Alternative stabilization policies in a Keynes-Kaldor-Tobin model of
business cycles

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Abstract

In this paper, we formulate a model of business cycles, based upon the ideas of Keynes (1936), Kaldor (1940) and Tobin (1975), (the Keynes-Kaldor-Tobin model) and discuss two alternative monetary policies, the quantity policy and the interest rate policy, to investigate which of them has more powerful in stabilizing the economy. Consequently, we find that the quantity policy can cause a periodic orbit (persistent fluctuations) by way of Hopf bifurcations while the interest rate policy does not have such a property and in this sense, we may conclude that the interest rate policy is superior to the quantity policy in stabilizing the economy.

Keywords: Business cycles, Keynesian economics, Monetary policies, Nonlinear analysis

JEL Classifications: E12, E21, E22, E31, E32, E41, E44

1 Introduction

Since the publication of Keynes' *General Theory* (1936), several economists have attempted to incorporate the basic ideas of Keynes (1936) (e.g., the principle of effective demand) in the theory of business cycles. Kaldor (1940) is one of the earliest contributions in the Keynesian theory of business cycles, and he argued that nonlinearity is necessary for reproducing persistent fluctuations in economic models.\(^1\) In particular, he emphasized the nonlinearity of the investment function and formulated the S-shaped investment function to describe persistent business cycles observed in reality.\(^2\) Although Kaldor (1940) was successful in modeling persistent business cycles in the Keynesian manner, he paid little attention to variations in prices or inflations / deflations. On the other hand, Tobin (1975) investigated the effects of price flexibility and expectations on inflations in a Keynesian model and demonstrated that both flexibility of prices and inflation-deflation expectations are destabilizing factors in the Keynesian world. His analysis may provide useful insights on the aggravating effects of price flexibility and inflation expectations in the Keynesian (non-neoclassical) world, but

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\(^1\)Hicks (1950) and Goodwin (1951) also insisted on the importance of nonlinearity in business cycle models.

\(^2\)Rigorous mathematical examinations and extensions of his model were given by, for example, Chang and Smyth (1971), Varian (1979) and Asada (1987) and Murakami (2014a).
his Keynesian model was short-run and did not accommodate variations in capital stock through (dis)investment activities. Thus, both Kaldor (1940) and Tobin (1975) analyzed economic fluctuations, following Keynes’ (1936) ideas, and provided useful insights, but their focuses were slightly different. In this sense, it is worth while to integrate their perspectives for more complete analysis of economic fluctuations in the Keynesian world.

The purpose of this paper is to formalize a Keynesian model of business cycles (Keynes-Kaldor-Tobin model) by unifying the basic ideas of Kaldor (1940) and Tobin (1975) and investigate the stabilizing effects of monetary policies. To discuss several kinds of monetary policies, we shall examine separately the quantity policy (adjustments of the money supply through changes in the monetary base) and the interest rate policy (like Taylor’s (1993) rule).

This paper is organized as follows. In section 2, we will explain the basic components of the Keynes-Kaldor-Tobin model and derive the conditions for generation of persistent business cycles. In section 3, we shall introduce the quantity policy and the interest rate policy in the Keynes-Kaldor-Tobin model and derive the stability conditions and the conditions for occurrence of persistent business cycles. In section 4, we shall conclude this paper.

2 The Keynes-Kaldor-Tobin model

In this section, we will set up a Keynes-Kaldor-Tobin model following the ideas of Keynes (1936), Kaldor (1940) and Tobin (1975). The model is composed of the following equations:

\[
\begin{align*}
\frac{M}{p} &= L(r, y, k), \\
\dot{y} &= \alpha[I(r, y, k, \pi^e) - S(r, y, k, \pi^e)], \\
\dot{k} &= I(r, y, k, \pi^e), \\
\frac{\dot{p}}{p} &= H(y, k, p) + \pi^e, \\
\dot{\pi}^e &= \beta\left(\frac{\dot{p}}{p} - \pi^e\right),
\end{align*}
\]

where \(M\) is a positive constant; \(\alpha\) and \(\beta\) are positive parameters.\(^3\)

In this model, \(y, r, k, p, \pi^e\) and \(M\) represent aggregate income, the (nominal) rate of interest, capital stock, the price level, the expected rate of inflation and the quantity of money, respectively; \(I, S, L\) and \(H\) are, respectively, the \textit{ex ante} (net) investment function, the (net) saving function, the liquidity preference function and the Phillips-inflation function; \(\alpha\) and \(\beta\) are the speed of income adjustments and the speed of expectation formations, respectively.

We shall explain the meanings of Eqs. \((2.1)-(2.5)\). Firstly, Eq. \((2.1)\) expresses the usual Keynesian money market equilibrium condition. Unlike in usual formulations, capital stock \(k\) is included in the liquidity preference

\(^3\)This model is basically identical with Murakami’s (2014a, p.76) System (F). These investment and saving functions \(I\) and \(S\) can be micro-economically founded in Murakami (2014b).
(money demand) function \( L \). This formalization reflects the fact that capital stock \( k \) is one component of total wealth, an increment of which is usually considered to cause a rise in liquidity preference.

Secondly, Eq. (2.2) is a standard representation of Keynesian income-quantity adjustments, and this means that aggregate output \( y \) is increased (resp. decreased) when aggregate excess demand \( I - S \) is positive (resp. negative).

Thirdly, Eq. (2.3) represents capital formation processes. It is assumed in (2.3) that actual capital formation is equal to \( ex \ ante \) net investment. To consider more general situations, we may replace (2.3) by:

\[
\dot{k} = \theta I(r, y, k, \pi^e) + (1 - \theta)S(r, y, k, \pi^e),
\]

where \( \theta \in [0, 1] \), which states that capital accumulation is equal to a weighted average of \( ex \ ante \) investment and savings.\(^4\) However, the formalization given in (2.3) is adopted for the sake of simplicity.

Fourthly, Eq. (2.4) is one kind of Phillips-Friedman-Phelps curve. Since the rate of (un)employment is influenced also by capital stock \( k \), the Phillips function \( H \) is function also of \( k \). The inclusion of \( p \) in \( H \) reflects the hypothesis that when \( p \) is extremely high (resp. low), the rate of inflation tends to be negative (resp. positive), i.e., that the price level \( p \) has a "self-adjusting" function.

Finally, Eq. (2.5) expresses adaptive expectation formations on inflation / deflation. The concept of adaptive expectation formations is usually viewed as non-rational, but as Muth (1961) explained, adaptive expectation formations can be "rational" under some reasonable assumptions. Moreover, as Lorenz (1993, pp.22-23) argued, nonlinear differential equations are generally unsolvable, so that even if economic agents learn the structures of the macro economy, which are usually nonlinear, they cannot form "rational" expectations in the new-classical sense. In this way, the formalization of expectation formations, given in (2.5), can be justified.

For analytical convenience, the domain \( D \) of the model shall be defined as follows:

\[
D \equiv \{(r, y, k, p, \pi^e) \in \mathbb{R}_+^4 \times \mathbb{R} : r \in [\underline{r}, \bar{r}], y \in [\underline{y}, \bar{y}], k \in [\underline{k}, \bar{k}], p \in [\underline{p}, \bar{p}], \pi^e \in [\underline{\pi}^e, \bar{\pi}^e]\},
\]

where \( \underline{x} \) and \( \bar{x} \) are positive constants with \( \underline{x} < \bar{x} \) for \( x = r, y, k, p \) and \( \pi^e \) and \( \pi^s \) are negative and positive constants, respectively. \( \underline{x} \) and \( \bar{x} \) may be regarded as the minimum value (lower bound) and the maximum value (upper bound) of \( x \), respectively for \( x = r, y, k, p, \pi^e \). Since our concern is in the medium-term analysis and economic variables can, in the medium-term analysis, be considered to vary with some bounded regions, the domain \( D \) may be viewed as a set of economically meaningful (feasible) combinations of \( (r, y, k, p, \pi^e) \).\(^5\)

To analyze this model in details below, we shall make the following conventional assumptions.

**Assumption 2.1.** The real-valued functions \( I, S, L \) and \( H \) are defined and twice continuously differentiable (at

\(^4\)This type of formalization can be found in, for instance, Stein (1969), Fischer (1972) and Flaschel and Sechi (1996). A more general formulation can be found in Murakami (2014a).

\(^5\)In the context of long-run growth, it is not appropriate to take \( y \) and \( k \) as bounded, but in our analysis, any factors of long-run growth are ignored, so we can safely assume that they are bounded. See also Murakami (2014a).
least) everywhere on $D$, and the following conditions are satisfied (at least) everywhere on $D$:

\begin{align}
L_r < 0, L_y > 0, L_k & \geq 0, \\
I_r < 0, I_y > 0, I_k < 0, I_{\pi^*} & > 0, \\
S_r & \geq 0, S_y > 0, S_k \leq 0, S_{\pi^*} \leq 0, \\
H_y & > 0, H_k \leq 0, H_p \leq 0, \\
I_r S_h > I_k S_r, I_r L_y > I_y L_r, (I_r - S_r) L_k < (I_k - S_k) L_r, \\
(I_y S_k - I_k S_y) L_r + (I_k S_r - I_r S_k) L_y + (I_r S_y - I_y S_r) L_k & < 0, \\
[(I_y S_k - I_k S_y) L_r + (I_k S_r - I_r S_k) L_y + (I_r S_y - I_y S_r) L_k] H_p p + [(I_r S_k - I_k S_y) H_y + (I_y S_r - I_r S_y) H_k] M & > 0.
\end{align}

Assumption 2.2. The following conditions are satisfied:

\begin{align}
\bar{p} L(\bar{\tau}, \bar{\eta}, \bar{\kappa}) \leq M \leq p L(\tau, y, k), \\
I(\bar{\tau}, \bar{\eta}, \bar{\kappa}, 0) - S(\bar{\tau}, \bar{\eta}, \bar{\kappa}, 0) & < 0 < I(\tau, y, k, 0) - S(\tau, y, k, 0), \\
I(\bar{\tau}, \bar{\eta}, \bar{\kappa}, 0) & < 0 < I(\tau, y, k, 0), \\
H(\bar{\eta}, \bar{\kappa}, \bar{p}) & < 0 < H(\eta, k, p).
\end{align}

We shall briefly check the validity of Assumptions 2.1 and 2.2. To begin, Assumption 2.1 is examined. Conditions (2.6)-(2.9) are fairly standard and conventional, so we do not mention the validity of them. Under (2.6)-(2.9), condition (2.10) is met unless $S_r = 0$ and $S_k = 0$ do not hold simultaneously and if the marginal effect of the rate of interest on liquidity preference $L_r$ is comparatively large in its magnitude (as in the case of the Keynesian liquidity trap). In this sense, this condition may be regarded as valid from Keynesian viewpoints. Under (2.6)-(2.10), condition (2.11) is fulfilled if the absolute values of $S_r$ and $S_k$ are sufficiently small. Since the marginal effects of $r$ and $k$ on $S$ can be considered comparatively weak in practice, condition (2.11) is realistic from this aspect. Moreover, condition (2.12) holds under (2.6)-(2.11) if the magnitude of $H_k$ is sufficiently small. Since the marginal effect of $k$ on the rate of (un)employment is not too large compared with that of $y$ on the rate of (un)employment, condition (2.12) does not seem unrealistic. It can thus be confirmed that Assumption 2.1 is realistic.

Next, the validity of Assumption 2.2 is checked. To see if Assumption 2.2 is realistic, Assumption 2.1 is assumed. Under Assumption 2.1, conditions (2.13)-(2.16) implies that for every $x_i \in [\underline{x}, \bar{x}]$ for $x = r, y, k, p$ and
\[ i = 0, 1, 2, 3, 4: \]
\[
L(r, y_0, k_0) \leq \frac{M}{p_0} \leq L(r, y_0, k_0). \tag{2.17}
\]
\[
I(r_0, \bar{y}, k_1, 0) - S(r_0, \bar{y}, k_1, 0) < 0 < I(r_1, \underline{y}, k_2, 0) - S(r_1, \underline{y}, k_2, 0), \tag{2.18}
\]
\[
I(r_2, y_1, \bar{k}, 0) < 0 < I(r_3, y_2, \bar{k}, 0), \tag{2.19}
\]
\[
H(y_3, k_3, \bar{p}) < 0 < H(y_4, k_4, \bar{p}). \tag{2.20}
\]

Condition (2.17) states that when \( r \) is at the lower bound \( \underline{r} \) (resp. at the upper bound \( \bar{r} \)), the money market is in excess demand (resp. in excess supply). This means that when the rate of interest \( r \) is extremely low (resp. extremely high), investors prefer to hold their wealth in the form of money because assets prices, except for money, are extremely high (resp. extremely low).\(^6\) Conditions (2.18)-(2.20) concern the case where the expected rate of inflation is \( \pi^e = 0 \), so to see the economic implications of them, we will consider, for the time being, the case of \( \pi^e = 0 \). Condition (2.18) means that when \( y \) is at the minimum value \( \underline{y} \) (resp. at the maximum value \( \bar{y} \)), the good market is in excess demand (resp. in excess supply) because of the deficiency of savings (resp. of redundant savings); condition (2.19) says that when \( k \) is at the minimum value \( \underline{k} \) (resp. the maximum value \( \bar{k} \)), capital stock is increased (resp. decreased) due to the deficiency of capital stock (resp. to the depletion of investment opportunities on account of abundance of capital stock); condition (2.20) indicates that when \( p \) is at the minimum value \( \underline{p} \) (resp. at the maximum value \( \bar{p} \)), the price level rises (resp. declines) because the “self-adjusting” function of the price level operates. Mathematically, under Assumption 2.1, condition (2.17) ensures that Eq. (2.1) can be solved for \( r \) and conditions (2.18)-(2.20) imply the existence and uniqueness of an equilibrium point of the model.\(^7\)

We will first confirm that, under Assumptions 2.1 and 2.2, \( r \) can be represented as a well-defined function of \( y, k, p \).

**Proposition 2.1.** Let Assumptions 2.1 and 2.2 hold. Then, there uniquely exists a real-valued twice continuously differentiable function \( R \) from \( D_R \) into \([\underline{r}, \bar{r}]\) that satisfies the following conditions everywhere on \( D_R \) for \( x = y, k \):

\[
\frac{M}{p} = L(R(y, k, p), y, k), \tag{2.21}
\]
\[
R_x = -\frac{L_x}{L_r} (R(y, k, p), y, k), \quad R_p = -\frac{M}{p L_r} (R(y, k, p), y, k), \tag{2.22}
\]

where \( D_R \) is defined as:

\[ D_R \equiv \{(y, k, p) \in \mathbb{R}^3_{++} : y \in [\underline{y}, \bar{y}], k \in [\underline{k}, \bar{k}], p \in [\underline{p}, \bar{p}] \}. \]

\(^6\)Condition (2.17) can be interpreted as a reflection of the description of the money-asset market by Keynes (1936, ch.15). For details, see Murakami (2014a, p.83, Appendix A).

\(^7\)For more detailed discussions of the meanings of Assumption 2.2, see Murakami (2014a).
Proof. One can easily obtain the conclusion by slightly modifying the proof in Murakami (2014a, p.83, Proposition A.1).

Since we have shown that $r$ can be defined as a twice continuously differentiable function $R$ of $y, k$ and $p$ and the range of $R$ is $[\underline{r}, \bar{r}]$, we can summarize (2.1)-(2.3) as:

\[\dot{y} = \alpha X(y, k, p, \pi^e), \quad (2.23)\]
\[\dot{k} = A(y, k, p, \pi^e), \quad (2.24)\]

where

\[X(y, k, p, \pi^e) \equiv I(R(y, k, p), y, k, \pi^e) - S(R(y, k, p), y, k, p, \pi^e),\]
\[A(y, k, p, \pi^e) \equiv I(R(y, k, p), y, k, \pi^e).\]

Note that Assumption 2.1 implies:

\[X_y = (I_y - S_y)R_y + (I_y - S_y) < 0, X_k = (I_k - S_k)R_k + (I_k - S_k) < 0, X_p = (I_p - S_p)R_p < 0, X_{\pi^e} = I_{\pi^e} - S_{\pi^e} > 0, \quad (2.25)\]
\[A_y = I_y R_y + I_y > 0, A_k = I_k R_k + I_k < 0, A_p = I_p R_p < 0, A_{\pi^e} = I_{\pi^e} > 0, \quad (2.26)\]
\[X_y A_y > X_k A_y, X_k A_p < X_p A_k, \quad (2.27)\]
\[(X_y A_k - X_k A_y)H_y + (X_k A_p - X_p A_k)H_y + (X_p A_y - X_y A_p)H_k < 0. \quad (2.28)\]

Furthermore, by substituting (2.4) in (2.5), Eq. (2.5) can be rewritten as:

\[\dot{\pi}^e = \beta H(y, k, p). \quad (2.29)\]

In what follows, we will investigate the system of Eqs. (2.4), (2.23), (2.24) and (2.29).

We shall call a point $(y, k, p, \pi^e)$ that satisfies $\dot{y} = \dot{k} = \dot{p} = \dot{\pi}^e = 0$ an equilibrium. Since the equilibrium value of $\pi^e$ is 0, an equilibrium can be defined as a point $(y, k, p) \in D_R$ that satisfies:

\[X(y, k, p, 0) = 0, \quad (2.30)\]
\[A(y, k, p, 0) = 0, \quad (2.31)\]
\[H(y, k, p) = 0. \quad (2.32)\]

The following proposition concerns the existence and uniqueness of an equilibrium. As proved by Murakami (2014a, p.78, Proposition 3.1), it is possible to verify the existence and uniqueness of an equilibrium.

**Proposition 2.2.** Let Assumptions 2.1 and 2.2 hold. Then, there uniquely exists an equilibrium $(y^*, k^*, p^*) \in \ldots$
$D_R^c$, where $D_R^c$ denotes the interior of $D_R$.


The Jacobian matrix evaluated at the unique equilibrium is given by:

$$J(y^*, k^*, p^*, 0) = \begin{pmatrix}
\alpha X_y^* & \alpha X_k^* & \alpha X_p^* & \alpha X_e^* \\
A_y^* & A_k^* & A_p^* & A_e^* \\
H_y^*p^* & H_k^*p^* & H_p^*p^* & p^* \\
\beta H_y^* & \beta H_k^* & \beta H_p^* & 0 \\
\end{pmatrix},$$

where $F_x^*$ denotes $F_x$ evaluated at the equilibrium for $F = X, A, H$ and $x = y, k, p, \pi^e$.

To simplify the argument below, we shall make the following simplifying assumption.

**Assumption 2.3.** The following condition is satisfied:

$$H_k^* = 0, H_p^* = 0. \quad (2.33)$$

Condition (2.33) indicates that, at the equilibrium, the marginal effects of capital stock on the rate of inflation and of the price level on the rate of inflation are zero. This condition may seem an oversimplification, but because of the weakness of the marginal effect of capital stock on the rate of (un)employment and due to the fact that the “self-adjustment” mechanism of the price level does not work when the price level is around the equilibrium (or within the “middle” range), it serves as a first approximation to the reality.\(^8\) Note that Assumption 2.3 is not inconsistent with Assumption 2.1.

Under Assumption 2.3, the characteristic equation associated with the Jacobian matrix is as follows:

$$\lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0, \quad (2.34)$$

where

$$a_1 \equiv -(\alpha X_y^* + A_y^*),$$
$$a_2 \equiv \alpha (X_y^* A_k^* - X_k^* A_y^* - X_p^* H_y^* p^* - \beta X_e^* H_y^*),$$
$$a_3 \equiv \alpha [(X_y^* A_k^* - X_k^* A_y^*) p^* - \beta (X_e^* A_e^* - X_p^* A_k^* + X_k^* A_p^*)] H_y^*,$$
$$a_4 \equiv \alpha \beta (X_p^* A_e^* - X_k^* A_p^*) H_y^* p^* > 0,$$

Assumption 2.1 ensures that $a_4 > 0$.

In what follows, we shall treat the parameter $\alpha$ as a positive constant and investigate how changes in the expectation formation parameter $\beta$ affect the properties of the model.

\(^8\)For details on the justification of this assumption, see Murakami (2014a, p.81).
To examine the possibility of persistent business cycles generated by a Hopf bifurcation, we shall introduce the following assumption.

**Assumption 2.4.** The following conditions are satisfied:

\[
\begin{align*}
\alpha X_y^* + A_k^* &< 0, \quad (2.35) \\
\alpha X_y^* X_{\pi e}^* + X_k^* A_{\pi e}^* + X_p^* p^* &> 0, \quad (2.36) \\
X_k^* A_{\pi e}^* + X_p^* p^* &> X_{\pi e}^* A_k^*, \quad (2.37) \\
\alpha X_y^* X_k^* H_y^* p^* &> (\alpha X_y^* + A_k^*)(X_y^* A_k^* - X_k^* A_y^*). \quad (2.38)
\end{align*}
\]

Under Assumption 2.1, conditions (2.35)-(2.38) are satisfied if the speed of quantity adjustments is relatively slow and the absolute value of $X_p^*$ is small. As easily seen from (2.25), the smallness of $X_p^*$ implies that the Keynes effect and the Pigou effect are sufficiently weak.

As proved by Murakami (2014a, p.81, Proposition 3.3), we can prove the existence of a periodic orbit (representing a persistent business cycle) by the Hopf bifurcation theorem.

**Proposition 2.3.** Let Assumptions 2.1-2.4 hold. Then, there exists a positive $\beta^*$ such that at least one periodic orbit exists, if $\beta$ is sufficiently close to $\beta^*$.

**Proof.** See Murakami (2014a, pp.81-82, Proposition 3.3).  

Proposition 2.3 implies that the greater the speed of inflation-deflation expectation formations, $\beta$, is, the more likely the equilibrium is to be (locally asymptotically) unstable. That is to say, the processes of inflation-deflation expectation formations function as a destabilizing factor.

In this section, we have presented a simple Keynes-Kaldor-Tobin model and briefly reviewed some properties of this model, especially the possibility of a persistent business cycle yielded. In the next section, we will discuss the possibility of stabilization through two kinds of monetary policies (the quantity policy and the interest rate policy) and compare the effectiveness of them.

## 3 Alternative monetary policies and economic stability in the Keynes-Kaldor-Tobin model

In this section, we shall analyze the effects of the two kinds of monetary policies (the quantity policy and the interest rate policy) in a simple version of the Keynes-Kaldor-Tobin model. The money supply policy, in which the quantity of money is the target variable of the monetary authority, will first be investigated and then the...
interest rate policy, in which the short-term rate of interest is controlled by the monetary authority, shall be examined.

### 3.1 The quantity policy

In this subsection, we shall simplify the Keynes-Kaldor-Tobin model offered in section 2 to take account of changes in the quantity of money \( M \) through the quantity policy conducted by the monetary authority.

For the purpose above, we shall modify the model presented in section 2 by introducing some counter-cyclical policy. The model is modified as follows:

\[
\begin{align*}
m &= L(r, y, k), \quad (3.1) \\
\dot{y} &= \alpha [I(r, y, k, \pi^e) - S(r, y, k, m, \pi^e)], \quad (3.2) \\
\dot{k} &= I(r, y, k, \pi^e), \quad (3.3) \\
\frac{\dot{p}}{p} &= H(y) + \pi^e, \quad (3.4) \\
\dot{\pi}^e &= \beta \left( \frac{\dot{p}}{p} - \pi^e \right), \quad (3.5) \\
\frac{\dot{M}}{M} &= \gamma (y^* - y), \quad (3.6)
\end{align*}
\]

where \( m \equiv M/p \); \( y^* \) is a positive constant that satisfies \( H(y^*) = 0 \); \( \gamma \) is a positive parameter.

Eq. (3.6) represents a simple counter-cyclical policy, in which the monetary authority sets the rate of changes in the supply of money \( M \) in accordance with the difference between the current income and the target income \( y^* \), which is assumed to be equal to the natural rate income. The positive parameter \( \gamma \) measures the sensitivity of the monetary policy on the current economic condition. Note that, in (3.4), the Phillips function \( H \) is assumed for simplicity to be a function only of \( y \), not of \( k \) or \( p \).

As in section 2, Eq. (3.1) may be solved for \( r \) under some reasonable assumptions. We will suppose henceforth that the assumptions are fulfilled so that we have:

\[
\begin{align*}
m &= L(R(y, k, m), y, k), \quad (3.7) \\
R_x &= \frac{L_x}{L_r} (R(y, k, m), y, k), R_m = \frac{m}{L_r} (R(y, k, m), y, k), \quad (3.8)
\end{align*}
\]

Putting (3.7) in (3.2) and (3.3) and (3.4) in (3.5) and combining (3.4) and (3.6), we obtain:

\[
\begin{align*}
\dot{y} &= \alpha X(y, k, m, \pi^e), \quad (3.9) \\
\dot{k} &= A(y, k, m, \pi^e), \quad (3.10) \\
\frac{\dot{m}}{m} &= -[H(y) + \gamma (y - y^*) + \pi^e], \quad (3.11) \\
\dot{\pi}^e &= \beta H(y), \quad (3.12)
\end{align*}
\]
where
\[
X(y, k, m, \pi^e) \equiv I(R(y, k, m), y, k, \pi^e) - S(R(y, k, m), y, k, m, \pi^e), \tag{3.13}
\]
\[
A(y, k, m, \pi^e) \equiv I(R(y, k, m), y, k, \pi^e). \tag{3.14}
\]

In what follows, we will consider the system of Eqs. (3.9)-(3.12) to examine the effects of the monetary policy parameter $\gamma$ on the stability.

We shall make the following assumption consistent with Assumption 2.1.

**Assumption 3.1.** The real valued functions $X, A$ and $H$ are defined and twice continuously differentiable for every $(y, k, m, \pi^e) \in \mathbb{R}^{3+} \times \mathbb{R}$, and the following conditions are satisfied for every $(y, k, m, \pi^e) \in \mathbb{R}^{3+} \times \mathbb{R}$:
\[
X_k < 0, X_m > 0, X_{\pi^e} > 0, \tag{3.15}
\]
\[
A_y > 0, A_k < 0, A_m > 0, A_{\pi^e} > 0, \tag{3.16}
\]
\[
H_y > 0, \tag{3.17}
\]
\[
X_y A_k > X_k A_y, X_k A_m > X_m A_k. \tag{3.18}
\]

Except for $X_k A_m > X_m A_k$, all the conditions (3.15)-(3.18) are consistent with Assumption 2.1. It follows from (3.13) and (3.14) that we can obtain:
\[
X_k A_m - X_m A_k = (I_r R_k + I_k) (S_r R_m + S_m) - I_r R_m (S_r R_k + S_k) > 0,
\]
where the inequality holds if $S_r$ is sufficiently small under (3.15) and (3.16). Since the magnitude of $S_r$ is negligible, this condition can be considered to be met in practice. Assumption 3.1 can thus be justified.

To make our analysis simpler, the following assumption concerning the existence and uniqueness of an equilibrium will be imposed.

**Assumption 3.2.** There uniquely exists a $(k^*, m^*) \in \mathbb{R}^{2+}$ that satisfies:
\[
X(y^*, k^*, m^*, 0) = 0, \tag{3.19}
\]
\[
A(y^*, k^*, m^*, 0) = 0. \tag{3.20}
\]

Because of the property of $H(y^*) = 0$, Assumption 3.2 ensures that $(y^*, k^*, m^*, 0) \in \mathbb{R}^{3+} \times \mathbb{R}$ is a unique equilibrium.
The Jacobian matrix evaluated at the unique equilibrium \((y^*, k^*, m^*, 0)\) is as follows:

\[
J(y^*, k^*, m^*, 0) = \begin{pmatrix}
\alpha X_y^* & \alpha X_k^* & \alpha X_m^* & \alpha X_{\pi^*}^* \\
A_y^* & A_k^* & A_m^* & A_{\pi^*}^* \\
-(H_y^* + \gamma)m^* & 0 & 0 & -m^* \\
\beta H_y^* & 0 & 0 & 0
\end{pmatrix}.
\]

The characteristic equation associated with the Jacobian matrix is calculated as:

\[
\lambda^4 + b_1 \lambda^3 + b_2 \lambda^2 + b_3 \lambda + b_4 = 0,
\]

where

\[
b_1 \equiv -(\alpha X_y^* + A_k^*),
\]

\[
b_2 \equiv \alpha(X_y^* A_k^* - X_k^* A_y^*)m^* + \beta X_m^* H_y^* + \gamma X_m^* m^*),
\]

\[
b_3 \equiv \alpha\{(X_k^* A_m^* - X_m^* A_k^*)m^* - \beta (X_m^* A_{\pi^*}^* - X_{\pi^*}^* A_m^* - X_{\pi^*}^* m^*\}) H_y^* + \gamma (X_k^* A_m^* - X_m^* A_k^*) m^*\},
\]

\[
b_4 \equiv \alpha \beta (X_k^* A_m^* - X_m^* A_k^*) H_y^* m^* > 0,
\]

Assumption 3.1 ensures that \(b_4 > 0\).

According to the Routh-Hurwitz criterion, the necessary and sufficient condition for the local asymptotic stability of the equilibrium is that the following conditions are all satisfied:

\[
b_1 > 0, b_3 > 0, b_4 > 0,
\]

\[
(b_1 b_2 - b_3) b_3 > b_1^2 b_4,
\]

which are, under Assumption 3.1, equivalent to:

\[
\alpha X_y^* + A_k^* < 0,
\]

\[
(X_k^* A_m^* - X_m^* A_k^*)(H_y^* + \gamma)m^* > \beta (X_m^* A_{\pi^*}^* - X_{\pi^*}^* A_m^* - X_{\pi^*}^* m^*) H_y^*,
\]

\[
g(\gamma) \equiv c_1 \gamma^2 + c_2 \gamma + c_3 < 0,
\]
Under Assumption 3.1, condition (3.22) is satisfied if the speed of quantity adjustments $\alpha$ is not large, and $c_1 < 0$ and $c_3 > 0$ hold if $\alpha$ and the slope of the Phillips curve $H_y^*$ are small enough. As long as $c_1 < 0$ holds, it follows from (3.23) and (3.24) that a large value of the policy parameter $\gamma$ is conductive to the (local asymptotic) stability of the equilibrium. If $c_3 > 0$, however, $g(\gamma)$ has positive and negative zeros. Letting $\gamma^*$ be the positive zero, there is some range of $\gamma$, $(0, \gamma^*)$, in which the stability of the equilibrium is lost.

To discuss the case where $c_1 < 0$ and $c_3 > 0$, we shall make the following assumption.

**Assumption 3.3.** The following conditions, as well as (3.22), are satisfied:

\[
\begin{align*}
\alpha X_y^* X_m^* + X_k^* A_m^* &< 0, \\
(X_k^* A_m^* - X_m^* A_k^*) m^* &> \beta (X_k^* A_m^* - X_m^* A_k^* - X_m^* m^*), \\
(\alpha X_y^* + A_k^*) (X_k^* A_m^* - X_m^* A_k^*) &+ \alpha (\alpha X_y^* + A_k^*) [(X_y^* A_k^* - X_k^* A_y^*) (X_k^* A_m^* - X_m^* A_k^*) m^* + (\alpha X_y^* A_m^* + X_k^* A_k^* - X_m^* m^*)] H_y^* \\
&+ [(\alpha X_y^* X_m^* + X_k^* A_m^*) m^* - \beta (\alpha X_y^* X_m^* + X_k^* A_k^* - X_m^* m^*)] \\
&\times [(X_k^* A_m^* - X_m^* A_k^*) - \beta (X_k^* A_m^* - X_k^* A_k^* - X_m^* m^*)] (H_y^*)^2 > 0.
\end{align*}
\]

Under Assumption 3.1, conditions (3.25)-(3.27) are all fulfilled if $\alpha, \beta$ and $H_y^*$ are comparatively small. In what follows, we shall restrict our attention to the case where Assumption 3.3 holds, more specifically, the case where $\alpha, \beta$ and $H_y^*$ are small enough.

To demonstrate that, if the policy parameter $\gamma$ is small, the equilibrium is devoid of stability, we can prove the existence of a periodic orbit (a persistent business cycle) by way of the Hopf bifurcation theorem.

**Proposition 3.1.** Let Assumptions 3.1-3.3 hold. Then, there exists a positive $\gamma^*$ such that at least one periodic orbit exists, if $\gamma$ is sufficiently close to $\gamma^*$.

**Proof.** One can easily know that, under Assumptions 3.1-3.3, there is a positive zero $\gamma^*$ of $g(\gamma) = b_1^2 b_4 - (b_1 b_2 - b_3) b_3$ and we have $b_1 > 0, b_3 > 0$ and $b_4 > 0$ for $\gamma = \gamma^*$. Moreover, since $\gamma^*$ is not a multiple root of $g(\gamma) = 0$, one can obtain $g'(\gamma^*) \neq 0$. Then, it follows from Asada and Yoshida (2003, p.527, Theorem 3) that a Hopf bifurcation occurs for $\gamma = \gamma^*$ and that a periodic orbit is generated by this bifurcation.
Proposition 3.1 implies that even if both \( \alpha \) and \( \beta \) are small, which is favorable for the stability of the model in section 2, the smallness of the monetary policy parameter \( \gamma \) destabilizes the model. This means that the counter cyclical policy can have a negative influence on the economic stability in situations where the stability is ensured in the absence of this policy, if the degree of the sensitivity of the monetary authority to the output gap \( \gamma \) is not large enough. In this sense, inadequate quantity policies can aggravate economic situations.

We can conclude from the analysis above that the counter-cyclical policy through changes in the quantity of money is contributory to the economic stability if the monetary authority is very sensitive to the output gap (the difference between the current output and the natural rate), while this policy can have a negative impact on the stability if it is inadequate, and that when \( \gamma \) is around some threshold value, a periodic orbit (a persistent economic fluctuation) is generated by this policy.

3.2 The interest rate policy

In this subsection, we shall formulate the interest policy conducted by the monetary authority and analyze its stabilization effects in a simple Keynes-Kaldor-Tobin model.

For this purpose, we will formalize a simpler version of the Keynes-Kaldor-Tobin model in which the rate of interest \( r \) varies in accordance with the interest rate policy carried out by the monetary authority. The model is composed of (3.12) and:

\[
\begin{align*}
\dot{y} &= \alpha[I(y, k, \rho^e) - S(y, k, \rho^e)], \\
\dot{k} &= I(y, k, \rho^e), \\
\dot{r} &= \delta(y - y^*),
\end{align*}
\]

where \( \rho^e \equiv r - \pi^e; y^* \) is a positive constant with \( H(y^*) = 0; \delta \) is a positive parameter.

In (3.28) and (3.29), it is assumed for simplicity that the expected rate of inflation affects investment and savings only through changes in the real rate of interest.

Eq. (3.30) is an expression of the feedback policy of the monetary authority through changes in the rate of interest. As in Taylor (1993), the monetary authority is supposed to change the rate of interest in response to the existing output gap. The parameter \( \gamma \) stands for the sensitivity of the monetary authority to the existing output gap.\(^{11}\)

Combining (3.12) and (3.30), we obtain the following differential equation on \( \rho^e \):

\[
\dot{\rho}^e = \delta(y - y^*) - H(y),
\]

We shall henceforth consider the system of Eqs. (3.28), (3.29) and (3.31).\(^{12}\)

\(^{11}\)Chiarella and Flashel (2000), Flaschel et al. (2001) and Flaschel et al. (2003) adopted similar formalizations on feedback interest rate policies. The formalization, given in (3.30), may be regarded as a simpler version of theirs.

\(^{12}\)This model is similar to Yoshikawa’s (1981) one but differs from his in that capital formation processes are taken into consideration in our model.
We shall make the following assumptions.

**Assumption 3.4.** The real valued functions \( I, S \) and \( H \) are defined and twice continuously differentiable for every \((y, k, \rho^e) \in \mathbb{R}^2_{++} \times \mathbb{R}\), and the following conditions are satisfied for every \((y, k, \rho^e) \in \mathbb{R}^2_{++} \times \mathbb{R}\):

\[
I_y > 0, I_k < 0, I_{\rho^e} < 0,
\]

(3.32)

\[
S_y > 0, S_k \leq 0, S_{\rho^e} \geq 0,
\]

(3.33)

\[
H_y > 0,
\]

(3.34)

\[
I_y S_k > I_k S_y, I_k S_{\rho^e} < I_{\rho^e} S_k
\]

(3.35)

**Assumption 3.5.** There uniquely exists a \((k^*, \rho_{\rho^e}^e) \in \mathbb{R}^2_{++} \times \mathbb{R}\) that satisfies:

\[
I(y^*, k^*, \rho_{\rho^e}^e) = S(y^*, k^*, \rho_{\rho^e}^e) = 0.
\]

(3.36)

The validity of Assumption 3.4, except for \(I_k S_{\rho^e} < I_{\rho^e} S_k\), has already been confirmed in section 2. The assumption of \(I_k S_{\rho^e} < I_{\rho^e} S_k\) is fulfilled under (3.32) and (3.33) unless \(S_k = 0\) and \(S_{\rho^e} = 0\) are simultaneously satisfied. In this sense, this assumption is not restrictive.

Assumption 3.5 means the existence and uniqueness of an equilibrium of the model discussed in this section.

The Jacobian matrix evaluated at the unique equilibrium is given by:

\[
J(y^*, k^*, \rho_{\rho^e}^e) = \begin{pmatrix}
\alpha(I_y^* - S_y^*) & \alpha(I_k^* - S_k^*) & \alpha(I_{\rho^e}^* - S_{\rho^e}^*) \\
I_y^* & I_k^* & I_{\rho^e}^* \\
\delta - \beta H_y^* & 0 & 0
\end{pmatrix}.
\]

Thus, the characteristic equation associated with this matrix is calculated as follows:

\[
\lambda^3 + d_1 \lambda^2 + d_2 \lambda + d_3 = 0,
\]

(3.37)

where

\[
d_1 \equiv -[\alpha(I_y^* - S_y^*)] + I_k^*,
\]

\[
d_2 \equiv \alpha[I_y^* S_k^* - I_k^* S_y^* - (\delta - \beta H_y^*)(I_{\rho^e}^* - S_{\rho^e}^*)],
\]

\[
d_3 \equiv \alpha(I_{\rho^e}^* S_k^* - I_k^* S_{\rho^e}^*)(\delta - \beta H_y^*).
\]

The necessary and sufficient condition for the local asymptotic stability of the equilibrium is that the following
conditions are all satisfied:

\[ \alpha(I_y^* - S_y^*) + I_k^* < 0, \]  
(3.38)

\[ \delta > \beta H_y^*, \]  
(3.39)

\[ [\alpha(I_y^* - S_y^*)(I_{\rho^e} - S_{\rho^e}) + (I_k^* - S_k^*)(I_{\rho^e})][(\delta - \beta H_y^*) > [\alpha(I_y^* - S_y^*) + I_k^*](I_y^* S_k^* - I_k^* S_y^*). \]  
(3.40)

Under Assumption 3.4, conditions (3.38)-(3.40) are satisfied if \( I_y^* < S_y^* \) and \( \delta > \beta H_y^* \). The hypothesis of \( I_y^* < S_y^* \) states that the marginal propensity to invest is less than that to save, and this hypothesis violates Kaldor’s (1940) condition but is consistent with the so-called Keynesian stability condition.

To establish the stability criterion, we shall make the following assumption.

**Assumption 3.6.** The following condition, as well as (3.38), is satisfied:

\[ \alpha(I_y^* - S_y^*)(I_{\rho^e} - S_{\rho^e}) + (I_k^* - S_k^*)(I_{\rho^e}) > 0. \]  
(3.41)

As discussed in the above, Assumption 3.6 is satisfied, under Assumption 2.1, if the marginal propensity to invest \( I_y^* \) is smaller than that to save \( S_y^* \). As in the Keynesian IS-LM analysis (e.g., Hicks 1937 and Modigliani 1944), this condition is conductive to the stability in the medium-term Keynesian analysis.

We are now ready to prove the following proposition on the local asymptotic stability of the equilibrium.

**Proposition 3.2.** Let Assumptions 3.4-3.6 hold. Then, if \( \delta \) is large enough to meet (3.39), the unique equilibrium \((y^*, k^*, \rho_{\rho^*}^*)\) is locally asymptotically stable.

**Proof.** The proof is obvious from the argument above. \( \square \)

The economic implication of Proposition 3.2 is that the economy can be stabilized when the monetary authority takes a strong attitude towards economic fluctuations by taking a high response to the exiting output gap.

To deny the possibility of a periodic orbit (a persistent business cycle) yielded by way of a Hopf bifurcation, we shall confirm that some of the necessary conditions for a Hopf bifurcation generated are violated. The Hopf bifurcation theorem requires that the characteristic equation has a pair of purely imaginary roots for some parameter value. According to Liu (1994), in our case, a necessary condition for the characteristic equation (3.37) to have a pair of purely imaginary roots is that both of the following conditions hold:

\[ d_2 > 0, \]

\[ d_1d_2 = d_3. \]
Thus, all we have to do is to prove that, under Assumptions 3.4-3.6, there exists no positive \( \delta \) that satisfies (3.42) and (3.43). For this purpose, we assume first that both condition (3.42) and condition (3.43) are fulfilled. Solving (3.43) for \( \delta \) and substituting it in (3.42), we obtain the following expression:

\[
\frac{(I_y^* S_k^* - I_k^* S_y^*)(I_k^* S_y^* - I_y^* S_k^*)}{\alpha (I_y^* - S_y^*)(I_y^* - S_y^*) + (I_k^* - S_k^*) I_y^*} > 0.
\]

But this contradicts Assumptions 3.4 and 3.6.

We will compare the effects of the quantity policy and the interest rate policy. The results obtained in sections 3.1 and 3.2 imply that if the monetary authority adopts strong stabilization policies either through the quantity policy or through the interest rate policy, the macro economy can be stabilized, but the interest rate policy is slightly more effective than the quantity policy in that inadequate quantity policies give rise to a persistent business cycle but interest rate policies do not even if they are inadequate. Note that the quantity of money itself cannot be controlled directly by the monetary authority, and this fact may be one of the inferior points of the quantity policy.

4 Concluding remarks

In this paper, we have analyzed two kinds of stabilization policies, the quantity policy and the interest rate policy, in the Keynes-Kaldor-Tobin model of business cycles. We will now summarize our analysis.

In section 2, we have formalized the Keynes-Kaldor-Tobin model of business cycles and reviewed its basic properties. The existence of a periodic orbit has been verified if the parameter of expectation formations \( \beta \) is within some range.

In section 3.1, we have investigated the effects of the quantity policy in which the monetary authority changes the quantity of money in response to aggregate output \( y \) in the Keynes-Kaldor-Tobin model and revealed that it is possible to achieve the (local) asymptotic stability of the equilibrium if the parameter of speed of response \( \gamma \) is sufficiently large, but, if \( \gamma \) is not large, the (local) asymptotic stability of the equilibrium is lost and a periodic orbit (a persistent business cycle) is generated.

In section 3.2, we have examined the effects of the interest rate policy in which changes in the rate of interest \( r \) are proportionate to the existing gap between the current output \( y \) and the target output \( y^* \) (identified with the natural-rate output) in the Keynes-Kaldor-Tobin model and found that if the parameter of speed of response \( \delta \) is large enough to satisfy (3.39), the equilibrium possesses the local asymptotic stability and that the possibility of generation of a periodic orbit by a Hopf bifurcation is denied.

or:

\[
I_y^* S_k^* - I_k^* S_y^* > (\delta - \beta H_y^*)(I_y^* - S_y^*), \quad (3.42)
\]

\[
[\alpha(I_y^* - S_y^*)(I_y^* - S_y^*) + (I_k^* - S_k^*) I_y^*](\delta - \beta H_y^*) = [\alpha(I_y^* - S_y^*) + I_k^*](I_y^* S_k^* - I_k^* S_y^*). \quad (3.43)
\]
Comparing the results of sections 3.1 and 3.2, we can conclude that the interest rate policy is superior to the quantity policy in that there is no possibility of a persistent business cycle (economic fluctuations) generated by economic policies in the case of the interest rate policy.

References


