A dynamic marketing model with best reply and inertia

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The Farris’s Market Model


\[ x_i(t + 1) = (1 - \lambda_i)x_i(t) + \lambda_i \left( \sqrt{B \frac{\sum_{j \neq i} a_j x_j(t)}{a_i}} - \sum_{j \neq i} a_j x_j(t) \right) \]
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It results from standard arguments in marketing modelling:

*customers’ attraction:* \( A_i = a_i x_i^{\beta_i} \)
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profits: \( \Pi_i(t) = Bs_i(t) - x_i(t) \)
The Farris’ Market Model

The resulting profits are

$$\Pi_i(t) = \frac{a_i x_i^{\beta_i}(t)}{a_i x_i^{\beta_i}(t) + \sum_{j \neq i} a_j x_j^{\beta_j}(t)} - x_i(t)$$
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Each firm maximizes its own profit function computing its gradient

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Setting $\beta_i = 1$, this leads to

$$x_i(t + 1) = \sqrt{\frac{\sum_{j \neq i} a_j x_j^{(e)}(t + 1)}{a_i} - \sum_{j \neq i} a_j x_j^{(e)}(t + 1)}$$
The Farris’ Market Model

Assuming naïve expectations, \( x_j^{(e)}(t + 1) := x_j(t) \), Farris et al. derive the following “Best Response” dynamic model with adaptive adjustment:

\[
x_i(t + 1) = (1 - \lambda_i)x_i(t) + \lambda_i \left( \sqrt{\frac{\sum_{j \neq i} a_j x_j(t)}{a_i}} - \sum_{j \neq i} a_j x_j(t) \right)
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\]

Setting \( N = 2 \) and the new (rescaled) variables

\[
x = a_1 a_2 x_1, \quad y = a_1 a_2 x_2
\]

we have

\[
Farris: \quad \begin{cases} 
  x' = (1 - \lambda_1)x + \lambda_1 a_2 \left( \sqrt{y} - y \right) \\
  y' = (1 - \lambda_2)y + \lambda_2 a_1 \left( \sqrt{x} - x \right)
\end{cases}
\]
Similarities with the Puu’ model

\[
\begin{align*}
Puu & : \\
q_1' &= (1 - \lambda_1) q_1 + \lambda_1 \left( \sqrt{\frac{q_2}{c_1}} - q_2 \right) \\
q_2' &= (1 - \lambda_2) q_2 + \lambda_2 \left( \sqrt{\frac{q_1}{c_2}} - q_1 \right)
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\text{new var} & : x = c_2 q_1, \quad y = c_1 q_2
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\end{align*} \]

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Similarities with the Puu’ model

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\text{new var} : & \quad x = c_2 q_1, \quad y = c_1 q_2
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\text{Puu'} : & \quad x' = (1 - \lambda_1)x + \lambda_1 \frac{c_2}{c_1} (\sqrt{y} - y) \\
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\]

We obtain the \textit{Puu’} from the \textit{Farris} setting \( a_2 = 1/a_1 \).
The equilibria’ abscissas follows from the fourth order algebraic equation:

\[
\eta \left[ \frac{(1 - a_1 a_2)^2}{a_1 a_2} \eta^3 + 2 (1 - a_1 a_2) \eta^2 + a_2 (a_1 + 1) \eta - a_2 \right] = 0
\]

where \( \eta := \sqrt{x} \). An analogous equation for \( \zeta := \sqrt{y} \) holds.

From Cardano’s formula, the number of real solutions is given provided the sign of the discriminant:

\[
D(a_1, a_2) : \begin{cases} 
> 0 & 1 \text{ real solution} \\
= 0 & 1 \text{ real solution and 2 coincident} \\
< 0 & 3 \text{ distinct real solutions}
\end{cases}
\]
Equilibria

Section of the function $D(a_1, a_2)$ with the plane $(a_1, a_2, 0)$
Section of the function $D(a_1, a_2)$ with the plane $(x, y, 0)$

$\begin{align*}
D(a_1, a_2) < 0 \\
(3 \text{ sol.})
\end{align*}$

$\begin{align*}
D(a_1, a_2) = 0 \\
(1 + 2 \text{ sol.})
\end{align*}$

$\begin{align*}
D(a_1, a_2) > 0 \\
(1 \text{ sol.})
\end{align*}$

$a_2 = 1/a_1$
The General case $a_1 \neq a_2$

**Proposition**

Besides $E_0 = (0, 0)$ a non vanishing fixed point always exists in the region $S = (0, 1) \times (0, 1)$. If $a_1 a_2 \neq 1$ then two further distinct fixed points exist in the region $S$ if the following inequality holds

$$D(a_1, a_2) = \frac{a_1^2 a_2^4}{108 (1 - a_1 a_2)^6} \left[ 27 + a_1 a_2 (4a_1 + 4a_2 - 18) - a_1^2 a_2^2 \right] < 0$$

and if $D(a_1, a_2) = 0$ these two further fixed points are merging, i.e. there are two real coincident solutions of the cubic equation. In the particular case $a_1 a_2 = 1$ the unique fixed point $E = \left( \frac{1}{(a_1+1)^2}, \frac{1}{(a_2+1)^2} \right)$ is get.
Bifurcation path: $a_2 = -a_1/9 + 13/3$
Bifurcation path: \( a_2 = a_1 + 0.5 \)
Bifurcation path: $a_2 = a_1 + 0.5$, $a_2 = a_1$ (gray)
Typical scenario (after the final bifurcation)
The symmetric case: \( a = a_1 = a_2 \)

Fixed points are:

\[
E_1 = \left( \frac{a^2}{(1 + a)^2}, \frac{a^2}{(1 + a)^2} \right) \in \Delta = \{(x, y) \in \mathbb{R}^2 | x = y\}
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For $a \geq 3$ the two further fixed points in symmetric positions with respect to $\Delta$:

$$E_2 = \frac{a^2}{2(a - 1)^2(a + 1)} \left( a - 1 + \sqrt{(a + 1)(a - 3)}, a - 1 - \sqrt{(a + 1)(a - 3)} \right)$$

$$E_3 = \frac{a^2}{2(a - 1)^2(a + 1)} \left( a - 1 - \sqrt{(a + 1)(a - 3)}, a - 1 + \sqrt{(a + 1)(a - 3)} \right)$$
The symmetric case: stability of $E_1 \in \Delta$

Local asymptotic stability for

$$a < a_p = 3$$
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Local asymptotic stability for

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Transverse instability (saddle) via pichfork, merging of two further fixed points $E_2$ and $E_3$, for

$$a \geq a_p$$
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Unstability via flip for

$$a \geq a_f = 1 + 2 \sqrt{1 - 2 \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) + \frac{4}{\lambda_1 \lambda_2}} \geq 3$$
Fixed points and their basin of attraction

(a) Codim. 2 bifurcation

(b) Just after the pichfork

(c) Just after the flip

(d) Just after the subcritical flip
The symmetric case: stability of $E_2$ and $E_3$

Local asymptotic stability for

$$a < a_h = 1 + \sqrt[3]{2 \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) + 2 \sqrt{\left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right)^2 - \frac{16}{27}}} + \sqrt[3]{2 \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) - 2 \sqrt{\left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right)^2 - \frac{16}{27}}}$$
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Unstability via Hopf for $a \geq a_h$
Profits

Computing the profits we get:

\[ \Pi_1 > \Pi_2 \iff (x - y)(x + y - a^2) < 0 \]
Conclusions: effect of heterogeneities

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  - Stress on the effect of heterogeneities (i.e. differences in parameters)
  - A rich dynamical scenario is observed
  - Heterogeneities and initial conditions: asymptotic behaviour of the system
Bibliography


