This paper is the second part of an earlier work devoted to the properties specific to maps of the plane characterized by the presence of a vanishing denominator, which gives rise to the generation of new types of singularities, called set of non-definition, focal points and prefocal curves. A prefocal curve is a set of points which are mapped (or “focalized”) into a single point, called focal point, by the inverse map when it is invertible, or by at least one of the inverses when it is noninvertible. In the case of noninvertible maps, the previous text dealt with the simplest geometrical situation, which is nongeneric. To be more precise this situation occurs when several focal points are associated with a given prefocal curve. The present paper defines the generic case for which only one focal point is associated with a given prefocal curve. This is due to the fact that only one inverse of the map has the property of focalization, but with properties different from those of invertible maps. Then the noninvertible maps of the previous Part I appear as resulting from a bifurcation leading to the merging of two prefocal curves, without merging of two focal points.

Keywords: Noninvertible maps; vanishing denominator; focal points; prefocal sets.

1. Introduction

In a previous paper [Bischi et al., 1999] (Part I henceforth), we studied some global dynamical properties of two-dimensional maps $T$, related to the presence of a denominator which vanishes in a one-dimensional subset of the plane. Differently from the present paper, which is limited to the case of noninvertible maps, in Part I we considered both invertible and noninvertible maps. We remind that the phase space of a noninvertible map is subdivided into open regions (or zones) $Z_k$, whose points have $k$ distinct rank-1 preimages, obtained by the application of $k$ distinct inverses of the map. A specific feature of noninvertible maps is the existence of the critical set $LC$ (from “Lignes Critique”, see [Gumowski & Mira, 1980; Mira et al., 1996]) defined as the locus of points having at least two coincident rank-1 preimages, located on the set of merging preimages denoted by $LC_{-1}$, $LC = T(LC_{-1})$. Segments of $LC$ are boundaries that separate different regions $Z_k$, but the converse is not generally true, that is, also in the two-dimensional case, as in the one-dimensional one, boundaries of regions $Z_k$ which are not portions of $LC$ may exist.
In Part I we have evidenced some new kinds of global bifurcations, due to the contacts of two singular sets of different nature, whose description requires the definition of new concepts which are specific of maps defined by functions having a vanishing denominator, like the set of nondefiniton, the focal points and the prefocal curves. Roughly speaking, a prefocal curve is a set of points which are mapped (or “focalized”, as we shall say for short) into a single point, called focal point, by the inverse function (if the map is invertible) or by at least one of the inverses (if the map is noninvertible). Such global bifurcations cause the creation of structures of the basins which are peculiar of maps with a vanishing denominator, called lobes and crescents, and have been explained in terms of contacts between basin boundaries and prefocal curves (see also [Mira, 1999; Bischi & Gardini, 1999; Bischi et al., 2001a]). These structures have been recently observed in discrete dynamical systems of the plane arising in different contexts, see e.g. [Yee & Sweby, 1994; Billings & Curry, 1996; Bischi & Naimzada, 1997; Brock & Hommes, 1997; Billings et al., 1997; Gardini et al., 1999; Bischi et al., 2001b].

These properties were studied by considering the image of an arc crossing through a focal point, and it has been shown that, in the generic case, given by a focal point which is simple (i.e. located at a transverse intersection of the curve of vanishing denominator and that of vanishing numerator) a one-to-one correspondence is obtained between the slopes of the arcs through a focal point and the points in which their images cross the corresponding prefocal set. This implies that the preimages of any curve crossing the prefocal set in two points includes a loop with a knot in the focal point, and this is the basic mechanism leading to the formation of lobes and crescents. In the case of noninvertible maps, the previous text was not sufficiently clear in the sense that it dealt with the simplest geometric situation which lead us to improperly indicate it as “generic” in the title. To be more precise, the situation described in Part I occurs when several focal points are associated with a given prefocal curve. This gives a simple geometric situation for the prefocal set, which may be considered as the result of a bifurcation characterized by the merging of two prefocal curves, without merging of focal points, as we shall see in the present paper. So, the noninvertible maps considered in Part I concern nongeneric situations in the family of maps having a vanishing denominator.

In the case of two-dimensional noninvertible maps, this paper will investigate more deeply the properties of two-dimensional iterated maps associated with the existence of simple focal points and related prefocal sets. The role played by the existence of two or more inverses at a point of a prefocal set is not obvious. In fact, from the definitions of focal points and prefocal sets it follows that at least one inverse must exist, and we shall call it the “focalizing inverse”. But which one is it? And what is its geometric action? Consider the simplest case in which only two inverses exist. We shall see that, generally, the focalizing inverse is only one along the whole prefocal set, but not always the same. If two focalizing inverses exist along a prefocal set (like in the maps studied in Part I) then this situation may be the result of the merging of two prefocal sets. Furthermore, when two distinct focal points exist, the related prefocal sets may be either different sets (generic case considered now) or the same set (bifurcation case considered in Part I). This qualitative difference is associated with a change of structure in the Riemann foliation of the plane (for the definitions and properties related to the Riemann foliation of two-dimensional noninvertible maps we refer to [Mira et al., 1996a; Mira et al., 1996b]).

So the present Part II essentially deals with the generic case of noninvertible maps for which only one focal point is associated with a given prefocal curve, i.e. when only one inverse of the map has the property of focalization, with properties different from those of invertible maps. More particularly, we shall see that the properties of the critical curves of the map are closely related to the presence of focal points and prefocal sets. As we shall see in the present paper, the focal points generally belong to the closure of the critical curve denoted by $LC_{-1}$, and consequently the critical curve $LC$ is tangent to the prefocal sets.

These properties of simple focal points and related prefocal sets are important in order to understand the dynamics of iterated maps which are not defined in the whole plane. For example, the appearance/disappearance of the particular structures called lobes and crescents, may be related to the critical curves of the map, as shown in Part I, but also to the simple crossing of invariant phase curves through prefocal sets, as we shall see in this
work. Other new results given in this paper concern the geometric action of the preimages, and the foliation structure, in the region between two prefocal sets, and several kinds of new bifurcation mechanisms related to arcs crossing the prefocal sets giving rise to loops and arcs through the focal points.

The paper is organized as follows. In Sec. 2 we remind some definitions related to the maps having a vanishing denominator, distinguishing between invertible maps and noninvertible ones. In particular, in Sec. 2.3 we shall see some relations between the critical set $LC_{-1}$ and other sets, such as the set in which the Jacobian determinant vanishes, and the set of points across which a change in sign of the Jacobian determinant may occur, as well as its relation with the focal points of a map. We show that in the generic case the focal points belongs to $LC_{-1}$, the closure of $LC_{-1}$. In Sec. 3 we stress some properties of the simplest noninvertible maps (of $Z_0 = Z_2$ type) having simple focal points. First, in Sec. 3.2, we discuss the case of focal points that do not belong to $LC_{-1}$. This may also be considered as a bifurcation case with respect to the situations described in Sec. 3.3, where maps with focal points belonging to $LC_{-1}$ are considered. In the latter case, which is the one mainly considered in the present paper, we show the particular role of the two inverses, which is at the basis of the particular geometric behavior associated with the focal points, and the foliation associated with the prefocal sets, which is peculiar of such kind of maps. Section 4 deals with a new kind of bifurcation which gives rise to the formation of crescents between two focal points. The results of the previous sections are then evidenced through the numerical examples of Sec. 5.

This paper will be followed by another one, Part III (to appear), where we shall consider the effects of the merging of two or more focal points, which thus satisfy the condition of “nonsimple” focal points, and some of the related bifurcations.

2. Definitions and Basic Properties

2.1. Data in common with invertible and noninvertible maps

In order to simplify the exposition, we assume that only one of the two functions defining the map $T$ has a denominator which can vanish. Let

$$ T : \begin{cases} x' = F(x, y) \\ y' = G(x, y) = \frac{N(x, y)}{D(x, y)} \end{cases} $$

where $x$ and $y$ are real variables, $F(x, y)$, $N(x, y)$, and $D(x, y)$ are continuously differentiable functions defined in the whole plane $\mathbb{R}^2$ (and without common factors). Then the set of nondefinition of the map $T$ (that is, the set of points where at least one denominator vanishes) is given by

$$ \delta_s = \{(x, y) \in \mathbb{R}^2 \mid D(x, y) = 0\}. $$

As in Part I we assume that $\delta_s$ is a smooth curve, or made up of pieces of smooth curves of the plane. The two-dimensional recurrence obtained by the successive iterations of $T$ is well defined provided that the initial condition belongs to the set $E$ given by $E = \mathbb{R}^2 \setminus \bigcup_{k=0}^{\infty} T^{-k}(\delta_s)$, where $T^{-k}(\delta_s)$ denotes the set of the rank-$k$ preimages of $\delta_s$, i.e. the set of points which are mapped into $\delta_s$ after $k$ applications of $T (T^0(\delta_s) = \delta_s)$. Indeed, in order to generate uninterrupted sequences by the iteration of the map $T$, the points of $\delta_s$, as well as all their preimages of any rank, constitute a set of zero Lebesgue measure which must be excluded from the set of initial conditions so that $T : E \to E$.

In order to analyze the role of the set of nondefinition $\delta_s$, we consider a bounded and smooth simple arc $\gamma$, parametrized as $\gamma(\tau)$, transverse to $\delta_s$, such that $\gamma(0) = (x_0, y_0)$ and $\gamma \cap \delta_s = \{(x_0, y_0)\}$, and we study its image $T(\gamma)$. As $(x_0, y_0) \in \delta_s$ we have, according to the definition of $\delta_s$, $D(x_0, y_0) = 0$, but in general $N(x_0, y_0) \neq 0$. Hence

$$ \lim_{\tau \to 0^\pm} T(\gamma(\tau)) = (F(x_0, y_0), \infty) $$

where $\infty$ means either $+\infty$ or $-\infty$. This means that the image $T(\gamma)$ is made up of two disjoint unbounded arcs asymptotic to the line of equation $x = F(x_0, y_0)$. A different situation may occur if the point $(x_0, y_0) \in \delta_s$ is such that not only the denominator but also the numerator of $G(x, y)$ vanishes in it, i.e. $D(x_0, y_0) = N(x_0, y_0) = 0$. In this case, the second component of the limit (3) takes the form $0/0$. This implies that this limit may give rise to a finite value, so that the image $T(\gamma)$ is a bounded arc crossing the line $x = F(x_0, y_0)$ at the point $(F(x_0, y_0), y)$, where

$$ y = \lim_{\tau \to 0} G(x(\tau), y(\tau)) $$

It is clear that the value $y$ in (4) must depend on the arc $\gamma$. Furthermore it may have a finite value
along some arcs and be infinite along other ones. This leads us to the following definition of the singular sets of focal point and prefocal curve (given in Part I):

**Definition.** A point \( Q = (x_0, y_0) \) is a focal point of a two-dimensional map \( T \) if at least one component of \( T \) takes the form \( 0/0 \) in \( Q \) and there exist smooth simple arcs \( \gamma(\tau) \), with \( \gamma(0) = Q \), such that \( \lim_{\tau \to 0} T(\gamma(\tau)) \) is finite. The set of all such finite values, obtained by taking different arcs \( \gamma(\tau) \) through \( Q \), is the prefocal set \( \delta_Q \).

In this paper we shall only consider simple focal points, i.e. points which are simple roots of the algebraic system

\[
N(x, y) = 0, \quad D(x, y) = 0
\]

Thus a focal point \( Q = (x_0, y_0) \) is simple if

\[
N_x D_y - N_y D_x \neq 0 \quad (5)
\]

where \( N_x = (\partial N/\partial x)(x_0, y_0) \) and analogously for the other partial derivatives. In the case of a simple focal point there exists a one-to-one correspondence between the point \( (F(Q), y) \), in which \( T(\gamma) \) crosses \( \delta_Q \), and the slope \( m \) of \( \gamma \) in \( Q \):

\[
m \to (F(Q), y(m)), \quad \text{with } y(m) = \frac{N_x + mN_y}{D_x + mD_y} \quad (6)
\]

and

\[
(F(Q), y) \to m(y) \text{ with } m(y) = \frac{D_y y - N_x}{N_y - D_y y} \quad (7)
\]

As shown in Part I, these relations can be obtained by using a method either based on a series expansion of the functions \( N(x, y) \) and \( D(x, y) \) in a neighborhood of \( Q = (x_0, y_0) \), or by considering the Jacobian determinant of the inverse map \( T^{-1} \) (or one of the inverses if the map is noninvertible).

If \( Q \) is simple, the point \( (F(Q), y) \) spans the whole line \( x = F(Q) \) as \( m \) varies. In fact, if \( D_y \neq 0 \) then the graph of \( y(m) \) is a hyperbola, so that for any \( m \in \mathbb{R}, m \neq -\frac{D_x}{D_y}, y(m) \in \mathbb{R} \setminus \{y^*\} \), with \( y^* = \frac{N_y}{D_y} \). Moreover, if \( m \to -\frac{D_x}{D_y} \) then \( y(m) \to \pm \infty \) (i.e. for \( \gamma \) tangent to \( \delta_s \), \( T(\gamma) \) is unbounded with \( \delta_Q \) as asymptote) and as \( m \to \pm \infty \) then \( y(m) \to y^* \) (i.e. the image of an arc \( \gamma \) with vertical tangent in \( Q \) is mapped by \( T \) into an arc which crosses the line \( \delta_Q \) in \( (F(Q), y^*) \)). Furthermore, if \( D_y = 0 \) then \( y(m) \) is a linear function of \( m \) so that its range is \( \mathbb{R} \). So, we can state the following proposition:

**Proposition.** Let \( T \) be the map defined in (1). If \( Q \) is a simple focal point of \( T \) then the related prefocal set \( \delta_Q \) is given by the whole line \( x = F(Q) \).

In this paper we shall consider only simple focal points and the structure of the plane foliation related to them, whereas the case of focal points which are not simple, as may occur in bifurcations which have been called class two in Part I, will be considered elsewhere (in Part III).

From the definition of the prefocal curve, it follows that the Jacobian determinant of the inverse of \( T \), \( \text{det}(DT^{-1}) \), if \( T \) is invertible, or of at least one of the inverses, if \( T \) is noninvertible, must necessarily vanish in the points of \( \delta_Q \). Indeed, if the map \( T^{-1} \) is defined in \( \delta_Q \), then all the points of the line \( \delta_Q \) are mapped by at least one inverse \( T^{-1} \) into the focal point \( Q \). This means that at least one inverse \( T^{-1} \) is not locally invertible in the points of \( \delta_Q \), it being a many-to-one map, and this implies that its Jacobian cannot be different from zero in the points of \( \delta_Q \).

From the relations (6) and (7) it follows that different arcs \( \gamma_j \), crossing through a focal point \( Q \) with different slopes \( m_j \), are mapped by \( T \) into bounded arcs \( T(\gamma_j) \) crossing \( \delta_Q \) at different points \( (F(Q), y(m_j)) \), see the qualitative picture in Fig. 1.

Interesting properties are obtained if the inverse (or inverses) of \( T \) is (are) applied to a curve that crosses a prefocal set. This is the basic mechanism which leads to the creation of lobes, a feature which often characterizes the structure of the basins of maps with denominator (see e.g. [Bischi & Gardini, 1997; Bischi & Naimzada, 1997; Billings et al., 1997;].

![Fig. 1. Qualitative sketch to describe how different arcs crossing through a focal point \( Q \), with different slopes, are mapped into bounded arcs crossing the corresponding prefocal curve \( \delta_Q \) at different points. \( \delta_s \) is the line of nondefinition (set where the denominator vanishes).](image-url)
Brock & Hommes, 1997; Gardini et al., 1999; Bischi et al., 1999). As we shall see, the properties of the inverses are different for invertible and noninvertible maps, so that we shall describe them separately. We first remind the properties of an invertible map, and then we shall consider those of a noninvertible one, which is the main object of this paper.

2.2. Properties specific to invertible maps

Let the map $T$ be invertible, and let $\delta_Q$ be a prefocal curve whose corresponding focal point is $Q$. As noted above, the inverse map is not locally invertible in the points of the prefocal set $\delta_Q$, thus we consider $T$ as an invertible map in its domain of definition ($\mathbb{R}^2 \setminus \delta_s$) and its range cannot include the prefocal set $\delta_Q$. However, it is clear that each point sufficiently close to $\delta_Q$ has its rank-1 preimage in a neighborhood of the focal point $Q$. Moreover, if the inverse $T^{-1}$ is defined and continuous along $\delta_Q$ (as generally occurs) then all the points of $\delta_Q$ are mapped by $T^{-1}$ in the focal point $Q$. Roughly speaking we can say that the prefocal curve $\delta_Q$ is “focalized” by $T^{-1}$ in the focal point $Q$, or, more concisely, that $T^{-1}(\delta_Q) = Q$. We stress again however that the map $T$ is not defined in $Q$, thus $T^{-1}$ cannot to be strictly considered as “an inverse of $T$ in the points of $\delta_Q$”, even if $T^{-1}$ is defined in $\delta_Q$.

The relation (7) implies that the preimages of different arcs crossing the prefocal curve $\delta_Q$ in the same point $(F(Q), y)$ are given by arcs crossing the singular set through $Q$, all with the same slope $m(y)$ in $Q$. Indeed, if we consider different arcs $\eta_j$ crossing $\delta_Q$ in the same point $(F(Q), y)$ with different slopes, then these arcs are mapped by the inverse $T^{-1}$ into different arcs $T^{-1}(\eta_j)$ through $Q$, all with the same tangent of slope $m(y)$, according to (7); they differ by the curvature at the point $Q$. Clearly, arcs crossing through $\delta_Q$ in different points $y_i$ have preimages which cross the focal point $Q$ in arcs with different slopes $m(y_i)$ according to (7), see the qualitative picture in Fig. 2(a). On the other hand, if we consider an arc crossing through $\delta_Q$ in two points $y_1$ and $y_2$ then its preimage is necessarily an arc with a knot in the focal point, with different tangents $m(y_i)$ according to (7), see the qualitative picture in Fig. 2(b). We can so state the following

**Proposition 2.1.** Let $T$ be an invertible map, $Q$ a focal point of $T$ with prefocal set $\delta_Q$; then any arc crossing the prefocal set $\delta_Q$ in two points has a preimage which includes a loop with knot in $Q$.

As shown in Part I, such kind of loops with knots in $Q$ are very important when studying, for example, the structure of basins and their boundaries. In fact, if the arc $\eta$ belongs to a basin boundary, then the loops are responsible for the formations of the particular areas called “lobes”, as we shall also see in the examples of the last section. Thus the contacts of a basin boundary with the prefocal set $\delta_Q$ may cause the occurrence of particular structures.

How these properties persist or change when $T$ is not invertible is the object of Sec. 3, while in the next subsection we give some basic definitions and properties concerning the case of noninvertible maps.
2.3. Properties specific to noninvertible maps

We remind here some definitions and properties specific to noninvertible maps and also give some new insight about the relations between critical sets and focal points. In the case of a noninvertible map, the phase space is divided into open regions (or zones) \( Z_k \), whose points have \( k \) distinct rank-1 preimages, obtained by the application of \( k \) distinct inverse maps (see e.g. [Gumowski & Mira, 1980; Mira et al., 1996a]). In other words, if \( (x, y) \in Z_k \) then \( k \) distinct points \( (x_j, y_j) \) are mapped into \( (x, y) \), i.e. \( T(x_j, y_j) = (x, y) \), \( j = 1, \ldots, k \), or equivalently we can say that \( k \) distinct inverse maps \( T_j^{-1} \) exist such that \( T_j^{-1}(x, y) = (x_j, y_j) \), \( j = 1, \ldots, k \). A specific feature of noninvertible maps is the existence of the critical set \( LC \) (from the French “Ligne Critique”, see [Gumowski & Mira, 1980]), defined as the locus of points having at least two coincident rank-1 preimages, which belong to the set of merging preimages, denoted by \( LC_{-1} \). In any neighborhood of a point of \( LC_{-1} \) there are at least two distinct points mapped by \( T \) into the same point, so that the map \( T \) is not locally invertible in the points of \( LC_{-1} \), and this implies that for differentiable maps the set \( LC_{-1} \) is included in the set \( J_0 \) of points in which the Jacobian of \( T \) vanishes:

\[
LC_{-1} \subseteq J_0 = \{(x, y) \in \mathbb{R}^2 | \det DT = 0 \}
\]

(8)

of course \( LC = T(LC_{-1}) \).

Segments of \( LC \) are boundaries that separate different regions \( Z_k \), but the converse is not generally true, that is boundaries of regions \( Z_k \), which are not portions of \( LC \), may exist. This happens, for example, in polynomial maps having an inverse function with a vanishing denominator. In fact, as shown in Part I, this may imply the existence of a set which is mapped by \( T \) in one point, and such a set must belong to \( J_0 \) even if it is not critical, so that we have a strict inclusion: \( LC_{-1} \subseteq J_0 \).

Another distinguishing feature in many noninvertible maps is the existence of a set of points, which we shall denote by \( J_C \), through which we have a change in the sign of the Jacobian of \( T \), \( \det DT \). From the geometric action of the foliation of the Riemann plane we can also say that the critical set \( LC_{-1} \) must belong to \( J_C \). In fact, a plane region \( U \) which intersects \( LC_{-1} \) is “folded” along \( LC \) into the side with more preimages, and the two folded images have opposite orientation. This implies that the map has the Jacobian with a different sign in the two portions of \( U \) separated by \( LC_{-1} \), so that

\[
LC_{-1} \subseteq J_C .
\]

Previous results (for example see [Mira et al., 1996a]) have shown that for a smooth map without a vanishing denominator we often have \( LC_{-1} = J_C \subseteq J_0 \), the strict inclusion occurring when there are sets mapped into a point (as shown in Part I of this paper), and when there are points \( X \) such that \( J_0(X) = 0 \) with the map \( T \) locally invertible at \( X \) (in the one-dimensional case such a point \( X \) is an inflection point). In the case of a smooth map with a vanishing denominator the situation is more complex. Indeed it can be seen (an example will be given in Sec. 5.3) that the set \( J_C \) may include also the singular set \( \delta_S \), although this can be considered as a bifurcation case, so that it occurs with the strict inclusion \( LC_{-1} \subseteq J_C \).

So it is plain that \( LC_{-1} \subseteq J_C \cap J_0 \). However, from the geometric properties of \( T \), we conjecture that the following relation holds:

\[
LC_{-1} = J_C \cap J_0 .
\]

(9)

With continuously differentiable maps without a vanishing denominator, we may have \( LC_{-1} = J_C \subset J_0 \), although \( LC_{-1} = J_C = J_0 \) often occurs in the simplest situations. In smooth maps with a vanishing denominator \( LC_{-1} = J_0 \subset J_C \) may occur. All these are particular cases of (9).

For the class of maps (1) considered in this paper, the Jacobian matrix is given by

\[
DT(x, y) = \begin{bmatrix}
    F_x & F_y \\
    N_y D - ND_y & N_x D - ND_x
\end{bmatrix}
\]

(10)

and we shall assume that \( F_y \) is not identically zero, i.e. we do not consider the particular case of a triangular map \( T \) (whose properties are often related to those of a one-dimensional map), because we are interested in true two-dimensional properties, and in particular we shall assume that \( F_y \) evaluated at a focal point \( Q \) is different from zero.

The Jacobian determinant is

\[
\det DT(x, y) = \frac{A(x, y)}{D^2}
\]

(11)

where

\[
A(x, y) = [F_x(N_y D - ND_y) - F_y(N_x D - ND_x)]
\]

(12)
DT is defined in \( \mathbb{R}^2 \setminus \delta_s \), and \( LC_{-1} \) is included in the set of points in which \( A(x, y) \) vanishes,

\[
J_0 = \{(x, y) \in \mathbb{R}^2 | A(x, y) = 0 \} \cap \{\mathbb{R}^2 \setminus \delta_s \}.
\]

As \( N = 0 \) and \( D = 0 \) at a focal point, from \( A(x, y) = 0 \) in (12) it follows that in general the focal points belong to the closure \( LC_{-1} \) of \( LC_{-1} \) (not defined at these points). Moreover, we can easily compute the slope of the tangent to \( LC_{-1} \) in a simple focal point \( Q \). In fact, the partial derivative with respect to \( y \), computed at \( Q \), of the expression \( A(x, y) \) in (12) is \( F_y(N_yD_x - N_xD_y) \), and the partial derivative with respect to \( x \), computed at \( Q \), is \( F_x(N_yD_x - N_xD_y) \). Hence, the slope of the tangent to \( LC_{-1} \) at a simple focal point \( Q \) (for which (5) holds) is

\[
\overline{m} = \frac{\overline{A}_x}{\overline{A}_y} = \frac{\overline{F}_x}{\overline{F}_y}.
\]

We also note that when \( F(x, \overline{y}) \) is a linear or an affine function, then the partial derivatives are constant and the slope in (14) is independent on the focal point \( Q \) (i.e. it has the same value for each focal point), so that the tangents to \( LC_{-1} \) in all the focal points are parallel.

The fact that the closure of \( LC_{-1} \) intersects the singular set \( \delta_s \) has some implications concerning the critical set \( LC = T(LC_{-1}) \). In fact, if a simple focal point \( Q_1 \in LC_{-1} \), then the critical curve \( LC \) has a contact with the corresponding prefocal line \( \delta_Q \) at a point \( V \) whose \( y \) coordinate can be easily computed by (6) with \( m \) given in (14):

\[
y_V = y(\overline{m}) = \frac{N_xF_y - F_xN_y}{D_xF_y - F_xD_y}.
\]

From (12) and (8) it is also evident that \( LC_{-1} \) may intersect \( \delta_s \) at a nonfocal point, i.e. a point \( (x_0, y_0) \) such that \( D(x_0, y_0) = 0 \) and \( N(x_0, y_0) \neq 0 \), provided that the equality

\[
F_yD_x - F_xD_y = 0
\]

holds in \( (x_0, y_0) \) so that \( A = N(F_yD_x - F_xD_y) = 0 \) at the point \( (x_0, y_0) \). This suggests that, generally, \( LC_{-1} \) may intersect \( \delta_s \) at isolated nonfocal points. However, it may occur, in particular cases, that (16) holds in all the points of \( \delta_s \). In this case, from (11) written in the form

\[
\det DT = \frac{1}{D^2}[D(F_xN_y - F_yN_x) + N(F_yD_x - F_xD_y)]
\]

we can see that when (16) holds for any point of \( \delta_s \) and \( D \) changes its sign crossing the set of non-definition, \( (F_xN_y - F_yN_x) \) being finite, the Jacobian determinant also changes sign on the two sides of \( \delta_s \), i.e. \( \delta_s \subset J_C \). So, even if the map \( T \) is not defined at the points of \( \delta_s \), in this case we are led to guess that \( \delta_s \) may have properties similar to those of \( LC_{-1} \). This is not so unusual if we consider that, as stated in Part I, \( \delta_s \) may be seen as the two-dimensional analogue of a vertical asymptote. Indeed, such cases remind us what occurs in one-dimensional maps like \( x' = 1/x^2 \), which is a \( Z_0 - Z_2 \) map such that for each \( x' > 0 \) the two preimages \( x = \pm 1/\sqrt{x'} \), are located at opposite sides with respect to the vertical asymptote of equation \( x = 0 \), so that the set of non-definition, \( x = 0 \), plays the role of a critical point of rank-0 with critical point of rank-1 “at infinity”.

In Sec. 5.3 we shall see that a similar geometric behavior also occurs in noninvertible maps not defined everywhere.


3.1. Description of two possible situations

When \( T \) is a continuous noninvertible map with focal points, then one of the following two cases arises:

(i) Several focal points may be associated with a given prefocal curve \( \delta_Q \), each with its own one-to-one correspondence between slopes and points, like the one defined in (6) and (7). It is the situation considered in Part I, for which no focal point belongs to \( LC_{-1} \).

(ii) Only one focal point \( Q \) is associated with a given prefocal curve \( \delta_Q \), because only one inverse focalizes, but with properties different from those recalled in Sec. 2.1 for invertible maps. In this second case the focal point is related to the existence of several focalizing inverses and a particular structure of the foliation.

The analysis of the properties related to noninvertible maps of the simplest form (i.e. maps of type \( Z_0 - Z_2 \)) is the object of this section. In the previous section we have seen, from the properties of a noninvertible map with a vanishing denominator, that generally a focal point \( Q \) belongs to the set \( LC_{-1} \cap \delta_s \), where \( LC_{-1} \) denotes the closure of
(b) If all the inverses are continuous along a pre-

is mapped by each 

the corresponding 

a point ( 

belongs to 

above, it is easy to see that an arc 

into different arcs through 

So this situation is nongeneric, in spite of the 

(improperly chosen) title of Part I which might lead 

to think it as “generic”.

3.3. All the focal points belong to \( \overline{LC}_{-1} \)

3.3.1. Contact between the critical curve 

and the prefocal set

When the focal points belong to \( \overline{LC}_{-1} \), the geometric properties of the phase plane, and the bifurca-

ctions which may occur are more complex with re-

spect to those described in Sec. 3.1. This is due to 

the fact that now \( \overline{LC} \) has contact points at finite 

distance with the prefocal curves. Moreover, the 

property \( Q_j = T_j^{-1}(\delta_Q), j = 1, \ldots, n, \) with \( n \leq k \), 

does not hold. Indeed, in the generic case a given 

prefocal curve \( \delta_Q \) is not associated with several focal 

points \( Q_j \) as in the case described in Sec. 3.1. Only 

one of the inverses \( T_j^{-1} \) maps a noncritical point of a 

prefocal curve into its related focal point, so that we 
can write \( Q = T_j^{-1}(F(Q), y) \), but the index \( j \), defin-
ing the focalizing inverse function, depends on the 

point \( (F(Q), y) \) considered on \( \delta_Q \). For this reason 

the previous situation of \( \delta_Q \) (i.e. with focal points 

not belonging to \( \overline{LC}_{-1} \)) appears as nongeneric (and 
in the examples of Sec. 5 we shall see that it may 

result from the merging of two prefocal curves \( \delta_{Q_1} \), 
and \( \delta_{Q_2} \) without merging of the corresponding focal 

points \( Q_1 \) and \( Q_2 \)).

Let us start by analyzing the properties associated 

with the arcs of \( \overline{LC}_{-1} \) which cross through a focal point \( Q \). As we have seen in Sec. 2, if an arc crosses the singular set \( \delta_S \) at a focal point, then 

its image by \( T \) must have a contact with the prefo-
cal curve at a specific point. This also holds for an 

arc of critical curve \( LC_{-1} \), and we can compute the 

contact point \( V = (F(Q), y_V) \) between \( \overline{LC} \) (image of \( \overline{LC}_{-1} \) and \( \delta_Q \) by using (15). However \( \overline{LC} \) generally does not intersect transversely the prefocal 

curve \( \delta_Q \) in \( V \). This can be easily explained in the 

case in which the map \( T : (x, y) \rightarrow (x', y') \) is of type \( Z_0 - Z_2 \). In fact, the image of an arc of \( LC_{-1} \) belongs to the critical curve \( LC \), which belongs to the boundary between the regions \( Z_0 \) and \( Z_2 \). If, by 

contradiction, we assume that an arc of \( \overline{LC} \) intersects 

transversely \( \delta_Q \) in \( V \), then we have a part of 

\( \delta_Q \) in \( Z_0 \), which is impossible, because by definition

\( LC_{-1} \) (this fact results from \( A(x, y) = 0 \) in (12) 

with \( N = D = 0 \)). Nevertheless in the particular 
cases in which \( \delta_S \) belongs to \( J_c \), it happens that a 
focal point \( Q \) may not belong to \( \overline{LC}_{-1} \). The geometric 

behavior and the plane foliation are different in 

these two cases. This leads to different situations, 

according to the fact that the focal points belong 
or not to the set \( \overline{LC}_{-1} \). Of course, it may occur 

that only some focal points belong to the set \( \overline{LC}_{-1} \), 

so that a map \( T \) may have mixed properties, with 

respect to the cases described below.

3.2. No focal point belongs to \( \overline{LC}_{-1} \)

The following properties of this nongeneric case 

have been shown in Part I.

(a) For each prefocal curve \( \delta_Q \) we have \( LC \cap \delta_Q = \emptyset \) 

(ruling out the points at infinity).

(b) If all the inverses are continuous along a pre-
focal curve \( \delta_Q \), then the whole prefocal set \( \delta_Q \) 

belongs to a unique region \( Z_k \) in which \( k \) inverse 

maps \( T_j^{-1}, j = 1, \ldots, k \), are defined.

It is plain that for a prefocal \( \delta_Q \) at least one 
inverse is defined that “focalizes” it into a focal 
point \( Q \). However, other inverses may exist that “fo-
calize” \( \delta_Q \) into distinct focal points, all associated 

with the same prefocal curve \( \delta_Q \). These focal points 

are denoted as \( Q_j = T_j^{-1}(\delta_Q), j = 1, \ldots, n, \) with 

\( n \leq k \). For each focal point \( Q_j \) the same results 
described in the previous sections can be obtained 

with \( T^{-1} \) replaced by \( T_j^{-1} \), so that for each \( Q_j \) a 

one-to-one correspondence \( m_j(y) \) in the form (6) is 
defined. Following arguments similar to those given 
above, it is easy to see that an arc \( \eta \) crossing \( \delta_Q \) at 
a point \( (F(Q), y) \), where \( F(Q) = F(Q_j) \) for any \( j \), 
is mapped by each \( T_j^{-1} \) into an arc \( T_j^{-1}(\eta) \), through 

the corresponding \( Q_j \) with the slope \( m_j(y) \). If differ-
ent arcs are considered, crossing \( \delta_Q \) at the same 
point, then these are mapped by each inverse \( T_j^{-1} \) 
into different arcs through \( Q_j \), all with the same 
tangent.

Considering a nonfocal point \( D = LC_{-1} \cap \delta_S \), 
then the critical set \( LC = T(LC_{-1}) \) includes an 

asymptote \( x = F(D) \). Examples given in Part I 
(e.g. the maps (17) and (28) in that paper) show 

that this asymptote is a prefocal curve. It will ap-

pear below that the case in which no focal point 

belongs to \( \overline{LC}_{-1} \), corresponds to a bifurcation 
of merging of two prefocal curves without merging 
of focal points, and it is also associated with the 

intersection between \( LC_{-1} \) and \( \delta_S \) in a nonfocal 
point \( D \). Then the fact that the asymptote \( x = 

F(D) \) is a prefocal curve will be deduced from the 

limit process giving the points \( LC \cap \delta_Q \) at infi-

nity. So this situation is nongeneric, in spite of the 

(improperly chosen) title of Part I which might lead 

to think it as “generic”.

3.3. All the focal points belong to \( \overline{LC}_{-1} \)

3.3.1. Contact between the critical curve 

and the prefocal set

When the focal points belong to \( \overline{LC}_{-1} \), the geometric 

properties of the phase plane, and the bifurca-

tions which may occur are more complex with re-

spect to those described in Sec. 3.1. This is due to 

the fact that now \( \overline{LC} \) has contact points at finite 

distance with the prefocal curves. Moreover, the 

property \( Q_j = T_j^{-1}(\delta_Q), j = 1, \ldots, n, \) with \( n \leq k \), 
does not hold. Indeed, in the generic case a given 

prefocal curve \( \delta_Q \) is not associated with several focal 

points \( Q_j \) as in the case described in Sec. 3.1. Only 

one of the inverses \( T_j^{-1} \) maps a noncritical point of a 

prefocal curve into its related focal point, so that we 
can write \( Q = T_j^{-1}(F(Q), y) \), but the index \( j \), defin-
ing the focalizing inverse function, depends on the 

point \( (F(Q), y) \) considered on \( \delta_Q \). For this reason 

the previous situation of \( \delta_Q \) (i.e. with focal points 

not belonging to \( \overline{LC}_{-1} \)) appears as nongeneric (and 
in the examples of Sec. 5 we shall see that it may 

result from the merging of two prefocal curves \( \delta_{Q_1} \), 
and \( \delta_{Q_2} \) without merging of the corresponding focal 

points \( Q_1 \) and \( Q_2 \)).

Let us start by analyzing the properties associated 

with the arcs of \( \overline{LC}_{-1} \) which cross through a focal point \( Q \). As we have seen in Sec. 2, if an arc crosses the singular set \( \delta_S \) at a focal point, then 

its image by \( T \) must have a contact with the prefo-
cal curve at a specific point. This also holds for an 

arc of critical curve \( LC_{-1} \), and we can compute the 

contact point \( V = (F(Q), y_V) \) between \( \overline{LC} \) (image of \( \overline{LC}_{-1} \) and \( \delta_Q \) by using (15). However \( \overline{LC} \) generally does not intersect transversely the prefocal 

curve \( \delta_Q \) in \( V \). This can be easily explained in the 

case in which the map \( T : (x, y) \rightarrow (x', y') \) is of type \( Z_0 - Z_2 \). In fact, the image of an arc of \( LC_{-1} \) belongs to the critical curve \( LC \), which belongs to the boundary between the regions \( Z_0 \) and \( Z_2 \). If, by 

contradiction, we assume that an arc of \( \overline{LC} \) intersects 

transversely \( \delta_Q \) in \( V \), then we have a part of 

\( \delta_Q \) in \( Z_0 \), which is impossible, because by definition
for each point of \( \delta_Q \) associated with a simple focal point \( Q \), at least one focalizing inverse must exist (see the qualitative picture in Fig. 3). We have so proved the following

**Proposition 3.1.** Let \( T \) be a noninvertible map of type \( Z_0-Z_2, Q \) a simple focal point of \( T \) with prefocal set \( \delta_Q \). Then an arc of \( L_{C-1} \) crossing the focal point \( Q \) has slope \( \overline{m} = -\frac{F_x}{F_y} \) in \( Q \), and is mapped by \( T \) into an arc of \( LC \) which is tangent to the prefocal set \( \delta_Q \) at the point \( V = (F(Q), y_V) \) where \( y_V \) is given in (15).

Being \( LC \) tangent to \( \delta_Q \) in the point \( V \), we have that the rank-1 preimages of any point \( (F(Q), y) \in \delta_Q \), with \( y \) different from \( y_V \), include two distinct points, one of which is necessarily focalized at the focal point \( Q \) and the other belongs to a curve which is tangent to \( L_{C-1} \) in \( Q \). This comes from the fact that \( LC \cap \delta_Q = V \) and the slope of arcs issuing from \( Q \) associated with the point \( V \) of the prefocal set is \( \overline{m} \) (see (14), (15), and (6) (7)), where \( \overline{m} \) is the slope of the tangent to \( L_{C-1} \) in \( Q \). Thus \( T^{-1}(LC \cup \delta_Q) \) is made up of two arcs, \( L_{C-1} \) and an arc \( \tau \), which have a common tangent of slope \( \overline{m} \) in \( Q \).

Noticing that both the preimages of the tangency point \( V \) are “focalized” at the same focal point, we are led to consider the tangency point \( V \) (which belongs to \( LC \) but not to \( LC' \)) as a particular point for the foliation structure of the two-dimensional noninvertible maps. In fact, \( V \) really behaves as a critical point (with two merging preimages) and crossing \( V \) we may have a change of role of the focalizing inverse on the prefocal set. We remark the difficulty in the definition of the critical sets for this class of maps. That is, in Sec. 2.3 we consider the critical points of \( T \) as those having two merging rank-1 preimages, then \( V \) ought to be a point of \( LC \), but the merging preimages are of a focal point \( Q \), where the map is not defined, thus this point is not a point of \( L_{C-1} \) and consequently also \( V \) is not a critical point.

### 3.3.2. Preimages of an arc crossing through a prefocal set

In order to clarify this particular geometric behavior, a qualitative illustration of the simplest case is given in Fig. 4. This case is related to a situation with two prefocal curves generated by a noninvertible map \( T : (x, y) \to (x', y') \) of type \( (Z_0-Z_2) \), i.e. the plane contains only regions whose points have no rank-1 preimages, and regions whose points have two rank-2 preimages. In other words, the inverse relation \( T^{-1}(x', y') \) has two components in the region \( Z_2 \), denoted by \( T_1^{-1} \) and \( T_2^{-1} \), and no real components in the region \( Z_0 \). The set of non-definition \( \delta_i \) is a simple straight line, and there are two prefocal lines, \( \delta_{Q_i} \), of equation \( x = F(Q_i) \), associated with the focal points \( Q_i \), \( i = 1, 2 \), respectively, and \( V_i = LC \cap \delta_{Q_i} = (F(Q_i), y(V_i)) \) (where \( y(V_i) \) is given in (15)) are the points of tangency between \( LC \) and the two prefocal curves. Let \( \delta'_{Q_i} \) be the segment of \( \delta_{Q_i} \) such that \( y > y(V_i) \) (pecked line in Fig. 4), and \( \delta''_{Q_i} \), the segment of \( \delta_{Q_i} \) such that \( y < y(V_i) \) (continuous line in Fig. 4). The “focalization” of the inverses occurs in the following way:

\[
T_1^{-1}(\delta'_{Q_i}) = Q_i, \quad T_2^{-1}(\delta''_{Q_i}) = Q_i, \quad i = 1, 2 \quad (17)
\]

while the other preimages give

\[
T_2^{-1}(\delta_{Q_i}) \cup T_1^{-1}(\delta_{Q_i}) = \pi_i, \quad i = 1, 2
\]

each \( \pi_i \) being a set crossing through the focal points \( Q_i \) and tangent to \( L_{C-1} \) at the focal point, and in the qualitative picture in Fig. 4 (corresponding to an example which will be explicitly described in Sec. 5) \( \pi_i \) is exactly the line tangent to \( L_{C-1} \) in \( Q_i \). It follows that the vertical strip between the two prefocal lines \( \delta_{Q_i} \) is entirely included in the region \( Z_2 \), so that each point \( (x', y') \) in this strip has two distinct rank-1 preimages. However, differently from what occurs in two-dimensional maps defined in the whole plane, the two preimages \( T_1^{-1}(x', y') \) and \( T_2^{-1}(x', y') \) are not located one on the right and one on the left of \( L_{C-1} \), as both are located in the
Fig. 4. Qualitative representation of the preimages of an arc crossing through a prefocal set in a situation with two prefocal curves generated by a noninvertible map of type \((Z_o - Z_2)\). The prefocal set \(\delta_Q\) is made up of two lines \(\delta_Q = \delta_{Q_1} \cup \delta_{Q_2}\), \(\delta_{Q_1} = \delta'_{Q_1} \cup \delta''_{Q_1}\), \(\delta_{Q_2} = \delta'_{Q_2} \cup \delta''_{Q_2}\). Every arc crossing through \(\delta'_{Q_j}, j = 1, 2\), is mapped by \(T_{1}^{-1}\) into an arc crossing through \(Q_j\) (focalization). Every arc crossing through \(\delta''_{Q_j}, j = 1, 2\), is mapped by \(T_{2}^{-1}\) into an arc crossing through \(Q_j\). Moreover one has \(T_{1}^{-1}(\delta'_{Q_i}) \cup T_{2}^{-1}(\delta''_{Q_i}) = \pi_i, i = 1, 2\). Segments \(\delta'_{Q_j}\) and \(\delta''_{Q_j}\) join at the tangency points \(V_j\) between the critical curve \(LC\) and the prefocal set.

strip bounded by the two lines \(\pi_i\) tangent to \(LC_{1}\) in \(Q_i\) which does not include any point of \(LC_{-1}\) (see Fig. 4). Moreover, the two rank-1 preimages are separated by the singular line \(\delta_s\), i.e. the two inverses of a point \((x', y')\) of the strip between the two prefocal curves are one above and one below \(\delta_s\), in the strip between the two \(\pi_i\). As an example, the two rank-1 preimages of a segment \(\eta\) connecting two points of \(\delta'_{Q_i}\) are shown in Fig. 4. As \(T_{1}^{-1}\) focalizes the points of \(\delta'_{Q_i}\), we have that \(T_{1}^{-1}(\eta)\) must be an arc belonging to the strip between \(\pi_1\) and \(\pi_2\) and connecting the two focal points \(Q_1\) and \(Q_2\). While \(T_{2}^{-1}\) has no focalizing properties, thus \(T_{2}^{-1}(\eta)\) is an arc belonging to the strip between \(\pi_1\) and \(\pi_2\) connecting the two preimages of the extrema of the segment \(\eta\). It is clear that if we take a point \((x', y')\) close to a prefocal line then one of the two preimages is close to a focal point. Thus we can say that the preimages of that strip are related to the focalization properties, in the sense that if we take an arc \(\eta\) in the strip, or an arc \(\omega\) crossing the strip, then their preimages \(T_{1}^{-1}\) and \(T_{2}^{-1}\) may have peculiar properties, specific to maps with focal points, as evidenced in the qualitative pictures of Fig. 5.

The bifurcation situations due to a tangential contact between an arc \(\eta\) or \(\omega\) with a prefocal line \(\delta_Q\), have not been evidenced in Fig. 5. However the geometric behavior is quite obvious, except for the fact that the preimage with a focalizing inverse is such that the contact with the corresponding focal point is \textit{nonsmooth} (i.e. at the bifurcation such a preimage has a cusp point at the focal point).

It can be noticed that there are some particular configurations of arcs, crossing a prefocal line at two distinct points, whose two rank-1 preimages have no loops with knot in any focal point (and thus cannot be associated with the formation of “lobes”). This occurs when an arc crosses only one prefocal curve and the intersection points are one above and
Fig. 5. Some qualitative sketches showing peculiar properties (specific to noninvertible maps with focal points) of the preimages of arcs crossing one or two prefocal lines. These sketches depend on the intersection locations with respect to the tangency points \( V_j \) between the critical curve \( LC \) and the prefocal set.
Fig. 5. (Continued)
one below the tangency point \( V \), as for the arc \( \eta \) in Fig. 5(b).

It is also worth noticing, however, that in any other situation, as evidenced from Figs. 5(a), 5(c)–5(e), at least one loop in one focal point is necessarily formed, and also that the formations of particular arcs [Figs. 5(a) and 5(e)], called “crescents” in Part I, associated with arcs and lobes crossing two focal points, may occur via a mechanism different from the one shown in Part I. In Part I we have seen that the formation of particular arcs through two focal points, giving rise to the special structure of “crescents”, occurs when two distinct loops, or “lobes”, approach a critical curve \( LC \) and have a contact on that curve, so that the reunion of two “lobes” gives rise to one “crescent” (see Fig. 9 of Part I), and now we may think that such a structure results from the merging of two prefocal sets without merging of focal points. Moreover, a new mechanism of formation of crescents is described in this paper. In fact, the preimage of the arc \( \omega \) in Fig. 5(a) (via \( T_1^{-1} \)) and in Fig. 5(e) (via \( T_2^{-1} \)) shows how crescents may be formed involving only one of the inverses and no critical point of \( LC \).

In Fig. 5(a), we see that the preimage \( T_1^{-1}(\eta) \) is a lobe issuing from the focal point \( Q_2 \) and as \( \eta \) approaches the second prefocal curve the preimage \( T_1^{-1}(\eta) \) approaches the second focal point \( Q_1 \). Thus the preimage of an arc \( \omega \) crossing both the two prefocal curves has \( T_1^{-1}(\omega) \) which includes a crescent connecting the two focal points \( Q_1 \) and \( Q_2 \), and a lobe issuing from \( Q_1 \): Similarly occurs with \( T_2^{-1} \) in Fig. 5(e).

3.3.3. Bifurcation case

Taking into account that the focal points \( Q_i \) belong to the closure \( LC \) and that the role of the singular set \( \delta_s \) in the segment between the two focal points is particular (it separates the preimages of the points belonging to the strip between the two prefocal lines) we are led to consider that the segment of \( \delta_s \) between \( Q_1 \) and \( Q_2 \) has particular geometric properties as for a segment of \( LC \) (as stated in Sec. 2.3). This particular geometric role played by the set of nondefinition \( \delta_s \) is specially evidenced in the bifurcation cases (we shall see an example in Sec. 5.3). That is, with the merging of the two prefocal sets, when \( \delta_{Q_1} \to \delta_{Q_2} \) (due to a parameter variation) without merging of the focal points
4. Consequences on the Geometric Properties of the Phase Curves

When the presence of a vanishing denominator induces the existence of focal points, important effects on the geometrical and dynamical properties of the map $T$ can be observed. Indeed, a contact between a curve segment $\gamma$ and the singular set $\delta_s$ causes remarkable qualitative changes in the shape of the image $T(\gamma)$. Moreover, a contact of an arc $\omega$ with a prefocal curve $\delta_Q$, gives rise to important qualitative changes in the shape of the preimages $T_j^{-1}(\omega)$. When the arcs $\omega$ are portions of phase curves of the map $T$, such as invariant closed curves, stable or unstable sets of saddles, basin boundaries, one has that contacts between singularities of different nature generally induce important qualitative changes, which constitute new types of global bifurcations that change the structure of the attracting sets, or of their basins. In particular, this concerns the formation of lobes and crescents for basin boundaries.

As we have reminded in Sec. 3.3.2, the formation of crescents has already been shown in Part I for maps having focal points not belonging to $LC_{-1}$. In such a case the contact on $LC_{-1}$ of two lobes issuing from two distinct focal points is the only mechanism giving rise to a crescent. That is, the mechanism requires first the formation of two lobes via a contact with a prefocal curve, followed by a contact with a critical curve $LC$ causing on its turn the merging of the two lobes on the critical curve $LC_{-1}$, thus leading to the formation of a crescent.

In the case of noninvertible maps with focal points belonging to $LC_{-1}$ a crescent in a basin boundary, or the stable set of a saddle fixed (or periodic) point, may even be formed starting from a single lobe, issuing from one focal point (due to the crossing of some arc with one prefocal curve at two points which are both on the same side with respect to the tangency point, say $V_i$), which is then conveyed through a different focal point (due to the contact of the same arc with a different prefocal curve on the same side with respect to the new tangency point, say $V_j$), as it is qualitatively shown in Figs. 5(a) and 5(e). This mechanism may also be frequently repeated giving rise to sequences of crescents issuing from the same two focal points.

In the following section, by using only one family of $Z_0 - Z_2$ maps $T$ with several parameter constellations, we shall see some numerical explorations to illustrate some global bifurcations induced by

(i.e. $Q_1 \neq Q_2$ and $F(Q_1) = F(Q_2)$), the tangency points $V_i$ on the prefocal curves tend to infinity, i.e. the two prefocal lines merge into a unique one $\delta_Q$ ($\delta_Q = \delta_{Q_1} \equiv \delta_{Q_2}$) which becomes an asymptote for $LC$, as it occurs in the case described in Sec. 3.2. This also corresponds to a qualitative change in the foliation of the two-dimensional map, i.e. a qualitative change of $LC_{-1}$ and a change in the role played by the two inverses. At the bifurcation, the merging of the two prefocal lines also causes the disappearance of the strip between them and of the strip between the two tangent lines $\pi_i$ as well, so that each point $(F(Q), y)$ of $\delta_Q$ has now two distinct inverses, both focalizing: $T^{-1}_1(F(Q), y) = Q_1$ and $T^{-1}_2(F(Q), y) = Q_2$. As we shall see in the example in Sec. 5.3, the critical curve $LC_{-1}$ is an hyperbola which degenerates into the asymptotes at the bifurcation of the prefocal curves, reducing to a line intersecting the set of nondefinition $\delta_s$ at a point between the two focal points $Q_i$, $i = 1, 2$, and the second degenerate line is $\delta_s$ itself (see the qualitative picture in Fig. 6), which turns out to verify condition (16) in all its points. It is thus clear why the cases described in Sec. 3.2 (as well as several examples in Part I) can be viewed as bifurcation cases of those described in Sec. 3.3, which are the generic ones. An explicit example and the description of some situations will better clarify the mechanisms.
the presence of a vanishing denominator, and the related properties. In particular, we shall consider the boundary between the basin of divergent trajectories and the basin of the attractor at finite distance.

5. Examples of Situations
Generated by a $Z_0 - Z_2$ Map

5.1. Focal points, prefocal lines, equation of the two inverses

Consider the following map

$$ T : \begin{cases} 
  x' = y + \varepsilon x \\
  y' = \frac{\alpha x^2 + \gamma x}{y - \beta + \sigma x} 
\end{cases} \quad (18) $$

not defined in the points of the line $\delta_s$ of equation $y - \beta + \sigma x = 0$, on which two focal points exist given by

$$ Q_1 = \left(-\frac{\gamma}{\alpha}, \beta + \frac{\gamma \sigma}{\alpha}\right) \quad \text{and} \quad Q_2 = (0, \beta) $$

whose corresponding prefocal curves, of equation $x = F(Q_i), i = 1, 2$, are

$$ x = \beta + (\sigma - \varepsilon)\gamma/\alpha(\delta_{Q_1}) \quad \text{and} \quad x = \beta(\delta_{Q_2}) \quad (19) $$

Being $(N_xD_y - N_yD_x) = 2\alpha x + \gamma$ we have that $(\overline{N_xD_y} - \overline{N_yD_x})|_{Q_1} = -\gamma$ and $(\overline{N_xD_y} - \overline{N_yD_x})|_{Q_2} = \gamma$ so that for $\gamma \neq 0$ both the focal points are simple. The one-to-one correspondence (6) associated with each simple focal point $Q_i$ is given by $y_i(m) = 2\alpha x Q_i + \gamma/\sigma + m$, that is

$$ y(m) = -\frac{\gamma}{\sigma + m} \quad \text{for} \quad \delta_{Q_1} $$

$$ y(m) = \frac{\gamma}{\sigma + m} \quad \text{for} \quad \delta_{Q_2} $$

from which the inverse relations $m(y)$ easily follow.

The map $T$ in (18) is a noninvertible map of $(Z_0-Z_2)$ type, i.e. a point $(x', y')$ can have two or no rank-1 preimages whose coordinates are obtained by solving the algebraic system (18) with respect to the unknowns $x$ and $y$. The solution of this system is equivalent to

$$ \begin{cases} 
  \alpha x^2 + (\gamma - (\sigma - \varepsilon)y')x + y'((\beta - x') = 0 \\
  y = x' - \varepsilon x
\end{cases} \quad (20) $$

for $y \neq -\sigma x + \beta$, and gives the inverses of the map from its two real and distinct solutions if

$$ \Delta(x', y') = (\gamma - (\sigma - \varepsilon)y')^2 - 4\alpha y'((\beta - x') > 0 \quad (21) $$

while no solution exists if the reverse inequality holds. Accordingly, the two regions $Z_0$ and $Z_2$ are defined as $Z_0 = \{(x, y)|\Delta(x, y) < 0\}$ and $Z_2 = \{(x, y)|\Delta(x, y) > 0\}$. From each point $(x', y') \in Z_2$ the two distinct rank-1 preimages are computed by the following two inverses (obtained by solving the second degree equation in (20)):

$$ T_1^{-1}(x', y') : \begin{cases} 
  x = \frac{1}{2\alpha}[(\sigma - \varepsilon)y' - \gamma - \sqrt{\Delta(x', y')}]
  \\
  y = x' - \varepsilon x
\end{cases} \quad (22) $$

$$ T_2^{-1}(x', y') : \begin{cases} 
  x = \frac{1}{2\alpha}[(\sigma - \varepsilon)y' - \gamma + \sqrt{\Delta(x', y')}]
  \\
  y = x' - \varepsilon x
\end{cases} \quad (23) $$

where $\Delta(x', y')$ is given in (21).

5.2. Critical curve and preimages of the prefocal lines

The boundary which separates the region $Z_0$ from $Z_2$, defined by the equation $\Delta(x, y) = 0$, is the critical set $LC$ (locus of points having merging rank-1 preimages), which can be expressed as

$$ LC : \begin{cases} 
  x = \frac{4\alpha \beta y - (\gamma - (\sigma - \varepsilon)y)^2}{4\alpha y} \quad (24)
\end{cases} $$

so that $LC$ is formed by the two branches of a hyperbola of asymptotes

$$ y = 0 $$

and

$$ \begin{cases} 
  y = -\frac{4\alpha x + 4\alpha \beta + 2(\sigma - \varepsilon)\gamma}{(\sigma - \varepsilon)^2} \quad \text{for} \quad \varepsilon \neq \sigma \\
  x = \beta \quad \text{for} \quad \varepsilon = \sigma
\end{cases} $$

and symmetry centre $s = (\beta + ((\sigma - \varepsilon)\gamma/2\alpha), 0)$. The locus of merging rank-1 preimages $LC_{-1}$ has the parametric equation obtained by (22), or (23), with $\Delta = 0$, i.e.

$$ \begin{cases} 
  x = \frac{1}{2\alpha}[(\sigma - \varepsilon)y' - \gamma]
  \\
  y = x' - \varepsilon x
\end{cases} $$

So, for $\varepsilon \neq \sigma$, $LC_{-1}$ is given by the hyperbola of cartesian equation

$$ LC_{-1} : y = \frac{-\alpha(\sigma + \varepsilon)x^2 + (2\alpha \beta - \varepsilon \gamma)x + \beta \gamma}{\gamma + 2\alpha x} \quad (25) $$

The same equation can also be obtained from the condition \( \det DT(x, y) = 0 \), where

\[
DT(x, y) = \begin{bmatrix} \varepsilon & 1 \\ (2\alpha x + \gamma)D(x, y) - \sigma N(x, y) & -\frac{N(x, y)}{D(x, y)} \end{bmatrix} D(x, y)^2
\]

In fact, \( \det DT(x, y) = (\sigma - \varepsilon)N(x, y) - (2\alpha x + \gamma)D(x, y)/D(x, y)^2 \) so that

\[
J_0 = \{(x, y) \in \mathbb{R}^2 | (\sigma - \varepsilon)N(x, y) - (2\alpha x + \gamma)D(x, y) = 0 \} \cap \{\mathbb{R}^2 \setminus \delta_s \}
\]

and for \( \varepsilon \neq \sigma \) we have \( LC_{-1} = J_0 = J_C \). The slopes of the tangent lines \( \pi_i \) to \( LC_{-1} \) in the focal points, given by (14), are \( m_1 = m_2 = -\varepsilon \) and the points of tangency between \( LC \) and the prefocal lines \( \delta_{Q_i} \), via (15), are given by

\[
V_1 = \left( \beta + \frac{(\sigma - \varepsilon)\gamma}{\alpha}, -\frac{\gamma}{\sigma - \varepsilon} \right), \\
V_2 = \left( \beta, -\frac{\gamma}{\sigma - \varepsilon} \right)
\] (26)

The rank-1 preimage of the prefocal lines can be explicitly computed. Let us consider first a generic point of \( \delta_{Q_2} \), say \( (\beta, \gamma') \). Then

\[
T_{1,2}^{-1}(\beta, \gamma') : \begin{cases} 
  x = \frac{1}{2\alpha}((\sigma - \varepsilon)\gamma' - \gamma) \\
  y = \beta - \varepsilon x 
\end{cases}
\]

so that

if \((\sigma - \varepsilon)\gamma' - \gamma > 0\) then

\[
T_1^{-1}(\beta, \gamma') = Q_2 : \begin{cases} 
  x = 0 \\
  y = \beta 
\end{cases}
\]

if \((\sigma - \varepsilon)\gamma' - \gamma < 0\) then

\[
T_2^{-1}(\beta, \gamma') \in \pi_2 : \begin{cases} 
  x = \frac{(\sigma - \varepsilon)\gamma' - \gamma}{\alpha} \\
  y = \beta - \varepsilon x 
\end{cases}
\]

The condition \((\sigma - \varepsilon)\gamma' - \gamma > 0\) is satisfied by the points of an half line of the prefocal \( \delta_{Q_1} \) issuing from the tangency point \( V_1 \), which one depending on the parameters, and which part of the tangent line \( \pi_1 \) issuing from \( Q_1 \) is obtained depends on the parameters as well. Also in this case for \((\sigma - \varepsilon)\gamma' + \gamma < 0\) the complementary half lines are involved.

Analogously, considering a generic point of \( \delta_{Q_1} \), say \((\beta + ((\sigma - \varepsilon)\gamma/\alpha), \gamma')\), we have

\[
T_{1,2}^{-1}(\beta, \gamma') : \begin{cases} 
  x = \frac{1}{2\alpha}((\sigma - \varepsilon)\gamma' - \gamma) \\
  y = \beta + \frac{(\sigma - \varepsilon)\gamma}{\alpha} - \varepsilon x 
\end{cases}
\]

so that

if \((\sigma - \varepsilon)\gamma' + \gamma > 0\) then

\[
T_1^{-1}(\beta, \gamma') = Q_1 : \begin{cases} 
  x = -\frac{\gamma}{\alpha} \\
  y = \beta + \frac{\sigma\gamma}{\alpha} 
\end{cases}
\]

if \((\sigma - \varepsilon)\gamma' + \gamma < 0\) then

\[
T_2^{-1}(\beta, \gamma') \in \pi_1 : \begin{cases} 
  x = \frac{(\sigma - \varepsilon)\gamma'}{\alpha} \\
  y = \beta + \frac{(\sigma - \varepsilon)\gamma}{\alpha} - \varepsilon x 
\end{cases}
\]

The map \( T \) in (18) has two fixed points, the origin \( O = (0, 0) \) and \( P = (x_p, (1 - \varepsilon)x_p) \), where \( x_p = (\beta(1 - \varepsilon) + \gamma)/(1 - \varepsilon)^2 + \gamma(1 - \varepsilon - \alpha)) \). In Fig. 7(a), obtained with (18) and parameters \( \alpha = 0.5, \gamma = 0.5, \beta = 1.4, \sigma = 0.3, \varepsilon = -0.2 \), we have a situation similar to the one shown in
Fig. 4. The red region represents the basin of the stable fixed point \( O = (0, 0) \), and the complementary white region denotes the basin of infinity, i.e. the set of diverging trajectories. In Fig. 7(a), as in Fig. 4, \( T_1^{-1} \) (resp. \( T_2^{-1} \)) focalizes the points of the prefocal lines \( \delta_{Q_i} \) above (resp. below) the tangency points \( V_i \).

5.3. Bifurcations by merging of the prefocal lines, without merging of the focal points

In the particular case occurring when \( \varepsilon = \sigma \) the two prefocal lines \( \delta_{Q_i} \) merge. Indeed from \( x = F(Q_i) \) given in (19) we obtain \( \delta_Q = (\delta_{Q_1} \equiv \delta_{Q_2}) \), and its equation is \( x = \beta \). We obtain the situation of the noninvertible maps of Part I, for which several focal points are associated with a given prefocal curve. The parametric equation of \( LC_{-1} \), obtained by (22) with \( \Delta = 0 \), reduces to:

\[
\begin{align*}
  x &= -\frac{\gamma}{2\alpha}, \\
  y &= x' - \varepsilon x
\end{align*}
\]

thus in this (bifurcation) case \( LC_{-1} \) is the line of equation \( x = -\gamma/2\alpha \). The Jacobian determinant becomes \( \det DT(x, y) = -(2x + \gamma)/D(x, y) \), so that we have \( LC_{-1} = J_0 \) and the hyperbola (25) degenerates into the two asymptotes \( x = -\gamma/2\alpha \) and \( \delta_s \). Notice that \( \delta_s \) now belongs to the set denoted by \( J_C \) in Sec. 2.3. In fact, the Jacobian determinant changes sign both crossing \( x = -\gamma/2\alpha \) (i.e. \( LC_{-1} \)) and crossing the line \( \delta_s \), at which \( D(x, y) = 0 \), so that \( J_C = \{(x, y)|x = -\gamma/2\alpha \} \cup \delta_s = LC_{-1} \cup \delta_s \). That is, at this bifurcation value we have \( LC_{-1} = J_0 \subset J_C \), the resulting “double” prefocal curve \( \delta_Q \) is an asymptote of \( LC \) because the tangency points \( V_i \) tend to the Poincaré Equator (as can be seen from the coordinates of \( V_i \) given in (26) for \( \varepsilon = \sigma \)), and the lines \( \pi_1 \) and \( \pi_2 \) degenerate at the two focal points, which now are not located on \( LC_{-1} \). In Fig. 7(b), obtained with \( \varepsilon = \sigma = 0.1 \) and the other parameters as in Fig. 7(a), the two focal points are still distinct and simple, and we can see that the two prefocal lines merge into a unique one, which is an asymptote of \( LC \). Note that \( LC_{-1} \) is degenerate only for \( \varepsilon = \sigma \). For \( \sigma < \varepsilon \) as in the case of Fig. 7(c), the two prefocal lines, \( LC_{-1} \) and \( LC \) have the same structure as for \( \sigma > \varepsilon \) [Fig. 7(a)], but the two branches of the hyperbola \( LC_{-1} \) have a different geometric configuration, the strip between \( \pi_1 \) and \( \pi_2 \) is rotated and (as it is evident from the inverses of the map) the role of the focalizing inverses is exchanged: now \( T_2^{-1} \) focalizes the points of the prefocal lines \( \delta_{Q_i} \) above the tangency points \( V_i \). Considering a segment \( \eta \) between the two prefocal lines above the points \( V_i \), then \( T_2^{-1}(\eta) \) is an arc through the two focal points, while \( T_1^{-1}(\eta) \) is not focalized [compare the arc \( \eta \) and its preimages in Figs. 7(a) and 7(c)]. Clearly, the two rank-1 preimages of the strip, say \( S \), between the two prefocal lines are still in the strip between the tangency lines \( \pi_i \), separated by the segment of \( \delta_s \), one being above and one below that segment. In Fig. 7(d) we have labeled the strip between the two prefocal lines as \( S = A \cup B \cup C \) (where \( A, B, C \) denote the portions of the basins: white, red and white, respectively).

In this figure it can be seen that \( T_1^{-1}(S) \) is in the strip between \( \pi_i \) on the left of \( \delta_s \) and \( T_2^{-1}(S) \) on the right. Moreover, it can also be seen that the role of the portion of \( \delta_s \) in the strip, although it separates preimages, has not the unfolding action (characteristic of a segment of \( LC_{-1} \)) because the preimages \( T_i^{-1}(A), T_i^{-1}(B), T_i^{-1}(C) \) go from left to right for both \( i = 1, 2 \) [see Fig. 7(d)]. This is due to the particular foliation structure existing in the strip between the two prefocal lines.

5.4. Foliations generated by the map between two prefocal curves

The particular foliation structure generated by the map (18) of \((Z_0, Z_2)\) type, in the cases shown in Fig. 7, is represented by the qualitative pictures in Figs. 8 and 9. The strip between the two lines \( \pi_i \) is separated in two parts, say \( P_1 \) and \( P_2 \), by the segment of \( \delta_s \) (Fig. 8(b)). We have \( T(P_1) = S \) and \( T(P_2) = S \) as well, \( S \) being the whole strip between the two prefocal lines. The arrows in Fig. 8 show the mechanism of the foliation of the forward image by \( T \), for both cases \( \sigma \geq \varepsilon \) [Fig. 8(a)] and \( \sigma < \varepsilon \) [Fig. 8(b)]. The points close to \( \delta_s \) on both sides, left and right parts are mapped by \( T \) into points close to the Poincaré Equator (P.E. henceforth), on opposite sides of \( S \). More precisely for \( \sigma \geq \varepsilon \), points close to \( \delta_s \) on the left side are mapped into points close to the P.E. on the upper part of \( S \), and points close to \( \delta_s \) on the right side are mapped into points close to the P.E. on the lower part of \( S \). For \( \sigma < \varepsilon \), points close to \( \delta_s \) on the left side are mapped into points close to the P.E. on the lower part of \( S \), and points close to \( \delta_s \) on the right side are mapped into points close to the P.E. on the upper part of \( S \).
Fig. 7. Basins of attraction obtained with the map (18). The red region represents the basin of the stable fixed point $O = (0, 0)$, and the complementary white region denotes the basin of infinity, i.e., the set of diverging trajectories. (a) Basins obtained with parameters $\alpha = 0.5, \gamma = 0.5, \beta = 1.4, \sigma = 0.3, \varepsilon = -0.2$. The situation is that of the qualitative Fig. 4. (b) The same parameters as in (a) and $\varepsilon = \sigma = 0.1$, the two prefocal lines $\delta_{Q_1}$ and $\delta_{Q_2}$ merge (see the situation described in the qualitative Fig. 6). (c) The same parameters as in (a) and $\varepsilon = -0.2, \sigma = 0.3$. The two branches of $LC_{-1}$ change their location with respect to the four sectors defined by their asymptotes. (d) A larger view of basins of (c).
In Fig. 9, similar pictures show the mechanism of the preimages of the unbounded strip $S$ ($T^{-1}_1(S)$ in white and $T^{-1}_2(S)$ in light blue). The focalizing inverse in the upper and lower parts of $S$, close to the P.E., is the one giving preimages close to $\delta_s$. Thus in Fig. 9(a), for $\sigma > \varepsilon$, the upper part of the prefocal lines $\delta_{Q_i}$ is focalized by $T^{-1}_1$ and the lower part by $T^{-1}_2$; while the vice versa occurs for $\sigma < \varepsilon$, i.e. $T^{-1}_2$ focalizes the upper part and $T^{-1}_1$ the lower one.

### 5.5. First example of formation of crescents

Consider the map $T$ given in (18) with the following parameter values: $\alpha = 0.5$, $\gamma = 0.5$, $\beta = -1.7$, $\sigma = -0.4$, $\varepsilon = 0$; in Fig. 10(a) we show the basin of attraction $B$ of the stable fixed point $O$, which is again represented by the red area, and separated by the set of divergent trajectories (white area) by
a frontier $\mathcal{F} = \partial \mathcal{B}$. At these parameter values the tangent lines to $LC_{-1}$ in the focal points have slope $\varepsilon = 0$ so that the lines $\pi_1$ have equation $y = y(Q_1)$ defining a horizontal strip (that is, $y = \beta$ through $Q_1$ and $y = \beta + (\gamma \sigma / \alpha)$ through $Q_2$). It is evident that a portion of the frontier is very close to the critical curve $LC$ in the lower part of Fig. 10(a) (see the arrow), and is very close also at the prefocus lines (below the two tangency points $V_i$, $i = 1, 2$). As $\alpha$ decreases, the frontier crosses $LC$ [see Fig. 10(b)] causing the appearance of a portion of basin $\mathcal{B}$ in the region $Z_2$, that is, creating a headland $d_0$ ($d_0$ is bounded by an arc of $LC$ and an arc of $\mathcal{F}$) made up of points having two distinct preimages, which give rise to the appearance of an island $d_{-1}$, a nonconnected part of the basin $\mathcal{B}$, located on the two sides of an arc of $LC_{-1}$ (this situation is due to the fact that $\mathcal{B} \cap Z_2$ is nonconnected, as shown in [Mira et al., 1994], see also [Mira et al., 1996a; Abraham et al., 1997]). That is, $d_{-1} = T^{-1}(d_0)$ is given by two components, $d_{-1} = d_{-1,1} \cup d_{-1,2} = T_{-1}^{-1}(d_0) \cup T_{-1}^{-1}(d_0)$, where $T_{-1}^{-1}(d_0)$ is on the left of $LC_{-1}$ in the figure, while $T_{-1}^{-1}(d_0)$ is on the right of the same segment of $LC_{-1}$. In Fig. 10(b), the island $d_{-1}$ has no contact with the focal point $Q_2$, neither with the tangent line $\pi_2$ but we can see that $d_0$ is not far from the

![Diagram](image)

Fig. 10. Lobes and crescent of a basin generated by the map $T$ given in (18) with $\varepsilon = 0$, $\gamma = 0.5$, $\beta = -1.7$, $\sigma = -0.4$, $\varepsilon = 0$. The basin of attraction of the stable fixed point $O$ is red colored, the set of divergent trajectories is the white area, whereas the yellow area in (d)-(f) represents the basin of a stable cycle of period 4. (a) Basins obtained with the parameter $\alpha = 0.5$. (b) Parameter $\alpha = 0.48$. As described in [Mira et al., 1994], $d_0$ (bounded by a $LC$ segment and an arc of the basin boundary) is a headland, whose preimage $d_{-1} = T^{-1}(d_0)$ is made up of two components $d_{-1} = d_{-1,1} \cup d_{-1,2} = T_{-1}^{-1}(d_0) \cup T_{-1}^{-1}(d_0)$, forming an island (nonconnected part of the red basin). (c) Parameter $\alpha = 0.43$. Just before attaining this value with $\alpha$ decreasing, the contact of the boundary $\partial d_0$ of $d_0$ with $\delta_{Q_2}$, causes at the same time the contact between $d_{-1,1}$ and the focal point $Q_2$ on one side (not smooth, i.e. with a cusp point in $Q_2$, as described in Part I) and a smooth contact between $d_{-1,2}$ and the line $\pi_2$ on the other side. Then for $\alpha = 0.43$ two lobes area is created. The crossing of $\delta_{Q_2}$ at two distinct points below $V_2$ causes the creation of a lobe issuing from the focal point $Q_2$, $T_{-1}^{-1}(d_0)$ now includes the lobe $L_2$. (d) Parameter $\alpha = 0.39$. This figure is obtained after the contact between $\partial d_0$ and $\delta_{Q_1}$, which leads to a nonsmooth contact of $L_2$ has the focal point $Q_2$, thus creating a crescent $c_1$ between $Q_1$ and $Q_2$, while on the other side $d_{-1,2}$ becomes tangent to $\pi_1$. (d) Shows that a $d_0$ part now crosses the prefocus $\delta_{Q_1}$ at two distinct points below $V_1$. This part has a rank-1 preimage (by $T_{-1}^{-1}$) which is a lobe $L_1$ issuing from $Q_1$. The rank-1 preimage (by $T_{-1}^{-1}$) is the portion of $d_{-1,2}$ below the tangent line $\pi_1$. (e) $\alpha = 0.35$; (f) $\alpha = 0.3$. (e) and (f) Show the birth of a new headland $d_0'$, a new island $d_{-1} = d_{-1,1} \cup d_{-1,2}$, and after crossing through $\delta_{Q_2}$ a new lobe $L_2'$. 
prefocal line $\delta_{Q_2}$, and as the parameter $\alpha$ is further decreased, a contact will occur between $\partial d_0$ ($\subset \mathcal{F}$) and $\delta_{Q_2}$ causing at the same time the contact between $d_{-1,1}$ and the focal point $Q_2$ on one side (not smooth, i.e. with a cusp point in $Q_2$, as described in Part I) and a smooth contact between $d_{-1,2}$ and the line $\pi_2$ on the other side. The crossing of $\delta_{Q_2}$ at two distinct points below $V_2$ causes the creation of
a lobe issuing from the focal point \(Q_2\), \(T_i^{-1}(d_0)\) now includes the lobe \(l_2\), as shown in Fig. 10(c). We can also see that \(\partial d_0\) is already close to the other prefocal line \(\delta_{Q_1}\), (and also here involving points below the tangency point \(V_1\)). Clearly, as \(d_0\) approaches \(\delta_{Q_1}\) the portion \(l_2\) of its preimage is approaching the focal point \(Q_2\) on one side (by \(T_1^{-1}(d_0)\)) and on the other side (by \(T_2^{-1}(d_0)\)) a part of \(d_{-1.2}\) is approaching the tangent line \(\pi_1\). At the contact between \(\partial d_0\) and \(\delta_{Q_1}\), the arc \(l_2\) has a nonsmooth contact with the focal point \(Q_1\), thus creating a crescent between \(Q_1\) and \(Q_2\), while on the other side \(d_{-1.2}\) becomes tangent to \(\pi_1\). In Fig. 10(d), we can see that a portion of \(d_0\) now crosses the prefocal \(\delta_{Q_1}\) at two distinct points below \(V_1\), and this portion has a rank-1 preimage (by \(T_1^{-1}\)) which is a lobe issuing from \(Q_1\) [see \(l_1\) in Fig. 10(d)]. The rank-1 preimage (by \(T_2^{-1}\)) is the portion of \(d_{-1.2}\) below the tangent line \(\pi_1\). As \(\alpha\) is further decreased, the lobe \(l_1\) issuing from \(Q_1\) again approaches the critical curve \(LC\) and crosses it creating a new headland \(d'_0\) and a new island \(d'_{-1}\), and after the crossing of \(\delta_{Q_2}\) a new lobe \(l'_{2}\), belonging to \(T_1^{-1}(d'_0)\) appears, issuing from \(Q_2\). In Fig. 10(e) we can see that \(d'_0\) is quite close to the other prefocal line \(\delta_{Q_1}\), so that the lobe \(l'_2\) is approaching the other focal point \(Q_1\) preparing a new crescent and a new lobe issuing from \(Q_1\), and so on, the process is repeated in the same way. In Fig. 10(f) we can also see that the second lobe issuing from \(Q_1\) crosses the critical curve \(LC\) and the prefocal lines, thus further lobes and crescents are created. We also note that in Figs. 10(d)–10(f) the frontier \(F\) separates the basin \(\mathcal{B}\) of the origin \(O\) from the basin (light blue points) of another attractor at finite distance, a stable cycle of period 4 which, along the path followed in the parameter space described in Fig. 10, appeared via a saddle-node bifurcation at \(\alpha \approx 4\).

The structure of the basins shown up to now is still quite simple, mainly because the focal points belong to the region \(Z_0\) so that they have no preimages. When the focal points belong to \(Z_2\) we may expect that the contact bifurcations associated with the prefocal curves and the focal points may give rise to more complex structures, as we shall see in the following example.

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**Fig. 11.** Another example of lobes and crescents generation by the map \(T\) given in (18) with \(\varepsilon = -0.2\), \(\gamma = 0.5\), \(\beta = 1\), \(\sigma = 0.3\). The basin of attraction of the stable fixed point \(O\) is red colored, the set of divergent trajectories is the white area. (a) Parameter \(\alpha = -1.5\). (b) Parameter \(\alpha = -1.25\); a lobe appears from \(Q_1\). (c) \(\alpha = -1.1\); (d) \(\alpha = -0.9\); (e) \(\alpha = -0.6\); (f) \(\alpha = -0.4\). (c)–(f) show the birth of lobes and crescents, as in Fig. 10.
5.6. Second example of formation of crescents

Consider the map $T$ given in (18) with the following parameter values: $\alpha = -1.5$, $\gamma = 0.5$, $\beta = 1$, $\sigma = 0.3$, $\varepsilon = -0.2$; in Fig. 11(a), the basin of attraction $B$ of the stable fixed point $O$ is again red colored, and separated by the set of divergent trajectories (white area) by a frontier $\mathcal{F} = \partial B$ which
includes the stable set of the unstable fixed point $P$ (located on $\mathcal{F}$). The two focal points belong to the region $Z_2$. At these parameter values we can see that $\mathcal{F}$ is approaching the prefocal line $\delta_{Q_1}$, and consequently also the focal point $Q_1$ (and its preimages of any rank as well). As the parameter $\alpha$ is increased, a contact occurs causing the crossing of $\delta_{Q_1}$ at two points located between the two tangency points $V_1$ and $V_2$. In Fig. 11(b), obtained after the crossing, we can see that a portion of the basin $\mathcal{B}$, denoted by $b_0$, is between the two prefocal lines (we are in a situation similar to that of the arc $\eta$ described in the qualitative picture in Fig. 5(c), so that one of its preimages issues from the focal point $Q_1$, given by $b_{-1,1} = T_1^{-1}(b_0)$, while the other preimage, $b_{-1,2} = T_2^{-1}(b_0)$, is a portion of the same basin $\mathcal{B}$ in the strip above the tangent line $\pi_1$ [see the arrow in Fig. 11(b)]. If Fig. 11(c), at a higher value of $\alpha$, we see that the portion of basin $b_0$ is approaching also the other prefocal line $\delta_{Q_2}$ below the point $V_2$ (where the inverse $T_2^{-1}$ focalizes), and inside the strip between the lines $\pi_i$, $i = 1, 2$, the preimage $T_2^{-1}(b_0)$ approaches the focal point $Q_2$. It is clear that a crossing of $\delta_{Q_2}$ will cause the appearance of a lobe issuing the focal point $Q_2$, given by the inverse $T_2^{-1}$, as shown in Fig. 11(d) [we are in a situation similar to that of the arc $\omega$ described in the qualitative picture in Fig. 5(c)].

As $\alpha$ is further increased, lobes and crescents are formed, and the structure of the basin is more complex, both when two focal points are in the region $Z_2$ [as shown in Fig. 11(e)] and when only one focal point belongs to $Z_2$ [as shown in Fig. 11(f)].

6. Conclusions

The first part [Bischi et al., 1999, Part I] of the study of two-dimensional maps, defined by at least one function with a vanishing denominator, concerned invertible maps in generic case and noninvertible maps in a nongeneric situation. It was shown that the presence of a vanishing denominator requires the introduction of new concepts like the set of nondefinition, the focal points and the prefocal curves. Their definitions imply that at least one inverse, called the “focalizing inverse”, must exist along a prefocal curve. These concepts allowed to give a geometric characterization to some new bifurcations which change the structure of the basin boundaries, and to describe a new mechanism for the occurrence of homoclinic bifurcations.

With respect to that paper, the present one is essentially devoted to some generic global dynamical properties of noninvertible two-dimensional maps. More particularly, considering the simplest case for which only two inverses exist (case of $(Z_0 - Z_2)$ maps), it is shown that the focalizing inverse is generally only one, but not always the same in the whole prefocal set. Then when two distinct focal points exist, their related prefocal sets may be either different sets (generic case considered in this Part II) or the same set (as considered in Part I). This qualitative difference is associated with a change in structure in the Riemann foliation of the plane. If two focalizing inverses at each point of a prefocal set exist, then this situation may be the consequence of the merging of two prefocal sets. In the generic case, when two distinct focal points exist, the fact that their related prefocal sets are different induces a more complex behavior of the preimage of a curve intersecting the prefocal curves at two points. In other words, the mechanism of formation of lobes and crescents issuing from the focal points, frequently met in basin boundaries, presents a higher level of complexity.

Another difference, between the properties of the noninvertible maps considered in Part I and those of the present paper, lies in the situation of the critical curve $LC$, and its rank-1 preimage $LC_{-1}$, with respect to the prefocal curves, and the focal points. In the cases considered in Part I, no focal points belong to the closure $\overline{LC_{-1}}$ of $LC_{-1}$, and the prefocal curve (common to two focal points) is an asymptote of $LC$. In this paper we considered the generic case of focal points belonging to $\overline{LC_{-1}}$, which implies that the critical curve $LC$ is tangent to the prefocal curves.

All these properties, characterizing the generic situation of two-dimensional noninvertible maps, lead to new bifurcations which change the structure of the basin boundaries. They also result in consequences for maps defined in the whole plane (for example polynomial maps) due to the presence of a vanishing denominator in at least one inverse map. Part I has shown that for noninvertible maps the locus of points where the denominator of some inverse vanishes may separate regions of the phase plane whose points have a different number of preimages. Such points are the two-dimensional analogue of a horizontal asymptote of a one-dimensional map (which corresponds to a vertical asymptote of at least one inverse) which separates zones with different numbers of preimages. Another property,
related to such maps defined in the whole plane, will be a more complex mechanism of formation of knots of a chaotic attractor, with respect to the one described in Part I for an invertible map (see Sec. 3.1). From this point of view, the study of examples of noninvertible maps defined in the whole plane, with a vanishing denominator in at least one inverse map in the generic case considered in this paper, presents a real interest.

Important bifurcations, related to two-dimensional maps, have not been considered in the present paper, and will be the object of further studies. It is the case of bifurcations: due to the merging of focal points, the merging of prefocal sets different from the one mentioned in this paper, the merging of focal points and fixed points, the contact between prefocal curves and critical curves. Extensions to the case of $m$-dimensional noninvertible maps, $m > 2$, also offer a wide field of new results. The fundamental concepts of a set of nondefinition, focal points, and prefocal curves, introduced for $m = 2$, can now be generalized. So the dimension $p$ of a focal set may be such that $m - 1 > p \geq 0$, and the dimension $q$ of a prefocal set such that $m > q \geq 1$, this depending on the numerator and the denominator forms. It results in a larger richness of the bifurcation situations.

Acknowledgments

This work has been performed under the auspices of CNR, Italy, and under the activity of the national research project “Modelli dinamici in economia e finanza: evoluzione, incertezza e previsioni”, MURST, Italy.

References


