Recent Developments in Asset Pricing with Heterogeneous Beliefs and Adaptive Behaviour of Financial Markets

Tony He

UTS Business School
University of Technology Sydney

Asset pricing under heterogeneous beliefs

Aims of this survey

- To what extend the asset pricing models with adaptively heterogeneous beliefs can explain
  - complex market behaviour—asset bubbles and market crashes;
  - stylized facts—excess volatility, volatility clusterings and various power law behavior;
  - market anomalies and puzzles—momentum, value premium, portfolio performance, equity premium and risk-free rate puzzles.
- To establish CAPM and time-varying betas under heterogenous beliefs;
- To provide a unified asset pricing framework for HAMs in continuous-time.
Outline

A HAM of Single Risky Asset and Its Estimation

CAPM under Heterogeneous Beliefs and Time-Varying Beta

HAMs in Continuous-Time

Disagreement, Market Efficiency and Characteristics

Summary and Challenges


Single Risk Asset HAMs

- One risk free asset with gross return $R = 1 + r/K$;
- $p_t$ price and $d_t$ dividend of the risky asset;
- $H$ types of traders: $q_{h,t}$, $h = 1, 2, \cdots, H$.
- $R_{t+1} := p_{t+1} + d_{t+1} - Rp_t$; $W_{h,t}$ be agent’s wealth;
- $E_{h,t}$ and $V_{h,t}$: the conditional expectation and variance of type $h$ agents.
- $\max_z U_h(W) = -\exp(-a_hW)$ leads to $z_{h,t} = E_{h,t}(R_{t+1})/(a_h V_{h,t}(R_{t+1}))$.
- The population weighted aggregate demand $z_{e,t} \equiv \sum_{h=1}^{H} q_{h,t} z_{h,t}$.
Asset Pricing and Market Fractions

- Equilibrium—Walrasian scenario, Brock and Hommes (1998):

\[ z_{e,t} + \tilde{\delta}_t = z_s, \]

- Partial equilibrium—market maker scenario, Chiarella and He (2003):

\[ p_{t+1} = p_t + \mu [z_{e,t} - z_s] + \tilde{\delta}_t, \]

Dieci et al (2006) model

- Two types: the fundamentalists and trend followers;
- The fixed proportions: $n_1$ and $n_2$;
  \[ n_0 := n_1 + n_2, \quad m_0 = (n_1 - n_2)/n_0; \]
- The switching proportions: $n_{1,t}$ and $n_{2,t} = 1 - n_{1,t}$ among $1 - (n_1 + n_2)$; $m_t := n_{1,t} - n_{2,t}$
- The market fractions

\[
q_{1,t} = \frac{1}{2} \left[ n_0 (1 + m_0) + (1 - n_0)(1 + m_t) \right],
\]
\[
q_{2,t} = \frac{1}{2} \left[ n_0 (1 - m_0) + (1 - n_0)(1 - m_t) \right].
\]
The fundamentalists:

\[ E_{1,t} (p_{t+1}) = p_t + (1 - \alpha)(p_{t+1}^* - p_t), \quad V_{1,t} (p_{t+1}) = \sigma_1^2. \]

The trend followers:

\[ E_{2,t} (p_{t+1}) = p_t + \gamma (p_t - u_t), \quad V_{2,t} (p_{t+1}) = \sigma_1^2 + b_2 v_t, \]

\( u_t \) and \( v_t \) are sample mean and variance, respectively, following the geometric decaying process

\[ u_t = \delta u_{t-1} + (1 - \delta) p_t, \quad v_t = \delta v_{t-1} + \delta (1 - \delta) (p_t - u_{t-1})^2. \]

\( b_2 \geq 0 \) measures the sensitive to the sample variance.
Adaptive Switching: He and Li (2011)

- Discrete-choice model:
  \[
  n_{h,t+1} = \frac{\exp[\beta (\pi_{h,t+1} - C_h)]}{\sum_i \exp[\beta (\pi_{i,t+1} - C_i)]}, \quad h = 1, 2.
  \]

- the market fractions and asset price dynamics
  \[
  p_{t+1} = p_t + \mu (q_{1,t} z_{1,t} + q_{2,t} z_{2,t}) + \tilde{\delta}_t, \quad \tilde{\delta}_t \sim \mathcal{N}(0, \sigma^2_\delta),
  \]
  \[
  u_t = \delta u_{t-1} + (1 - \delta) p_t,
  \]
  \[
  v_t = \delta v_{t-1} + \delta (1 - \delta) (p_t - u_{t-1})^2,
  \]
  \[
  m_t = \tanh \left\{ \frac{\beta}{2} [(z_{1,t-1} - z_{2,t-1}) (p_t + d_t - Rp_{t-1}) - (C_1 - C_2)] \right\}.
  \]
Estimation: He and Li (2012)

- The fundamental price:
  \[ p_{t+1}^* = p_t^* \exp\left(-\frac{\sigma^2}{2} + \sigma\epsilon\tilde{\epsilon}_t\right), \quad p_0^* = \bar{p} > 0, \]

- Estimation of the power-law behavior of the DAX 30:

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \gamma )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( \mu )</th>
<th>( n_0 )</th>
<th>( m_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.488</td>
<td>1.978</td>
<td>7.298</td>
<td>0.320</td>
<td>1.866</td>
<td>0.313</td>
<td>-0.024</td>
</tr>
<tr>
<td>( \delta )</td>
<td>( b )</td>
<td>( \sigma )</td>
<td>( \sigma\delta )</td>
<td>( \beta )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.983</td>
<td>3.537</td>
<td>0.231</td>
<td>3.205</td>
<td>0.954</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure: (a) Autocorrelations of $r_t$, $r_t^2$ and $|r_t|$ for the model. (b) The ACs for the estimated model and the DAX 30.
The CAPM assumes that all agents have the same expectations about the means, variances and covariances of future returns, and hence the beta of the CAPM is assumed to be constant over time and is estimated via ordinary least squares (OLS).

Models: CAPM and time-varying betas

Provide a behavioural explanation for time-varying betas, by incorporating the two most commonly used types of agents, fundamentalists and chartists, into the model;

Show that there is a systematic change in the market portfolio, risk-return relationships, and time varying betas when agents change their behaviour.

The commonly used rolling window estimates of time-varying betas may not be consistent with the ex-ante betas.
Portfolio problem

- Wealth:
  \[
  W_{i,t+1} = z_i^T (p_{t+1} + d_{t+1}) + (1 + r_{f,t})[W_{i,t} - c_{i,t} - z_i^T p_t] \\
  = \zeta_i^T [r_{t+1} - r_{f,t} 1] + (1 + r_{f,t})[W_{i,t} - c_{i,t}],
  \]

- Portfolio problem:
  \[
  \max_{c_{i,t}, \zeta_{i,t}} \left[ E_i(t(u_i(c_{i,t}))) + \beta E_i(t(u_i(W_{i,t+1}))) \right]
  \]
The optimal consumption rule

\[ c_{i,t} = \frac{1}{2 + r_{f,t}} \left[ (1 + r_{f,t}) W_{i,t} - \frac{1}{\theta_i} \ln(\beta(1 + r_{f,t})) \right. \]

\[ + \left. \frac{1}{2\theta_i} (E_{i,t}(r_{t+1}) - r_{f,t}\mathbf{1})^T \Omega^{-1}_{i,t}(E_{i,t}(r_{t+1}) - r_{f,t}\mathbf{1}) \right] \]

and optimal portfolio

\[ \zeta_{i,t} = \frac{1}{\theta_i} \Omega^{-1}_{i,t}(E_{i,t}(r_{t+1}) - r_{f,t}\mathbf{1}). \]
Market Equilibrium

- The (dollar) demand vector of the risky assets at time $t$ is given by

$$\zeta_t := \sum_{h=1}^{H} n_h \zeta_{h,t} = \sum_{h=1}^{H} n_h \theta_{h}^{-1} \Omega_{h,t}^{-1} [E_{h,t}(r_{t+1}) - r_{f,t} 1].$$

- The market clearing condition becomes

$$Sp_t = \sum_{h=1}^{H} n_h \theta_{h}^{-1} \Omega_{h,t}^{-1} [E_{h,t}(r_{t+1}) - r_{f,t} 1] + \xi_t.$$  \hspace{1cm} (1)
The Consensus Belief

- The “average” risk aversion coefficient

\[ \theta_a := \left( \sum_{h=1}^{H} n_h \theta_h^{-1} \right)^{-1} \]

- The aggregate beliefs:

\[ \Omega_{a,t} = \theta_a^{-1} \left( \sum_{h=1}^{H} n_h \theta_h^{-1} \Omega_{h,t}^{-1} \right)^{-1} \]

\[ E_{a,t}(r_{t+1}) = \theta_a \Omega_{a,t} \sum_{h=1}^{H} n_h \theta_h^{-1} \Omega_{h,t}^{-1} E_{h,t}(r_{t+1}). \]
The Equilibrium

- The market equilibrium prices
  \[ p_t = S^{-1}\theta_a^{-1}\Omega_{a,t}^{-1}[E_{a,t}(r_{t+1}) - r_{f,t}1] + S^{-1}\xi_t. \] (2)

- The equilibrium risk-free rate
  \[ r_{f,t} = \frac{(E_{a,t}(r_{t+1}))^T\Omega_{a,t}^{-1}1 - \theta_a(W_m,t - c_t)/I}{1^T\Omega_{a,t}^{-1}1}. \] (3)

- A CAPM under heterogeneous beliefs,
  \[ E_{a,t}(r_{t+1}) - r_{f,t}1 = \beta_{a,t}[E_{a,t}(r_{m,t+1}) - r_{f,t}], \]

- The ex-ante “aggregate” beta
  \[ \beta_{a,t} = \frac{[E_{a,t}(r_{t+1}) - r_{f,t}1]^T\Omega_{a,t}^{-1}1}{[E_{a,t}(r_{t+1}) - r_{f,t}1]^T\Omega_{a,t}^{-1}[E_{a,t}(r_{t+1}) - r_{f,t}1]}[E_{a,t}(r_{t+1}) - r_{f,t}1]. \]
The Heterogeneous Beliefs

- Fundamentalists:

\[ E_{f,t}(r_{t+1}) = E_{f,t}(r_{t+1}) + \alpha P_t^{*-1}(p_t^* - p_{t-1}), \quad \Omega_f = \Omega_0. \]

The fundamental price \( p_{j,t}^* \)

\[ p_{t+1}^* = p_t^* + \epsilon_{t+1}, \quad E_{f,t}(\epsilon_{t+1}) = 0. \]

The dividend yield \( \rho_{j,t+1} = d_{j,t+1}/p_{j,t}^* \sim \mathcal{N}(\rho_j). \)

\[ E_{f,t}(r_{t+1}) = \rho + \alpha(1 - P_t^{*-1}p_{t-1}). \]

- The trend followers:

\[ E_{c,t}(r_{t+1}) = u_{t-1}, \quad u_{t-1} = \delta u_{t-2} + (1 - \delta)r_{t-1} \]

\[ \Omega_{c,t} = \Omega_0 + \lambda V_{t-1}. \]

\[ V_{t-1} = \delta V_{t-2} + \delta(1 - \delta)(r_{t-1} - u_{t-2})(r_{t-1} - u_{t-2})^\top. \]
Figure: The fluctuations of prices in the deterministic model for $\delta = 0.865$ (left panel) and $\delta = 0.840$ (right panel).
Figure: The equilibrium prices and the market portfolio proportions.
Consistency of Beta

**Figure:** The ex-ante and rolling estimates of the betas.
Chiarella, C., R. Dieci, X. He and K. Li (2012), An evolutionary CAPM under heterogeneous beliefs, working paper:

- Adaptive switching;
- Diversification effect and nonlinear spill-over effect;
- Time-varying betas;
- Volatility and trading volumes, and their ACs.
Limitations of HAMs in Discrete-Time

- *Wildness of heterogeneity:* Faces a limitation when dealing with expectations formed from lagged prices over different time horizons.

- Different dimensions of the systems need to be analyzed individually, in particular, when the time horizon of historical information used is long, the resulting models are high dimensional systems.
Continuous-time HAMs

- the time horizon of historical price information used by chartists is simply presented by a time delay.
- leading to systems of delay differential equations;
- provides an uniform treatment on various time horizons used in the discrete-time model.

Models: HAMs in Continuous-Time

- He, X. and Li, K. (2012), Heterogeneous beliefs and adaptive behaviour in a continuous-time asset price model, *Journal of Economic Dynamics and Control*
The Heterogeneous Demand

- The demand of the fundamentalists:
  \[ Z_f(t) = \beta_f [F(t) - P(t)]. \]

- The demand of the chartists
  \[ Z_c(t) = \tanh (\beta_c [P(t) - u(t)]). \]
  \[ u(t) = \frac{k}{1 - e^{-k\tau}} \int_{t-\tau}^{t} e^{-k(t-s)}P(s)ds, \]
  where \( \tau \in (0, \infty), k > 0. \)
The Price Trend

- When $k \to 0$,
  \[
  u(t) = \frac{1}{\tau} \int_{t-\tau}^{t} P(s) ds,
  \]

- When $\tau \to \infty$,
  \[
  u(t) = \frac{1}{k} \int_{-\infty}^{t} e^{-k(t-s)} P(s) ds,
  \]

- For $0 < k < \infty$,
  \[
  du(t) = \frac{k}{1 - e^{-k\tau}} \left[ P(t) - e^{-k\tau} P(t - \tau) - (1 - e^{-k\tau}) u(t) \right] dt.
  \]
The Performance Measure

Population fractions:

\[ n_f(t) = \frac{N_f(t)}{N}, \quad n_c(t) = \frac{N_c(t)}{N}. \]

The net profits over a short time interval \([t - dt, t]\):

\[ \pi_f(t) dt = Z_f(t) dP(t) - C_f dt, \quad \pi_c(t) dt = Z_c(t) dP(t) - C_c dt. \]

The performance measure—a cumulated and weighted profit over a time interval \([t - \tau, t]\) by

\[ U_i(t) = \frac{\eta}{1 - e^{-\eta \tau}} \int_{t-\tau}^{t} e^{-\eta(t-s)} \pi_i(s) ds, \quad i = f, c, \]

Consequently,

\[ dU_i(t) = \eta \left[ \frac{\pi_i(t) - e^{-\eta \tau} \pi_i(t - \tau)}{1 - e^{-\eta \tau}} - U_i(t) \right] dt, \quad i = f, c. \]
The Adaptive Switching

- The evolution dynamics of the market populations—replicator dynamics

\[
dn_i(t) = \beta n_i(t)[dU_i(t) - d\bar{U}(t)], \quad i = f, c, \quad (6)
\]

where

\[
d\bar{U}(t) = n_f(t)dU_f(t) + n_c(t)dU_c(t)
\]

is the average performance of the two strategies and \( \beta > 0 \) is a constant, measuring the intensity of choice.

- Consistent to the discrete choice model:

\[
dn_f(t) = \beta n_f(t)(1 - n_f(t))[dU_f(t) - dU_c(t)], \quad (7)
\]

leading to

\[
n_f(t) = \frac{e^{\beta U_f(t)}}{e^{\beta U_f(t)} + e^{\beta U_c(t)}}, \quad (8)
\]
The market maker according to the aggregate market excess demand,

$$dP(t) = \mu [n_f(t)Z_f(t) + n_c(t)Z_c(t)] dt + \sigma_M dW_M(t),$$

The fundamental steady state is stable for either small or large time delay when the market is dominated by the fundamentalists measured by $\gamma_f$ and the decay parameter $k$.

Otherwise, the fundamental steady state becomes unstable through Hopf bifurcations when time delay increases.

A very interesting phenomenon of the continuous time model—the system becomes unstable as time delay increases initially, but the stability can be recovered when the time delay becomes large enough.
Figure: The bifurcation and the market price for $\tau = 3$ and 16.
Rational Routes to Randomness and Excess Volatility

Figure: The bifurcation in $\beta$, the market price and population.
Stochastic Price Dynamics

- When the fundamental steady state is stable, the market price follows the fundamental price closely and there is no significant difference for the market prices with and without switching.

- When the fundamental steady state is unstable, the market price fluctuates around the fundamental price in cyclic way, and the price fluctuations of the stochastic model are high with switching.
Figure: The stochastic price dynamics underlying by the deterministic dynamics.
The Stylized Facts—Volatility Clustering

Figure: The market price and return.
Figure: The return distribution and the ACs of returns.
Figure: The ACs of the absolute and squared returns.
Figure: The ACs of the absolute and squared returns: with and without switching.
A growing literature on the behaviour of equilibrium asset prices under heterogeneous beliefs

- Resolving the equity premium puzzle and the risk-free rate puzzle, excess volatility puzzle, term structure of interest rates, trading volume of stocks and options, the over(under)-reaction and momentum and the survival of the irrational agents: Basak (2000), Hong and Stein (2007), Cao and Ou-Yang (2009), Xiong and Yan (2010).
Miller’s hypothesis and cross-sectional returns

- Miller’s hypothesis: dispersion in agents’ beliefs is not a risky factor.
Related Papers

With heterogeneous agents, the optimal portfolios are MV inefficient in general.

The traditional geometric relation of the MV frontiers with and without the riskless asset under homogeneous beliefs does not hold under heterogeneous beliefs.

Provide some explanations on the risk premium puzzle, Miller’s hypothesis, and under-performance of managed funds.

Diversification effect of the heterogeneous beliefs—Diversity in beliefs can improve the Sharpe and Treynor ratios of the market portfolio and the optimal portfolios of agents.
Disagreement and Risk Premium

- With the MPS, agents perceive the objective belief on average.
- There is a spill-over effect of dispersion in beliefs.
- A positive correlation between optimism and the belief about the correlation between asset return dramatically increases the market risk premium and decreases the risk-free rate.
- This is the case even when the market is optimistic and confident.
- Also, a positive correlation between risk tolerance and pessimism/doubt is no longer necessary to generate a high market risk premium and a low risk-free rate.
<table>
<thead>
<tr>
<th></th>
<th>BM</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{E}[r_m - r_f]$</td>
<td>1.49%</td>
<td>4.49%</td>
<td>1.75%</td>
<td>4.00%</td>
<td>6.09%</td>
</tr>
<tr>
<td>$\mathbb{E}_a[r_m - r_f]$</td>
<td>1.49%</td>
<td>9.51%</td>
<td>1.75%</td>
<td>7.60%</td>
<td>7.76%</td>
</tr>
<tr>
<td>$r_f$</td>
<td>4.62%</td>
<td>3.46%</td>
<td>4.63%</td>
<td>3.81%</td>
<td>1.64%</td>
</tr>
<tr>
<td>$P$</td>
<td>0.00%</td>
<td>-5.02%</td>
<td>0.00%</td>
<td>-3.60%</td>
<td>-1.67%</td>
</tr>
<tr>
<td>$D$</td>
<td>0.00%</td>
<td>0.00%</td>
<td>-0.18%</td>
<td>-0.49%</td>
<td>-0.15%</td>
</tr>
<tr>
<td>$\pi_{m,2}$</td>
<td>0.038</td>
<td>0.65</td>
<td>0.127</td>
<td>0.602</td>
<td>0.575</td>
</tr>
<tr>
<td>$\Delta V$</td>
<td>0.00%</td>
<td>4.01%</td>
<td>0.32%</td>
<td>3.54%</td>
<td>3.29%</td>
</tr>
<tr>
<td>$\Delta R$</td>
<td>0.00%</td>
<td>1.84%</td>
<td>0.27%</td>
<td>1.69%</td>
<td>1.62%</td>
</tr>
</tbody>
</table>

**Table:** Effects of heterogeneity on the market risk premium $\mathbb{E}[r_m - r_f]$, the market risk premium under the consensus belief $\mathbb{E}_a[r_m - r_f]$, risk-free rate $r_f$, pessimism premium $P$, doubt premium $D$, the market portfolio $\pi_m$, changes in market variance $\Delta V$ and expected return $\Delta R$. 
Summary

- Surveys these developments, of which the author and several coauthors have contributed in several papers.
- The interaction of heterogeneous agents and the role of adaptive behaviour on asset pricing;
- The HAMs shed lights on the complex financial market behaviour.
- The continuous time HAMs provide a uniform approach in dealing the impact of price history.
- The dynamic CAPM can be used to explain empirical evidence on the time variation of beta.
- Heterogeneous beliefs can provides explanations to some market anomalies and puzzles.
Challenges

- To examine the spill-over, contagion, and diversification effects and to estimate the HAMs with multiple risky assets.
- To address issues on the profitability of momentum and contrarian strategies widely observed in empirical literature.
- To extend the analysis to an intertemporal model, incorporate expectations feedback mechanism into the beliefs, and study how learning and adaptive behaviour of heterogeneous agents contribute to the survivability of agents and market volatility.
- To deal with ambiguity and heterogeneity.
- To develop a general asset pricing theory to incorporate heterogeneity, adaptivity, and bounded rationality.