + \frac{c(b + 2c)}{(b + 2C)w}(A_1 + A_2)\right] \tag{4.23}$$

If, given A_1 and A_2, the amount the two firms jointly produce according to (4.23) happens to be greater than $\hat{Q}$, we assume that they will be forced to pay an extra-wage equal to $\theta(Q^* - \hat{Q})$, where θ is a positive constant. We can think of $\hat{Q}$ as potential output, i.e. the amount producible in full employment at the going working schedule (working hours per day). If firms want to produce more than $\hat{Q}$, they have to pay an extra-wage to induce people to work extra-hours.

We can distinguish two regimes. In the underemployment regime, $Q^* < \hat{Q}$, the nominal wage is constant at w and the price level is $a - bQ^* > 0$. In the full employment regime $Q^* \geq \hat{Q}$, the nominal wage is $w + \theta(Q^* - \hat{Q})$, the firms produce Q^* but they will be able to sell only $\hat{Q}$ at the price $a - b\hat{Q} = 0$. In other words, the nominal wage obeys the following schedule:

$$\text{wage} = \begin{cases} w & \text{if } Q^* < \hat{Q} \\ w + \theta(Q^* - \hat{Q}) & \text{if } Q^* \geq \hat{Q} \end{cases}$$

which is reminiscent of a Philips curve in wage-output space.

When $Q^* < \hat{Q}$ (underemployment regime), the expected profit of each firm, which governs the evolution over time of the equity base, is (4.6). When $Q^* \geq \hat{Q}$, (full employment regime), the expected profit is:

$$E(\tilde{\Pi}_i) = (a - b\hat{Q})q_i^* - R\{[w + \theta(Q^* - \hat{Q})]q_i^* - A_i\} =$$
$$RA_i - R[w + \theta(Q^* - \hat{Q})]q_i^*$$

If we make the technical assumption that $\theta = b/R$, the expression above becomes:

$$E(\tilde{\Pi}_i) = (a - b\hat{Q})q_i^* - Rwq_i^* - b(Q^* - \hat{Q})q_i^* + RA_i = (a - bQ^* - Rw)q_i^* + RA_i$$

which is identical to (4.6). In the end, in our model the firm has the same expected profit function regardless of the employment regime it is experiencing. This conclusion greatly simplifies the analytic structure of the model.

Given the characterization discussed above, the dynamics of the equity bases are governed by the system (4.19) in all the points of the state space, regardless of the financial or employment regime. The trajectories generated by (4.19), however, may cross areas of the state space characterized by different financial or employment regimes.

4.4. General dynamical properties

The first step in the study of the properties of a dynamical system is the computation of the steady states, i.e. the trajectories characterized by $A_{i,t+1} = A_{i,t}$, $i = 1, 2$, for each t. The steady states are the fixed points of map T, i.e. the solutions of the algebraic system obtained from (4.19) with $A_1' = A_1$ and $A_2' = A_2$. We recall that, according to (3.5), the fixed points of (4.19) are points of zero expected profit for both firms. From (4.7) and (4.14) it is straightforward to conclude that each equation $E(\Pi_i) = 0$ represents an hyperbola in the plane (A_1, A_2). The fixed points of T therefore are located on the intersections of the two hyperbolas $E(\Pi_i) = 0$, $i = 1, 2$. The existence and the stability properties of the fixed points are stated in the following proposition, which is proved in Appendix A.

Proposition 1. *The map T defined by* (4.19) *has two and only two fixed points, located on the diagonal Δ of equation $A1 = A2$, given by*

$$S = (s, s), \quad \textit{with} \quad s = \frac{-(h_1 + h_2) - \sqrt{(h_1 + h_2)^2 - 8h_0h_5}}{4h_5} > 0 \qquad (4.24)$$

and

$$N = (n, n), \quad \textit{with} \quad n = \frac{-(h_1 + h_2) + \sqrt{(h_1 + h_2)^2 - 8h_0h_5}}{4h_5} < 0 \qquad (4.25)$$

The negative fixed point N is a repelling node, with eigenvalues $\lambda_2(n) > \lambda_1(n) > 1$; the positive fixed point S is an attracting node for sufficiently low values of v_1 or v_2, and it *may lose stability through a period doubling* (*or flip*) *bifurcation*[4] *at which S becomes a saddle point, with* $\lambda_1(s) < -1 < \lambda_2(s) < 1$, *and a stable cycle of period* 2 *is created near S.*

The coordinates of the fixed points are functions of the *h*-parameters which in turn are polynomials of the parameters characterizing the aggregate demand function (2.1) (a and b), the bankruptcy cost/solidity bonus (c) and the interest rate augmented nominal wage (Rw). Notice that the coordinates of the fixed points are independent of the retention ratios. In other words, the fact that the two firms can differ in terms of their dividend policy is irrelevant for the determination of the steady state equity bases.

Of course, only the positive fixed point $S = (s, s)$ is economically meaningful. Since it belongs to the bisector Δ, in the steady state the financial conditions of the two firms, as well as their equilibrium output levels, are identical. In fact, according to (4.5), at the steady state S we have a Symmetric Steady state Nash Equilibrium (SSNE) $E^* = (q^*, q^*)$, with

$$q^* = \frac{1}{3b + 2c}(a - Rw + cs) \tag{4.26}$$

The steady state S is stable if, starting from an initial configuration characterized by heterogeneous financial conditions, i.e. $A_{1,0} \neq A_{2,0}$, the endogenous dynamics lead to identical equity bases and outputs (4.26), provided that the initial condition (4.20) belongs to the basin of attraction $\mathcal{B}(S)$ of the stable fixed point S. The *basin* of S is the set of initial conditions that generate a trajectory converging to E^*:

$$B(S) = \{(A_{1,} A_2) | (A_{1,t}, A_{2,t}) = T^t(A_{1,0}, A_{2,0}) \to S \quad \text{as} \quad t \to +\infty\}$$

In order to explore the dynamic behavior of the model we have to determine the domain, in the parameters' space, for which the steady state S is locally stable. Moreover, for a set of parameters for which S is stable, we must detect the boundaries of the basin $\mathcal{B}(S)$ in the phase (A_1, A_2).

Due to the high number of parameters of the model, and the complicated dependence of the eigenvalues of the Jacobian matrix ((A.2) in Appendix A) on these parameters, a complete study of the local stability of S as the parameters are varied is not an easy task. Thus in the following we run some numerical simulations, based on the local stability analysis given in Appendix A, in order to explore the dynamic patterns of the model. A more complete numerical exploration of the influence of the retention ratios on the dynamical behavior of the equity bases (and consequently of the temporary Nash equilibrium) is performed in Section 5.

Setting the parameter values at $a = 10$, $b = 0.2$, $R = 1.05$, $w = 1$, $c = 1$, $v_1 = 0.3$, $v_2 = 0.6$ we obtain the steady state $S = (64.82\ldots, 64.82\ldots)$, which is represented in Fig. 2a on the phase plane (A_1, A_2) together with a typical trajectory converging

[4] A period doubling bifurcation occurs when, by varying a parameter, an eigenvalue of a fixed point crosses the unit circle with value $\lambda = -1$. At such bifurcation a cycle of period 2 is created near the fixed point. The bifurcation is *supercritical* if a stable two-cycle is created around the unstable fixed point, *subcritical* if an unstable two-cycle exists around the stable fixed point (see Lorenz, 1993, p. 111; Guckenheimer and Holmes, 1983, p. 158).

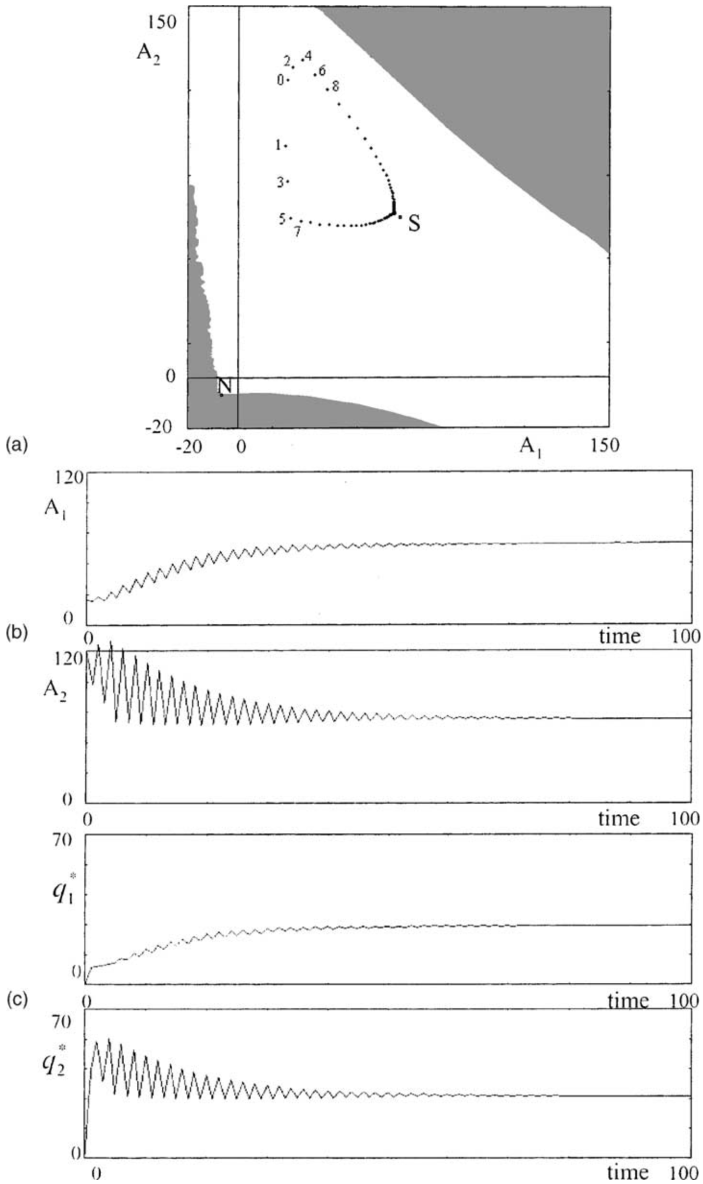


Fig. 2. (a) For the parameters' values $a = 10$, $b = 0.2$, $R = 1.05$, $w = 1$, $c = 1$, $v_1 = 0.3$, $v_2 = 0.6$, the trajectory starting with the initial condition $(A_{1,0}, A_{2,0}) = (20, 120)$ is represented on the phase plane (A_1, A_2). The white region represents the set of points which generate trajectories converging to the steady state $S = (64.82\ldots, 64.82\ldots)$, i.e. the basin $\mathcal{B}(S)$, whereas the grey region represents the basin of infinity $\mathcal{B}(\infty)$, defined as the set of points that generate diverging trajectories. (b) The same trajectory represented in Fig. 1a is plotted vs. time. (c) For the same trajectory $(A_1(t), A_2(t))$ shown in figures (a) and (b) the time evolution of the temporary Nash equilibrium $(q_1^*(t), q_2^*(t))$, driven by the corresponding evolution of the equity bases according to (4.5), is represented.

to it. For this set of parameters the fixed point S is locally stable, with eigenvalues $\lambda_1(s) = -0.91\ldots, \lambda_2(s) = 0.98\ldots$. Its basin $\mathcal{B}(S)$ is represented by the white region, whereas the grey region represents the basin of infinity $\mathcal{B}(\infty)$, defined as the set of points that generate diverging trajectories. Even if only points (A_1, A_2) with positive coordinates are meaningful, in Fig. 2a we have also represented a portion of negative orthants in order to show that the unstable (and negative) fixed point N belongs to the boundary that separates the grey basin of infinity from the white basin of bounded trajectories. The trajectory represented in Fig. 2a starts from the initial condition $(A_{1,0}, A_{2,0}) = (20, 120) \in B(S)$ (the black dot labelled by 0) and then converges to the steady state S through oscillations of decreasing amplitude (because $-1 < \lambda_1(s) < 0$). This can be more easily seen in Fig. 2b, where the same trajectory is plotted versus time. Of course, the same kind of evolution holds for the outputs of the two firms, computed according to (4.5). The time evolution of the temporary Nash equilibrium (4.5), driven by the corresponding evolution of the equity bases, is shown in Fig. 2c.

In the situation shown in Fig. 2 even if the initial conditions $A_{1,0}$ and $A_{2,0}$ are different (i.e. the two firms are *heterogeneous* with respect to the equity bases), the system spontaneously evolves towards a steady state characterized by long run homogeneity of the equity base and output, provided the initial condition $(A_{1,0}, A_{2,0})$ is in the basin of attraction of S. This conclusion only applies when S is stable. In fact if the parameters are changed until the fixed point S loses stability (via a period doubling bifurcation, according to Proposition 1) then the system tends to long run heterogeneity provided the retention ratios are different, $v_1 \neq v_2$. In order to see this, we consider the same set of parameters a, b, R, w, c and v_2 as in Fig. 2, and we increase the retention ratio v_1. For this set of parameters the flip bifurcation at which S loses stability occurs at $v_1 = v_1^f = 0.34\ldots$, and for v_1 slightly greater than v_1^f a stable cycle of period 2 occurs, say $C_2 = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$, with periodic points $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ very close to S. As v_1 is further increased, the cycle C_2 moves far from S: numerical simulations show that if $v_1 < v_2$ (respectively: $v_1 > v_2$) then C_2 belongs to the region of the phase space below (above) the diagonal Δ, i.e. its periodic points are characterized by $A_1 > A_2 (A_1 < A_2)$; if $v_1 = v_2$ then C_2 belongs to Δ, i.e. its periodic points are characterized by $A_1 = A_2$. For example, Fig. 3a is obtained with $v_1 = 0.4 > v_1^f$, so S is a saddle point with eigenvalues $\lambda_1(s) = -1.12\ldots$ and $\lambda_2(s) = 0.98\ldots$, and the generic bounded trajectory,[5] obtained by an initial condition in the white region, converges to the stable cycle of period two $C_2 = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \simeq ((69.8, 45.8), (82.7, 58.6))$. Being $v_1 = 0.4 < v_2 = 0.6$, C_2 is characterized by $A_1 > A_2$. In other words, starting from a generic initial condition in the basin of attraction $\mathcal{B}(C_2)$ of the cycle of period 2 (the white region in Fig. 3a), even with an homogeneous initial situation, i.e. $(A_{1,0}, A_{2,0}) = (A, A) \in \Delta$, the equity bases of the two firms become heterogeneous "in the long run". The same will be true, of course, for the output levels. A typical trajectory is plotted against time in Fig. 3b. It is worth noting that even starting from an initial condition of equity bases uniform across firms and very close to the steady state, namely $A_{1,0} = A_{2,0} = 64.82$, in the long run the

[5] By the term "generic" we mean the trajectories starting from almost all the points of the white region (i.e. excluded a subset of zero measure). The zero-measure subset includes, for example, the unstable fixed point S.

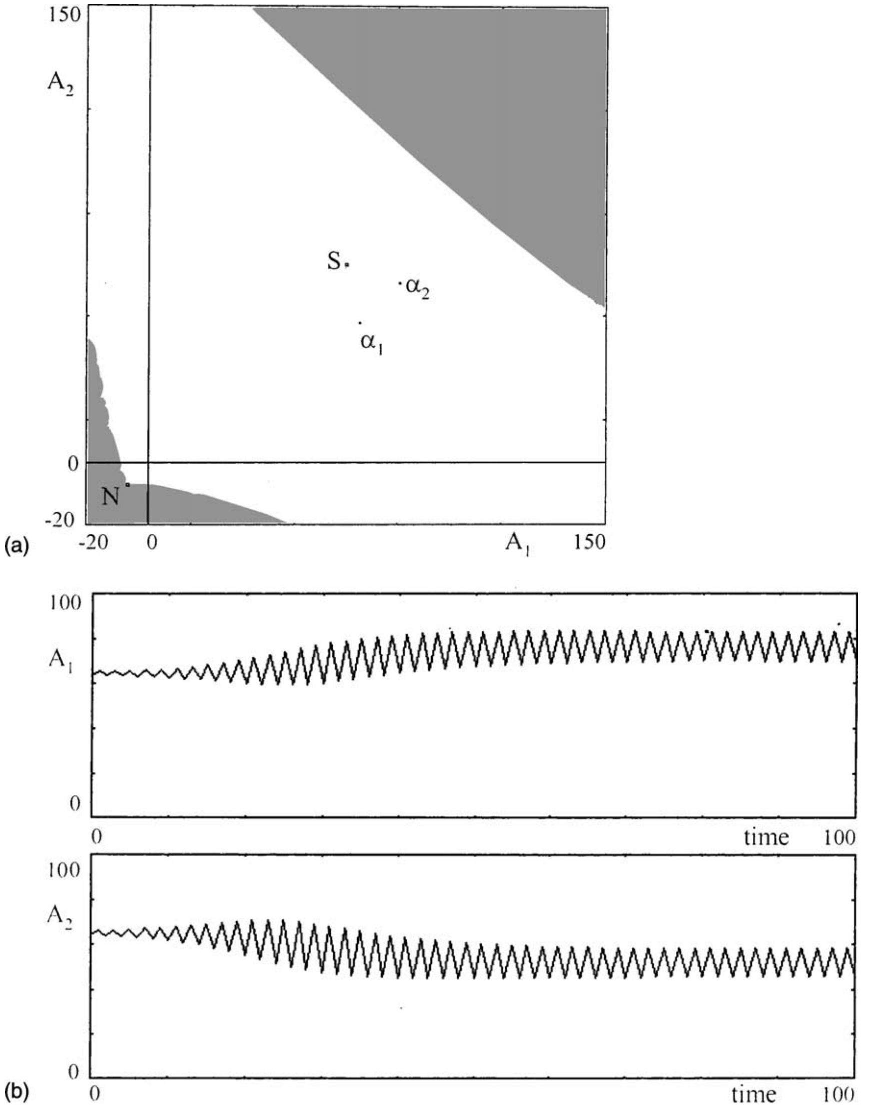


Fig. 3. (a) For the same set of parameters a, b, R, w, c and v_2 as in Fig. 1a and $v_1 = 0.4$ a generic trajectory starting from a point of the white region converges to the stable cycle of period 2 $C_2 = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \simeq ((69.8, 45.8), (82.7, 58.6))$. (b) A typical trajectory is plotted against time starting from the homogeneous initial condition $A_{1,0} = A_{2,0} = 64.8$.

equity base of the first firm oscillates in the range (69.8, 82.7) while that of the second firm oscillates in the range (45.8, 58.6). Therefore, in the long run the first firm is bigger (its average equity base is about 76) than the second firm (whose average equity base is about 52).

The situation is reversed if the stable cycle C_2 occurs when $v_1 > v_2$, due to the symmetric form of map (4.19), that remains the same swapping both the parameters v_1 with v_2 and the dynamic variables A_1 and A_2. This implies that if we exchange v_1 with v_2 the trajectories of the new dynamical system are simply obtained from the previous ones by exchanging A_1 with A_2.

Numerical simulations show that if (at least) one of the retention ratios is increased, the first period doubling bifurcation is followed by a sequence of other period doubling bifurcations that create stable cycles of period 2^k, $k = 2, 3, \ldots$ (similar to the well known Feigenbaum period doubling route to chaos occurring in one-dimensional maps). Such a sequence of period doublings leads to chaotic attractors, i.e. invariant two-dimensional sets inside which the trajectories of map (4.19) exhibit sensitive dependence on initial conditions. Fig. 4a shows the attracting set obtained with the same parameters a, b, R, c, and v_2 as in Figs. 2 and 3, and $v_1 = 0.67$. The dynamic behavior of a trajectory like that of Fig. 4a, a part of which is plotted versus time in Fig. 4b, appears to be rather irregular, apparently chaotic. It can be noticed, however, that the attracting set consists of two disjoint pieces. A trajectory moving on it cyclically visits these two pieces as shown in Fig. 4b, where it is evident that a sort of cyclicality of period 2 occurs in the long run. Such an attractor is also called *two-cyclic chaotic attractor*. We notice that in this case the average value of A_2 is greater than the average value of A_1, because $v_1 > v_2$.

If v_1 is further increased the two pieces of the chaotic area expand until they merge into a one-piece chaotic area, as shown in Fig. 5, obtained with $v_1 = 0.75$. This is a remarkable global bifurcation,[6] after which the generic bounded trajectory moves erratically along the bigger chaotic area, so that no cyclic pattern can be detected. Another consequence of the global bifurcation at which the two pieces form a one-piece chaotic area is that the boundary of the chaotic area is more "fuzzy", in the sense that "rare points" appear around a more densely covered central part. In the versus time representation of the trajectory, the existence of such rare points implies that some sudden jumps occur, which are of greater amplitude with respect to the majority of the irregular oscillations. This kind of attracting set is called *mixed chaotic area* in Mira et al. (1996).

In Fig. 5 we also notice that the boundary which separates the basin of bounded trajectories from the basin of infinity is rather complex. This kind of complexity is another sources of unpredictability, related to the fact that small changes in the initial conditions may yield a completely different asymptotic evolution if such changes cause a crossing of the basin boundaries. In fact, if a point is very close to a basin boundary (and many points are in such a situation in the presence of complex basin boundaries) a small perturbation has a high probability to cause a crossing of the boundary.

If v_1 is further increased, the chaotic area expands until it has a contact with the boundary of the basin. This contact marks the occurrence of a global bifurcation, called *final bifurcation* in Mira et al. (1996) and Abraham et al. (1997), or *boundary crises* in Grebogi et al. (1983), which makes the chaotic area disappear. After this bifurcation, the chaotic

[6] We call "global" the bifurcations which cannot be explained in terms of the linear approximation of the dynamical system, as opposed to the local bifurcations, which are revealed through the study of the eigenvalues of the Jacobian matrix (used to represent the linearization of the dynamical system).

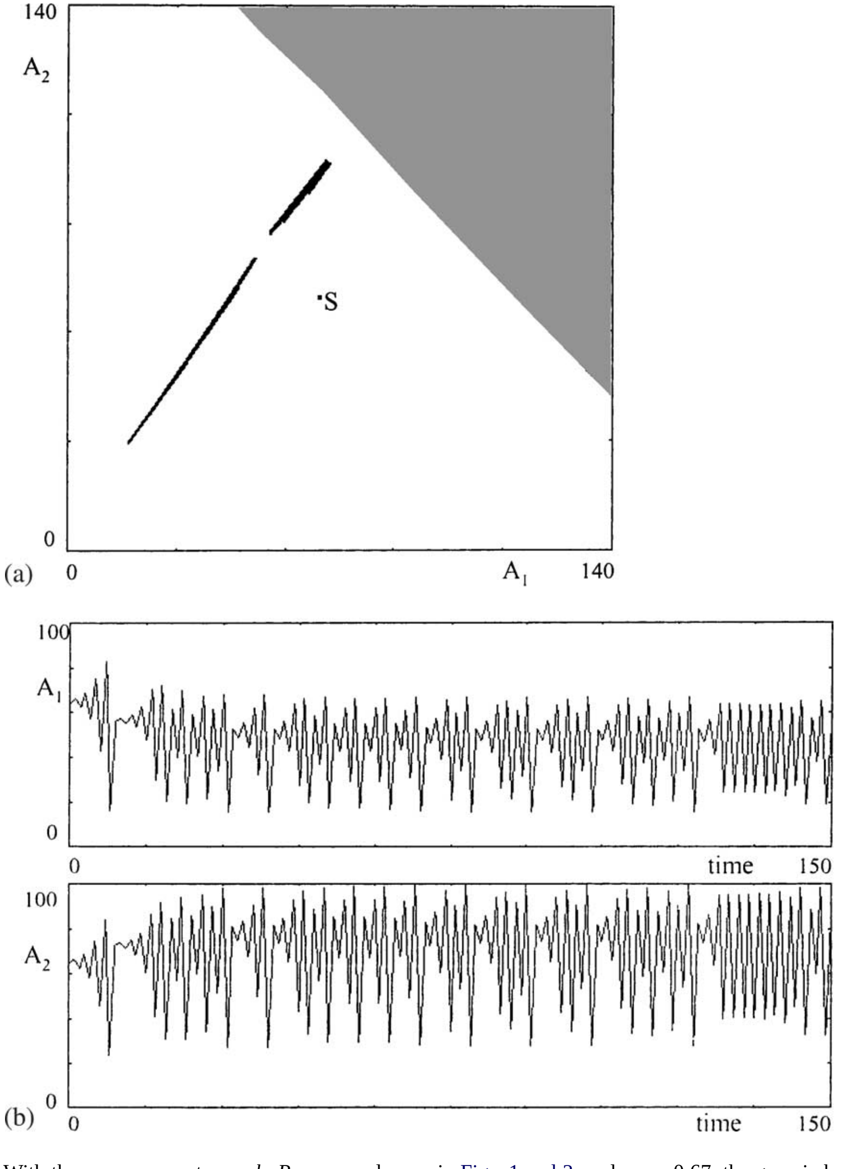


Fig. 4. With the same parameters a, b, R, w, c, and v_2 as in Figs. 1 and 2, and $v_1 = 0.67$, the generic bounded trajectory converges to a two-cyclic chaotic attractor. In (a) such attracting set is represented by plotting a typical trajectory in the plane (A_1, A_2), in (b) a part of the same trajectory is plotted vs. time.

attractor is transformed into a chaotic repellor, whose "skeleton" is formed by the dense set of repelling periodic points which where inside the chaotic area that just disappeared Grebogi et al. (1983), and the generic initial condition generates a trajectory which diverges after a chaotic transient.

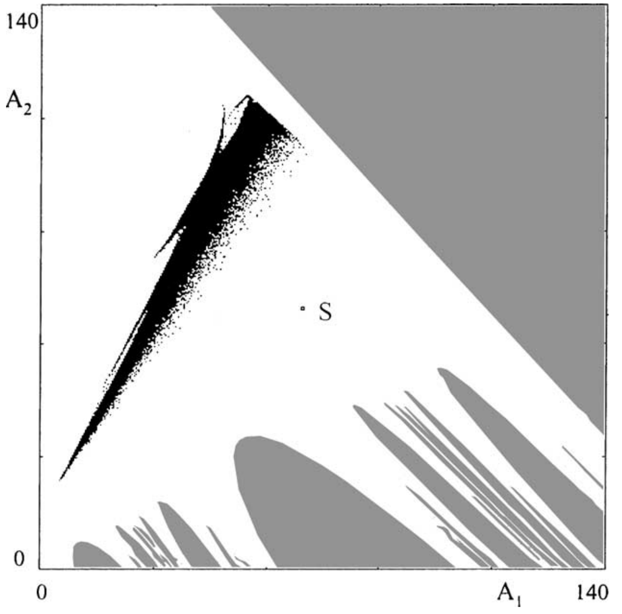
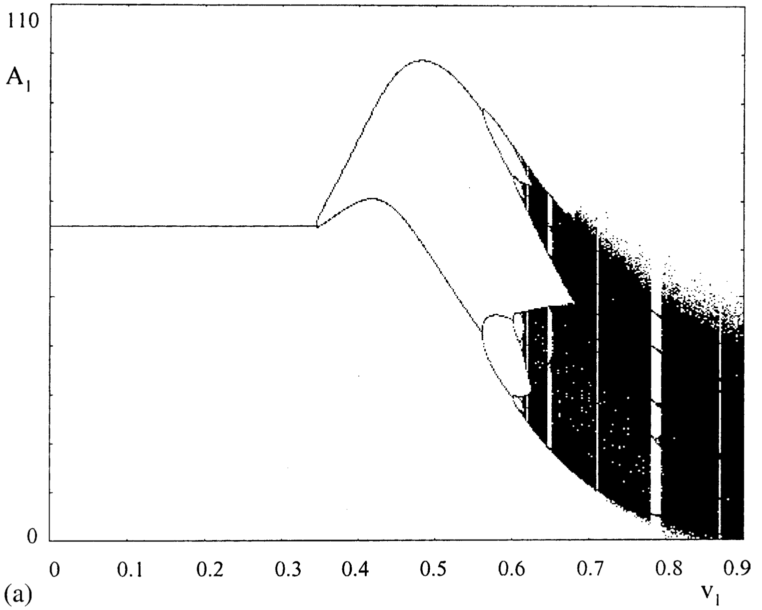


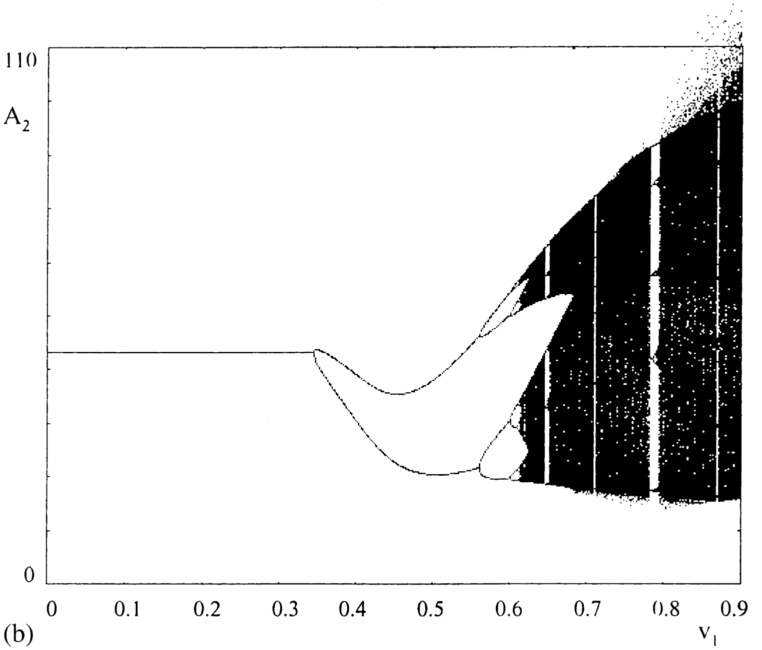
Fig. 5. With the same parameters a, b, R, w, c, and v_2 as in Figs. 1 and 2, and $v_1 = 0.75$, the generic bounded trajectory converges to a one-piece chaotic attractor.

5. Changes of retention ratios

In order to explore the effects of increasing retention ratios on the dynamics of the equity bases of the two firms, we consider a fixed set of values of the parameters a, b, R, w, c, and v_1, namely $a = 10$, $b = 0.2$, $R = 1.05$, $w = 1$, $c = 1$ and $v_2 = 0.6$ and vary the retention ratio v_1 in the range (0, 1). Due to the symmetry in the roles of v_1 and v_2, we would obtain analogous results with a fixed value of v_1 by varying the other retention ratio v_2. The impact on long run dynamics can be evaluated by means of a bifurcation diagram in which v_1 is measured on the horizontal axis, and a given number of asymptotic values of a dynamic variable are plotted on the vertical line. For example, the bifurcation diagram of Fig. 6a is obtained as follows: for each value of v_1 in the range (0, 1) we generate a trajectory starting with an initial condition near S: we discard a transient of the early 300 iterations and plot the subsequent 700 values of A_1 on the vertical line through v_1. In Fig. 6b we adopt the same procedure to represent the asymptotic values of A_2. These two bifurcation diagrams show that the steady state S is stable for small values of v_1 and the values of both equity bases are independent of v_1 (we recall that, in S, $A_1 = A_2 = s$, where s is given in (4.24)). As v_1 is increased, the steady state S loses stability at $v_1 \cong 0.4$ via a period doubling bifurcation at which a stable cycle of period 2 occurs. It can be noticed that the amplitude of the oscillations increases with v_1. Moreover, for $v_1 < v_2 = 0.6$ the values of A_1 are greater than the values of A_2, and the opposite holds true for $v_1 > v_2$. The sequence of period doubling bifurcations which occurs increasing v_1 is clearly visible in



(a)



(b)

Fig. 6. Bifurcation diagram obtained with increasing values of the parameter v_1 and fixed values of the other parameters: $a = 10$, $b = 0.2$, $R = 1.05$, $w = 1$, $c = 1$ and $v_2 = 0.6$. (a) In the vertical axis the asymptotic values of the equity base A_1 are reported. (b) In the vertical axis the asymptotic values of the equity base A_2 are reported.

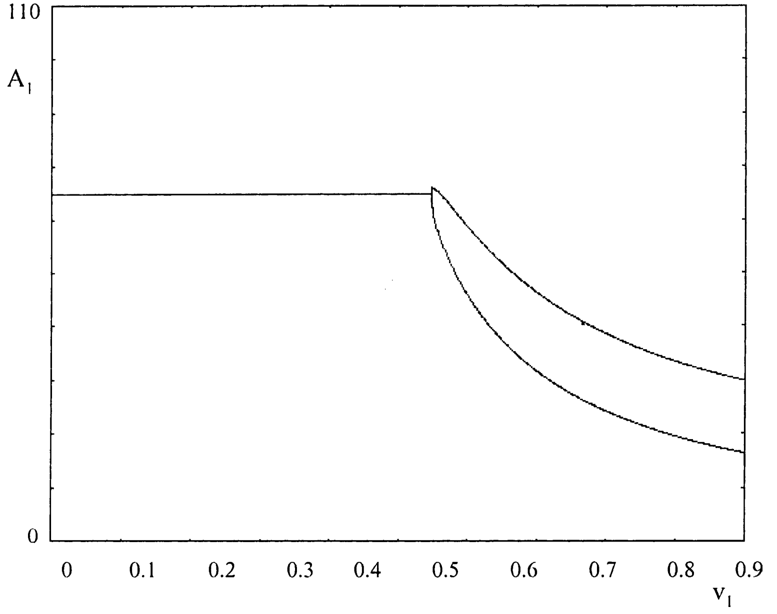


Fig. 7. Bifurcation diagram obtained with increasing values of the parameter v_1 and fixed values of the other parameters: $a = 10$, $b = 0.2$, $R = 1.05$, $w = 1$, $c = 1$ and $v_2 = 0.4$. In the vertical axis the asymptotic values of the equity base A_1 are reported.

Fig. 6. It is similar to the well known period doubling cascade that marks the route to chaos in one-dimensional maps. A noticeable global bifurcation occurs at $v_1 = v_h \cong 0.8$. Before this bifurcation the attractor is two-cyclic, i.e. the periodic or chaotic attractors observed for $v_1 < v_h$ are located inside a trapping region formed by two disjoint portions, whereas for $v_1 > v_h$ the trajectories move inside a unique larger region (see Figs. 4 and 5). Such bifurcation increases the complexity in dynamic behavior of the equity bases and leads to a loss of predictability, since no cyclicality can be revealed after it. Moreover, for $v_1 > v_h$ the attracting set is given by a mixed chaotic area (see Mira et al., 1996), i.e. the boundaries of the region inside which the two-dimensional chaotic attractors are included are not well defined, in the sense that a cloud of "rare points" surrounds the more dense part of the chaotic area (see also Fig. 5). At $v_1 \simeq 0.9$ the *final bifurcation* occurs, due to a contact between the chaotic attractor and the boundary of its basin, after which the chaotic attractor disappears and the generic trajectory goes to infinity.

In the bifurcation diagram shown in Fig. 7, obtained with the same values of the parameters a, b, R, w, and c as in Fig. 5a and $v_2 = 0.4$ instead of $v_2 = 0.6$ we have no chaos whatever the value of v_1 in the range (0, 1). For lower values of v_2 the fixed point S remains stable for each $v_1 \in (0, 1)$, i.e. complex dynamics are lost if at least one of the retention ratios is sufficiently small.

On the basis of the analytical and numerical results of the present section and of Section 4.4 we can sum up the discussion on the effects of changes in retention ratios on dynamics as follows:

1. If the retention ratios are "sufficiently low", the positive fixed point *S* is stable. Even if the initial (financial) conditions and the dividend policy of the two firms are different (i.e. $A_{1,0} \neq A_{2,0}$ and $v_1 \neq v_2$), the equity bases of the two firms will converge to the SSNE characterized by $A_1 = A_2$, provided that the initial conditions are inside the basin of attraction of *S*. In the steady state, the two firms become homogeneous as far as the equity base and the level of output are concerned.
2. Increasing the value of at least one retention ratio, the fixed point *S* becomes unstable, the long run dynamics is characterized by oscillations of period 2 such that $A_1 > A_2$ ($A_1 < A_2$) on average, as long as $v_1 < v_2$ ($v_1 > v_2$). In this case homogeneity is lost in the long run, since the equity base oscillates in a range which is different from on firm to the other.
3. Only if the retention ratio is uniform across firms, i.e. $v_1 = v_2$, the long run dynamics is characterized by oscillations such that $A_1 = A_2$, on average. In fact, if the retention ratios are uniform across firms the attractors are located on the 45° diagonal. In this case, and only in this case, long run homogeneity is preserved also in the case of dynamics which are more complex than the simple convergence to a stable steady state.
4. An increase of the retention ratios yields more complex, and consequently less predictable, dynamics of the equity bases. Moreover, also the boundary that separates the set of points that generate bounded trajectories from the basin of infinity becomes more complex, thus generating a greater uncertainty also with respect to the choice of initial conditions, or, equivalently, with respect to the possible effects of random shocks.

Therefore, the firms are heterogeneous in the long run if the retention ratios are not uniform across firms and not too small. Moreover, when heterogenous long run dynamics occur, the firm with the higher retention ratio ends up with the lower equity base (on average). This is broadly consistent with the stylized facts. In the real world, in fact, the corporate sector is heterogeneous, the retention ratios are different from one class of firms to the other and the average retention ratio is relatively high. Moreover, "small" firms usually retain a higher proportion of their profits than "large" ones. For example, according to Fazzari et al. (1988, p. 147), on average the retention ratio of the US manufacturing firms in the 1970–1984 period has been 60%, with "small" firms retaining up to 80% of their profits while the retention ratio of "big" firms was approximately 50%.

The standard theoretical explanation of this stylized fact is based on *the financing hierarchy* (*or pecking order*) *assumption*. When capital markets are affected by informational imperfections such as asymmetric information, a financing hierarchy can be envisaged: internal finance is the most preferred source of finance while credit has a cost advantage over the issue of new equities as far as external sources of finance are concerned. Small firms, which have limited access to the credit and equity markets, rely first and foremost on retentions to fill their financing gap while large firms are less financially constrained and therefore have a lower retention ratio.

The theoretical explanation of the same stylized fact in the present model is different. Let's start from a benchmark case in which both firms have the same retention ratio. The equity bases of the two firms therefore converge to the SSNE or—if they are not too low—to a uniform two-period cycle. Therefore they are homogeneous. Assume now that one of the two firms increases its retention ratio. The direct effect of an increase of the retention ratio on the accumulation of the firm's net worth is necessarily positive. However, the more rapid increase of the equity base may have a negative impact on the expected profit of the firm and therefore may depress the accumulation of net worth (see the discussion in Section 4.2) in the subsequent rounds of the time evolution of the equity base. If this is the case, the firm which tries to accumulate more rapidly ends up with a lower level of the equity base in the long run.

Increasing the retention ratios, the corporate sector remains heterogeneous (small and big firms coexist), but the dynamics of the equity bases become complex, i.e. observationally equivalent to the dynamics generated by random shocks.

6. Conclusions

In this paper we have explored the properties of a simple oligopolistic set-up in which firms run the risk of bankruptcy or gain a solidity bonus according to their financial conditions. We present, first of all, a benchmark case in which firms do not incur a bankruptcy cost/solidity bonus. In this case the quantity produced in the Cournot–Nash equilibrium would be the same for each and every oligopolist (Symmetric Nash Equilibrium) and would be independent of the financial condition (i.e. the level of the equity base) of the firm and of its rivals. Moreover, the accumulation of net worth by each firm would be independent of the accumulation of net worth on the part of rival firms.

In the general case in which firms face a bankruptcy cost/solidity bonus, the scale of production is affected by financial conditions. In this case, the quantity produced in the Cournot–Nash equilibrium would be different from one firm to the other and would depend on the net worth of the firm and of the competitors. Moreover, the motion over time of the equity bases of the firms may follow a wide range of dynamic patterns: convergence to a Symmetric Steady state Nash Equilibrium, periodic or non-periodic orbits, complex (even chaotic) trajectories. Endogenous fluctuations of the equity bases of the oligopolists also imply that the Cournot–Nash equilibrium fluctuates.

If the retention ratios are "sufficiently low", in the long run firms become homogeneous as far as the equity ratio and the output level are concerned. Increasing the value of at least one retention ratio, the equity base of each firm oscillates in a range which is different from one firm to the other.

In particular, the firm with the lower (higher) retention ratio ends up with the higher (lower) equity base on average. In a sense we have here a "paradox of thrift" applied to the corporate sector: if a firm tries to accumulate its equity base more rapidly by increasing its retention ratio, it obtains the opposite result and becomes "poorer". Further increases in at least one of the retention ratios yield more complex, and consequently less predictable, dynamics of the equity bases.

The present set-up represents a first modest step to model the generation of financially driven endogenous fluctuations in a game-theoretic context. In order to maintain the possibility to get some insights concerning the dynamic properties of the model, we stick to the simple analytic structure of a two-dimensional non-linear dynamical system by making a number of mainly technical assumptions in the characterization of the state space. An obvious extension of the present model consists in relaxing at least some of these assumptions and modifying the model accordingly. For instance, we could give up the idea of a solidity bonus in the full collateralization regime while keeping the bankruptcy cost in the incomplete collateralization case. This relaxation, however, leads to a more cumbersome dynamical system, defined by different analytic expressions in the different regions of the state space (due to the multiplicity of financial regimes the firms are experiencing) and makes the analysis much more complicated: as often occurs in dynamic modeling, there is an obvious trade-off between realism of the hypotheses and mathematical manageability of the dynamical system.

A different extension consists in framing the model in a different game-theoretic context. In the present set-up the quantities produced by the two firms are strategic substitutes: a higher volume of output by one firm makes the price go down—other things being equal—and negatively affects the expected profit of the other firm via the market demand function. We implicitly ignore the positive impact that an increase in the output of one firm can have on the expected profit of the other firm via income effects. Taking these effects into account, i.e. conceiving of the quantities produced as strategic complements instead of strategic substitutes, can modify the results in interesting directions such as multiple Pareto-ranked equilibria.

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Appendix A. Proof of Proposition 1

We first consider the existence of the fixed points, given by the solutions of the algebraic system $E(\Pi_i) = 0$, $i = 1.2$. with $E(\Pi_i)$ given by (4.7) and (4.14). This system is symmetric, because it remains the same by exchanging A_1 and A_2: this means that the hyperbola $E(\Pi_2) = 0$ is the symmetric of the hyperbola $E(\Pi_1) = 0$ with respect to the diagonal Δ of equation $A_1 = A_2$. If Eqs. (4.7) and (4.14) are subtracted, then one either has $A_1 = A_2$ or $A_1 + A_2 = (h_2 - h_1)/(h_3 - h_4)$. However, the latter condition leads to the equation

$$(h_3 + h_4 - h_5)A_1\left(A_1 + \frac{h_1 - h_2}{h_3 - h_4}\right) + h_0\frac{h_2 - h_1}{h_3 - h_4}(h_2h_3 - h_1h_4) = 0$$

which has no solutions in general, being $h_3 + h_4 - h_5 = 0$. Hence, the two symmetric curves $E(\Pi_1) = 0$ and $E(\Pi_2) = 0$ only cross at two points located on the diagonal Δ, whose coordinates are given by the solutions of the equation

$$2h_5x^2 + (h_1 + h_2)x + h_0 = 0.$$

Since $h_5 < 0$ and $h_0 > 0$ this equation always has two real solutions of opposite sign, say $s > 0$ and $n < 0$. Hence the map (4.19) has two fixed points, $S = (s, s)$ and $N = (n, n)$, both belonging the diagonal, with positive and negative coordinates, respectively, given by (4.24) and (4.25).

The study of the local stability of these two fixed points S and N is performed through the localization, on the complex plane, of the eigenvalues of the Jacobian matrix of (4.19) computed in S or N, i.e. the solutions of the characteristic equation

$$P(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \mathrm{Det}(A) = 0, \tag{A.1}$$

where $\mathrm{Tr}(A)$ and $\mathrm{Det}(A)$ denote the trace and the determinant of the Jacobian matrix.

$$DT(A, A) = \begin{bmatrix} 1 + v_1(h_1 + A(3h_3 + h_4)) & v_1(h_2 + A(3h_4 + h_3)) \\ v_2(h_2 + A(3h_4 + h_3)) & 1 + v_2(h_1 + A(3h_3 + h_4)) \end{bmatrix} \tag{A.2}$$

given by

$$\mathrm{Tr}(A) = 2 + (v_1 + v_2)(h_1 + A(3h_3 + h_4)) \tag{A.3}$$

$$\mathrm{Det}(A) = \mathrm{Tr}(A) - 1 + v_1v_2H_1(A)H_2(A) \tag{A.4}$$

with

$$H_1(A) = h_1 + h_2 + 4A(h_3 + h_4); \quad H_2(A) = h_1 - h_2 + 2A(h_3 - h_4), \tag{A.5}$$

where $A = s$ or n, respectively. Since $\mathrm{Tr}(A)^2 - 4\,\mathrm{Det}(A) = (h_1 + A(3h_3 + h_4))^2(v_1 - v_2)^2 + 4v_1v_2(h_2 + A(3h_4 + h_3))^2 > 0$ the eigenvalues of the matrix (A.2) are always real and can be written as

$$\lambda_1(A) = \frac{1}{2}\left(\mathrm{Tr}(A) - \sqrt{(\mathrm{Tr}(A) - 2)^2 - 4v_1v_2H_1(A)H_2(A)}\right) \tag{A.6}$$

$$\lambda_2(A) = \frac{1}{2}\left(\mathrm{Tr}(A) + \sqrt{(\mathrm{Tr}(A) - 2)^2 - 4v_1v_2H_1(A)H_2(A)}\right) \tag{A.7}$$

A sufficient condition for the stability of a fixed point is that both the eigenvalues belong to the interval $(-1, 1)$ whereas if at least one of them is greater than 1 or less that -1 the fixed point is unstable.

Let us first consider the fixed point $N = (n, n)$. Being $n < 0$, we have $(\mathrm{Tr}(n) - 2) > 0$, because $h_1 > 0$ and $3h_3 + h_4 < 0$. Moreover we have $H_1(n) > 0$ and $H_2(n) > 0$, because the first of (A.5), with n given by (4.25), gives $H_1(n) = \sqrt{(h_1 + h_2)^2 - 8h_0h_5} > 0$, and from the second of (A.5) easily follows $H_2(n) > 0$ being $h_1 - h_2 = R + (c(a - Rw)/(3b + 2c)) > 0$ ($a > Rw$ thanks to (3.4)) and $h_3 - h_4 = -bc^2/((b + 2c)(3b + 2c)) < 0$. These arguments imply that the condition $\lambda_1(n) > 1$ is equivalent to $\mathrm{Tr}(n) - 2 > \sqrt{(\mathrm{Tr}(n) - 2)^2 - 4v_1v_2H_1(n)H_2(n)}$ which is evidently true.

Instead, for the point $S = (s, s)$ we have $\lambda_2(s) \leq 1$ if and only if $h_1 + s(3h_3 + h_4) < 0$ and $v_1v_2H_1(s)H_2(s) \geq 0$. Hence $\lambda_2(s) = 1$ if $v_1 = 0$ or $v_2 = 0$, whereas we have

$\lambda_1(s) < \lambda_2(s) < 1$ for each $v_1 > 0$ and $v_2 > 0$. Instead, the second stability condition, $\lambda_1 > -1$ is satisfied if and only if $4 + 2(v_1 + v_2)(h_1 + s(3h_3 + h_4)) + v_1 v_2 H_1(s) H_2(s) \geq 0$.

This condition is satisfied for small values of v_1 or v_2, whereas it is not satisfied for large values of v_1 and v_2. this means that the fixed point S may have the eigenvalues $\lambda_1(s)$ and $\lambda_2(s)$ both inside the interval (−1, 1), thus giving local asymptotic stability of S, at least for sufficiently low values of the retention ratios v_1 or v_2, and for increasing values of the parameters v_1 and v_2 the smaller eigenvalues $\lambda_1(s)$ can exit the interval (−1, 1) through the values $\lambda_1(s) = -1$, so that the positive fixed point S loses stability via a flip (or period doubling) bifurcation at which S becomes a saddle point and a stable cycle of period two is created near it[7] (see e.g. Lorenz, 1993, Chapter 3, or Guckenheimer and Holmes, 1983, p. 158).

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[7] The rigorous proof that the flip bifurcation is supercritical, so that a stable cycle of period 2 is created, requires a center manifold reduction and the evaluation of higher order derivatives (up to the third order). This is rather boring and we claim numerical evidence for the existence of the stable cycle.