Heterogeneous adaptive expectations and cobweb phenomena

D. Colucci and V. Valori

Dipartimento di Matematica per le Decisioni
Università degli Studi di Firenze

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Questions

- How does the evolution in the number / types of agents affect the long run dynamics of a given economy?
- How does expectations’ heterogeneity influence local stability?
- What can we expect when markets integrate?
- Can we make predictions on stability when only the probability distribution of types is known?
- Can we say anything about transitional dynamics / speed of convergence based on the “amount” of expectations’ heterogeneity?
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Two sources of (potential) instability are identified:

- a structural source, linked to the market’s fundamentals
- a behavioural source, embedded in the average profile of expectations.

We find a simple relation connecting these factors to stability/instability.
Can predict outcome of market integration, under (stronger than elsewhere in the paper) qualifications.
Study random selection of firms from a continuous distribution and document a form of polarisation of convergence probabilities when number of market’s participants is increased.
Give conditions that ensure monotone and fastest convergence towards steady state.
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Preview of results

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(Closely) Related Literature

- M. Nerlove - *QJE 1958*: introduced adaptive exp. into Cobweb model
- J.A. Carlson - *RES 1968*
  - E. Barucci - *J. Ev. Econ. 1999*: studies the $n = 2$ case
  - G. Negroni - *JEDC 2003*: considers the $n = 2$ case allowing for heterogeneity in fundamentals
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(Mildly) Related Literature

- ARED stream of literature - Brock-Hommes
  *ECONOMETRICA 1997*
- Lasselle et al. - *MACRO. DYN. 2005*
- T. Puu - *JEBO 2008*
In a nutshell

- A standard Cobweb model with $n$ firms
- Firms supply a commodity with a one-period production lag
- Output decisions are based on expectations about future prices
- At each period, given aggregate supply, the price is determined by the demand
The model: details

- Supply and demand are monotonic
- The optimal supply is proportional to firm’s size, $\psi_i > 0$ hence $S_i (p_i^e) = \psi_i s (p_i^e)$
- All form adaptive expectations, gain parameter is firm-specific
  $$p_{t+1,i}^e = p_{t,i}^e + \alpha_i (p_t - p_{t,i}^e) \quad i = 1, \ldots, n$$
- Demand, $D(p)$ and aggregate supply are smooth and intersecting at a point $p^*$
Price equation

- Let $\Psi = \sum_i \psi_i$ the industry scale factor, $S(\cdot) = \Psi s(\cdot)$ and $\phi_i = \frac{\psi_i}{\Psi}$ the firm’s relative weight.
- Market clearing requires that $D(p_t) = \sum_{i=1}^n \phi_i S(p^e_{t,i})$ hence

$$p_t = D^{-1}\left(\sum_{i=1}^n \phi_i S(p^e_{t,i})\right) = F(p^e_{t,1}, \ldots, p^e_{t,n})$$

with the property $p^* = F(p^*, \ldots, p^*)$.
Plugging price equation into expectations gives the following system of $n$ difference equations:

$$p_{t+1,1}^e = p_{t,1}^e + \alpha_1 (F(p_{t,1}^e, \ldots, p_{t,n}^e) - p_{t,1}^e)$$

$$\ldots = \ldots$$

$$p_{t+1,n}^e = p_{t,n}^e + \alpha_n (F(p_{t,1}^e, \ldots, p_{t,n}^e) - p_{t,n}^e)$$

Point $p^*$ is unique steady state.
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\begin{align*}
    p_{t+1,1}^e &= p_{t,1}^e + \alpha_1 \left( F \left( p_{t,1}^e, \ldots, p_{t,n}^e \right) - p_{t,1}^e \right) \\
    \cdots &= \cdots \\
    p_{t+1,n}^e &= p_{t,n}^e + \alpha_n \left( F \left( p_{t,1}^e, \ldots, p_{t,n}^e \right) - p_{t,n}^e \right)
\end{align*}
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Point \( p^* \) is unique steady state.
Special case: one representative firm

- With a single firm price equation reduces to

\[ p_t = D^{-1} (\psi_s (p_t^e)) = D^{-1} (S(p_t^e)) \]

- Therefore the system evolves according to

\[ p_{t+1}^e = p_t^e + \alpha (D^{-1} (S(p_t^e)) - p_t^e) \]

- and stability requires \(-1 < 1 - \alpha + \alpha \frac{S'(p^*)}{D'(p^*)} < 1\)

- Defining \(\delta = -\frac{S'(p^*)}{D'(p^*)}\) and \(\beta = \frac{\alpha}{2-\alpha}\), can write this as

\[ \delta \beta < 1 \]

- Label \(\delta\) as structural degree of instability and \(\beta\) as behavioural degree of instability
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  \[ p_t = D^{-1}(\Psi s(p^e_t)) = D^{-1}(S(p^e_t)) \]

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  \[ p^e_{t+1} = p^e_t + \alpha (D^{-1}(S(p^e_t)) - p^e_t) \]

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The Jacobian matrix of the system evaluated at steady state is

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\begin{bmatrix}
1 - \alpha_1 (\phi_1 \delta + 1) & -\alpha_1 \phi_1 \delta & \cdots & -\alpha_1 \phi_1 \delta \\
-\alpha_2 \phi_2 \delta & 1 - \alpha_2 (\phi_2 \delta + 1) & \cdots & -\alpha_2 \phi_2 \delta \\
\cdots & \cdots & \cdots & \cdots \\
-\alpha_n \phi_n \delta & -\alpha_n \phi_n \delta & \cdots & 1 - \alpha_n (\phi_n \delta + 1)
\end{bmatrix}
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Define \( \bar{\beta}_n = \sum_{i=1}^{n} \phi_i \beta_i \) the *market degree of behavioural instability* for the \( n \) heterogeneous firms case

**Proposition 1**: The steady state of the system is locally stable and hyperbolic if and only if \( \delta \bar{\beta}_n < 1 \).
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**Proposition 1**: The steady state of the system is locally stable and hyperbolic if and only if \( \delta \bar{\beta}_n < 1 \)
How does this compare with the $n = 1$ case?

**Proposition 2:** Consider an $n$-firms market with gains $\alpha_1, \ldots, \alpha_n$ and weights $\phi_1, \ldots, \phi_n$ and an average-single-firm market with gain $\alpha = \sum_{i=1}^{n} \phi_i \alpha_i$. Conditions for stability in the heterogeneous market are sufficient but not necessary for the average homogeneous market.

Heterogeneity matters, from the dynamic stability/instability viewpoint: can’t be safely sterilized by using an average representation instead of the whole heterogeneous picture.
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Heterogeneity matters, from the dynamic stability/instability viewpoint: can’t be safely sterilized by using an average representation instead of the whole heterogeneous picture.
What is the role of $n$ ceteris paribus?

Proposition 3: Consider economy A and economy B where B has some extra firms in the supply side, given the same industry scale factor. Economy B’s extra firms have a weight $1 - \rho$ and a given $\bar{\beta}_{\text{extra}}$. Then if economy A is stable so is economy B if $\delta \bar{\beta}_{\text{extra}} < 1$. If instead $\delta \bar{\beta}_{\text{extra}} > 1$ then economy B is stable if and only if $\rho > \frac{\delta \bar{\beta}_{\text{extra}} - 1}{\delta \bar{\beta}_{\text{extra}} - \delta \bar{\beta}_A}$.

Stability (or instability) persists when a larger span of jointly stable (or unstable) firms is allowed for.
Comparative statics on $n$

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Market Integration

- What happens when two previously separated markets are integrated?
- (In progress) Basically things are straightforward if steady state does not move.
- Results in more general case require stronger conditions on supply and demand (e.g. linearity or concavity/convexity).
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How about the path of convergence to the steady state?

Propositions 4-5: The system shows monotonic local convergence to the steady state if and only if
\[ \sum_{i=1}^{n} \phi_i \frac{\alpha_i}{1-\alpha_i} < \frac{1}{\delta}. \]
If \( \phi_1 = \cdots = \phi_n = 1/n \) then the maximum speed of convergence to the steady state is
\[ \ln \left( \frac{\delta+2}{\delta} \right) \]
and it is attained if and only if \( \alpha_1 = \cdots = \alpha_n = \frac{2}{\delta+2} \).
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Stability when firms are sampled

- Know very little about actual expectations
  - Assume firms’ behavioural parameter results from a random choice, given a distribution, e.g. uniform on unit interval
  - Define a *stable sample* of behavioural parameters one for which the corresponding system has a locally stable steady state
  - Then probability of a stable sample will look like this:
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Probabilities as $\delta$ varies
Conclusions

- We study the effect of varying the level of market’s heterogeneity in a Cobweb model with adaptive expectations.
- We fully characterize the local stability properties for the generic $n$-firms case.
- We discuss the case of market integration giving conditions which grant stability in the resulting, integrated, market.
- We study the possibility of making predictions about the properties of market dynamics when firm’s types are unknown. We show that when types are uniformly distributed the probability of having a stable system polarizes towards 0 or 1 depending on the structural characteristics of the market.