Oligopolies with Contingent Workforce and Unemployment Insurance Systems

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Motivation
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Classical oligopoly analysis has provided several insights

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Given the persistent economic scenario we assume the oligopolists need to take into account the workforce cost.

- contingent workforce
- unemployment insurance systems
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The Model with Contingent Workforce
Contingent Workforce

Main features

- $N$ firms industry
- identical product
- $x_k$ firm $k$ output
- $X = \sum_{k=1}^{N} x_k$

Inverse demand function: $p(X) = A - BX$

Cost function: $C_k(x_k) = c_k + d_k x_k$

The output adjustment cost at time period $t$

$$\bar{C}_k(x_k, x_k(t-1)) = \begin{cases} 0 & \text{if } x_k \leq x_k(t-1) \\ \gamma_k (x_k - x_k(t-1)) & \text{otherwise.} \end{cases}$$

$\gamma_k > 0$, Matsumoto, Merlone, Szidarovszky (2014)
Contingent Workforce

The profit of firm $k$ at time period $t$

$$\Pi_k = \begin{cases} 
  x_k(A - Bx_k - BX_k) - (c_k + d_k x_k) & \text{if } x_k \leq x_k(t-1) \\
  x_k(A - Bx_k - BX_k) - (c_k + d_k x_k) - \gamma_k(x_k - x_k(t-1)) & \text{otherwise,}
\end{cases}$$

where $X_k = \sum_{l \neq k} x_l$ is the output of the rest of the industry.

Some assumptions

- $A > d_k$
- $L_k$ maximum possible output level for firm $k$
- $0 < x_k(t-1) < L_k$
The possible shapes of the profit functions

If $\partial \Pi_k / \partial x_k \leq 0$ at $x_k = 0$
The possible shapes of the profit functions

that is, if \( X_k \geq \frac{A-d_k}{B} \)

then the best response of firm \( k \) is

\[
R_k (X_k, x_k(t-1)) = 0
\]
The possible shapes of the profit functions

If $\partial \Pi_k / \partial x_k > 0$ at $x_k = 0$ and $\partial - \Pi_k / \partial x_k \leq 0$ at $x_k = x_k(t - 1)$
The possible shapes of the profit functions

that is, if \( \frac{A - d_k}{B} - 2x_k(t - 1) < X_k \leq \frac{A - d_k}{B} \)

then the best response of firm \( k \) is

\[
R_k(X_k, x_k(t - 1)) = \frac{A - d_k - BX_k}{2B}
\]
The possible shapes of the profit functions

If $\frac{\partial - \Pi_k}{\partial x_k} > 0$ at $x_k = x_k(t-1)$ and $\frac{\partial + \Pi_k}{\partial x_k} \leq 0$ at $x_k = x_k(t-1)$. 

![Graph showing the profit function shapes](image-url)
The possible shapes of the profit functions

that is, if \( \frac{A-d_k-\gamma_k}{B} - 2x_k(t-1) < X_k \leq \frac{A-d_k}{B} - 2x_k(t-1) \)

then the best response of firm \( k \) is

\[ R_k(X_k, x_k(t-1)) = x_k(t-1) \]
The possible shapes of the profit functions

If $\frac{\partial - \Pi_k}{\partial x_k} > 0$ at $x_k = x_k(t - 1)$, $\frac{\partial + \Pi_k}{\partial x_k} > 0$ at $x_k = x_k(t - 1)$, and $\frac{\partial \Pi_k}{\partial x_k} \leq 0$ at $x_k = L_k$.
The possible shapes of the profit functions

That is, if \( \frac{A - d_k - \gamma_k}{B} - 2L_k < X_k \leq \frac{A - d_k - \gamma_k}{B} - 2x_k(t - 1) \)

then the best response of firm \( k \) is

\[
R_k (X_k, x_k(t - 1)) = \frac{A - BX_k - d_k - \gamma_k}{2B}
\]
The possible shapes of the profit functions

If \( \frac{\partial - \Pi_k}{\partial x_k} > 0 \) at \( x_k = x_k(t - 1) \), \( \frac{\partial + \Pi_k}{\partial x_k} > 0 \) at \( x_k = x_k(t - 1) \), and \( \frac{\partial \Pi_k}{\partial x_k} > 0 \) at \( x_k = L_k \).
The possible shapes of the profit functions

That is, if \( X_k \leq \frac{A - d_k - \gamma_k}{B} - 2L_k \)

then the best response of firm \( k \) is

\[
R_k (X_k, x_k (t - 1)) = L_k
\]
Best response of firm $k$ as function of the output of the rest of the industry

Putting together

$$R_k (X_k, x_k (t - 1)) =$$

$$= \begin{cases} 
L_k & \text{if } X_k \leq \frac{A-d_k-\gamma_k}{B} - 2L_k \\
\frac{A-Bx_k-d_k-\gamma_k}{2B} & \text{if } \frac{A-d_k-\gamma_k}{B} - 2L_k < X_k \leq \frac{A-d_k-\gamma_k}{B} - 2x_k (t - 1) \\
x_k (t - 1) & \text{if } \frac{A-d_k-\gamma_k}{B} - 2x_k (t - 1) < X_k \leq \frac{A-d_k}{B} - 2x_k (t - 1) \\
\frac{A-d_k-Bx_k}{2B} & \text{if } \frac{A-d_k}{B} - 2x_k (t - 1) < X_k \leq \frac{A-d_k}{B} \\
0 & \text{if } \frac{A-d_k}{B} \leq X_k
\end{cases}$$
Best response of firm $k$ as function of the output of the rest of the industry

$$R_k (X_k, x_k (t - 1))$$
Dynamic extension and steady states

Discrete time dynamics

\[ x_k(t) = x_k(t-1) + K_k \left( R_k \left( \sum_{l \neq k} x_l(t-1), x_k(t-1) \right) - x_k(t-1) \right) \]

where \( K_k \) denote the speed of adjustment of firm \( k, k = 1, 2, \ldots, N \). As usual

- \( K_k = 0 \implies \) constant trajectories,
- \( K_k = 1 \implies \) best response dynamics.
Dynamic extension and steady states

**Definition**

A vector \( \bar{x} = (\bar{x}_k) \) is a steady state of this system if and only if for all \( k \),

\[
\bar{x}_k = R_k \left( \sum_{l \neq k} \bar{x}_l, \bar{x}_k \right)
\]

Given special forms and conditions of the best response functions, for each component of the steady state we have three possibilities:

1. \( \bar{x}_k = 0 \), if \( \frac{A-d_k-\gamma_k}{B} \leq \bar{X}_k \);
2. \( 0 < \bar{x}_k < L_k \), if \( \frac{A-d_k-\gamma_k}{B} - 2\bar{x}_k \leq \bar{X}_k \leq \frac{A-d_k}{B} - 2\bar{x}_k \); (1)
3. \( \bar{x}_k = L_k \), if \( \bar{X}_k \leq \frac{A-d_k}{B} - 2L_k \),

where \( \bar{X}_k = \sum_{k \neq l} \bar{x}_l \).
Best response of firm $k$ as function of the total output of the industry

\[ \bar{x}_k = L_k \]

where
- $\bar{X}$ on the horizontal axis with domain \( [0, \sum_{i=1}^{N} L_i] \),
- $\bar{x}_k$ on the vertical axis,
- the orizontal line is $\bar{x}_k = L_k$.  

Steady states

For each value of $\bar{X}$,

- $\bar{x}_k$ is an interval $[m_k(\bar{X}), M_k(\bar{X})]$ eventually 0 or $L_k$
- functions $m_k(\bar{X})$ and $M_k(\bar{X})$ are nonincreasing and continuous

Define next

- $m(\bar{X}) = \sum_{k=1}^{N} m_k(\bar{X})$
- $M(\bar{X}) = \sum_{k=1}^{N} M_k(\bar{X})$

We have

$$0 \leq m(0), M(0) \text{ and } m(L), M(L) \leq L = \sum_{k=1}^{N} L_k$$

Therefore there are unique values $\bar{X}^{(1)}$ and $\bar{X}^{(2)}$ from interval $[0, L]$ such that $m(\bar{X}^{(1)}) = \bar{X}^{(1)}$ and $M(\bar{X}^{(2)}) = \bar{X}^{(2)}$. 
The set of all steady states can be described as follows. Let $\bar{X}$ be an arbitrary value from interval $[\bar{X}^{(1)}, \bar{X}^{(2)}]$, then the corresponding steady state coordinates form the set

$$S(\bar{X}) = \{ (\bar{x}_1, \ldots, \bar{x}_N) | \sum_{k=1}^{N} \bar{x}_k = \bar{X}, m_k(\bar{X}) \leq \bar{x}_k \leq M_k(\bar{X}), k = 1, 2, \ldots, N \}$$
Example
Example: symmetric duopoly

\[ A = 20, \ B = 1, \ c_1 = c_2 = 0, \ d_1 = d_2 = \gamma_1 = \gamma_2 = 1 \] and \( L_1 = L_2 = 10 \)

In this case

\[
m_k(\bar{X}) = \begin{cases} 
10 & \text{if } \bar{X} \leq 8 \\
18 - \bar{X} & \text{if } 8 \leq \bar{X} \leq 18 \\
0 & \text{if } \bar{X} \geq 18
\end{cases}
\]

\[
M_k(\bar{X}) = \begin{cases} 
10 & \text{if } \bar{X} \leq 9 \\
19 - \bar{X} & \text{if } 9 \leq \bar{X} \leq 19 \\
0 & \text{if } \bar{X} \geq 19
\end{cases}
\]

\( k = 1, 2 \)
Example: symmetric duopoly

By symmetry, \( m(\bar{X}) = 2m_1(\bar{X}) \) and \( M(\bar{X}) = 2M_1(\bar{X}) \)
Example: symmetric duopoly

General duopoly

(i) \( \bar{x}_k = 0, \) if \( \frac{A-d_k-\gamma_k}{B} \leq \bar{x}_l; \)

(ii) \( 0 < \bar{x}_k < L_k, \) if \( \frac{A-d_k-\gamma_k}{B} - 2\bar{x}_k \leq \bar{x}_l \leq \frac{A-d_k}{B} - 2\bar{x}_k; \)

(iii) \( \bar{x}_k = L_k, \) if \( \bar{x}_l \leq \frac{A-d_k}{B} - 2L_k \)

with \( k = 1, 2 \) and \( l \neq k. \)

In the case of the previous example

\( \bar{x}_k = 0, \) if \( 18 \leq \bar{x}_l; \)
\( 0 < \bar{x}_k < 10, \) if \( 18 - 2\bar{x}_k \leq \bar{x}_l \leq 19 - 2\bar{x}_k; \)
\( \bar{x}_k = 10, \) if \( \bar{x}_l \leq -1 \)
Example: symmetric duopoly

General duopoly

(i) \( \bar{x}_k = 0 \), if \( \frac{A-d_k-\gamma_k}{B} \leq \bar{x}_l \);

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(iii) \( \bar{x}_k = L_k \), if \( \bar{x}_l \leq \frac{A-d_k}{B} - 2L_k \)

with \( k = 1, 2 \) and \( l \neq k \).

In the case of the previous example

\( \bar{x}_k = 0 \), if \( 18 \leq \bar{x}_l \);

\( 0 < \bar{x}_k < 10 \), if \( 18 - 2\bar{x}_k \leq \bar{x}_l \leq 19 - 2\bar{x}_k \);

\( \bar{x}_k = 10 \), if \( \bar{x}_l \leq -1 \)
Set of steady states for Example 1

It can be proved that all the steady states are internal

\[ x_1 - \bar{x}_1 = \frac{18 - x_1}{19^2} \]

\[ 18 - 2\bar{x}_1 \]

\[ 19 - 2\bar{x}_1 \]
Asymptotic behavior
Asymptotic behavior

Nonempty simplex with usually infinitely many points → there is no reason to examine analytically local or global asymptotical stability:

If $\bar{x}$ is a steady state and the initial state of the system is selected in its neighborhood as another steady state, then the trajectory will stay there for all $t > 0$, so it does not converge back to $\bar{x}$.

The asymptotic properties of the system are therefore examined by using computer simulation.

- semisymmetric case of $N$ firms ($N > 1$)
- $p(X) = 20 - 2X$,
- cost functions:
  - $C_k(x_k) = x_k$, for $k = 1, 2, \ldots, N - 1$  
  - $C_N(x_N) = 2x_N$, for $N$-th firm
- $\gamma_k = 1$
- $L_k = 10$
- identical initial output quantities. for firms $k = 1, 2, \ldots, N - 1$
Asymptotic behavior

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- identical initial output quantities. for firms \( k = 1, 2, \ldots, N - 1 \)
The case of $N = 4$

- $K = 0.2$: steady state $A$
- $K = 0.92$: 2-cycle $B_1 - B_2$
Basins with different values of parameter $K$

(a) $K \approx 0.8352941$

(b) $K \approx 0.835295$

(c) $K \approx 0.86$

(d) $K \approx 1.0$
The case of $N = 9$

- $K = 0.07$: steady state $A$
- $K = 0.52$: 2-cycle $B_1 - B_2$
- $K = 0.696$: steady state $C$
- $K = 0.92$: 2-cycle $D_1 - D_2$
The case of $N = 12$

- $K = 0.30$, steady state: $A$
- $K = 0.40$, 2-cycle: $B_1 - B_2$
- $K = 0.45$, 4-cycle: $C_1 - C_4$
- $K = 0.66$, chaotic trajectory
- $K = 0.72$, 2-cycle: $E_1 - E_2$
- $K = 0.92$, 4-cycle: $F_1 - F_4$
Bifurcation diagrams with $N = 13, 20$

(a) $N = 13$

(b) $N = 20$
Unemployment Insurance Systems
We assume that each firm $k$ pays a certain proportion of the lost wages to the unemployed workers:

- with all the workers employed, each firm would be able to produce the maximum amount $L_k$
- the number of unemployed workers is proportional to the output difference $L_k - x_k$
- the total amount of unemployed compensation is also proportional to $L_k - x_k$.

So the profit of firm $k$ can be formulated as

$$\Pi_k = x_k (A - Bx_k - BX_k) - (c_k + d_kx_k) - s_k (L_k - x_k).$$
Unemployment Insurance Systems

Again semisymmetric case

- firms 1, 2, \ldots, N - 1

\[ c_k \equiv c, \quad d_k \equiv d, \quad s_k \equiv s, \quad L_k \equiv L, \quad K_k \equiv K \]

- firm N

\[ c_N = \bar{c}, \quad d_N = \bar{d}, \quad s_N = \bar{s}, \quad L_N = \bar{L}, \quad K_N = \bar{K} \]

In this case the dynamic behavior of the firms can be described by the two-dimensional system

\[
\begin{align*}
    x(t+1) &= x(t) + K \left( -\frac{1}{2} (y(t) + (N - 2) x(t)) + \frac{A-d+s}{2B} - x(t) \right) \\
y(t+1) &= y(t) + \bar{K} \left( -\frac{1}{2} (N - 1) x(t) + \frac{A-\bar{d}+\bar{s}}{2B} - y(t) \right)
\end{align*}
\]

by assuming interior best responses.

This model is equivalent to the well known semisymmetric linear oligopoly model.
Again semisymmetric case
- firms $1, 2, \ldots, N - 1$

\[
c_k \equiv c, \; d_k \equiv d, \; s_k \equiv s, \; L_k \equiv L, \; K_k \equiv K
\]

- firm $N$

\[
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Again semisymmetric case

- firms 1, 2, . . . , N − 1

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- firm N

\[ c_N = \bar{c}, \ d_N = \bar{d}, \ s_N = \bar{s}, \ L_N = \bar{L}, \ K_N = \bar{K} \]

In this case the dynamic behavior of the firms can be described by the two-dimensional system

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\end{align*}
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by assuming interior best responses.

This model is equivalent to the well known semisymmetric linear oligopoly model.
Unemployment Insurance Systems

- $N = 2$: the system is asymptotically stable
- $N = 3$: $0 < K, \bar{K} \leq 1$ the system is asymptotically stable (if $K = \bar{K} = 1$, then the steady state is marginally stable)
- $N \geq 4$, the condition for asymptotical stability is

$$K < \frac{16 - 8\bar{K}}{4N - \bar{K}(N + 1)}$$
since the system is linear the asymptotical stability is global
Conclusion
contingent workforce
- allows greater flexibility to the firms
- more complex dynamics for higher values of adjustment speeds
- may be unstable
- the dynamics becomes complex with increasing adjustment costs, since the flexibility given by the contingent workforce is damped by the searching and training costs

unemployment insurance system
- simpler dynamics
- no cycles

maybe relying too much on contingent workforce is not such a great idea
Conclusion

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